The Debt Capacity of a Government*

Bernard Dumas
INSEAD, NBER and CEPR

Paul Ehling
BI Norwegian Business School

Chunyu Yang
BI Norwegian Business School

October 3, 2023

Abstract

Fiscal costs of excessive government debt should be assessed dynamically reflecting anticipated growth, deficits and interest rates. Studying their joint behavior with overlapping generations and structural deficits through social security, we define debt capacity as the level of debt that can be just sustained without changing policy all the way to an unstable steady state, in which primary deficits last forever. Below capacity, debt converges to a stable steady state. Above capacity, debt unravels. Exceeding capacity without unraveling requires higher taxes (or reduced expenditures), which is the true "fiscal cost." With money, monetary policy and seigniorage alter capacity. When the government stimulates growth by subsidizing innovation, benefits hardly surpass costs.

---

*Adresses are: bernard.dumas@insead.edu, paul.ehling@bi.no, chunyu.yang@bi.no. We thank Nicola Borri (the discussant) and participants in the 2021 Franco-German Fiscal seminar as well as participants in a 2022 workshop at the Vienna School of Finance. The paper also benefited from discussions by Burton Hollifield at the 2022 Western Finance Association meeting, by Mindy Xiaolan at the 2022 China International Conference in Finance, by Stijn Van Nieuwerburgh at the 2022 European Finance Association meeting and by Vytautas Valaitis at the 2023 Oxford Saïd – ETH Zurich Macro-finance conference. We also received useful comments from Raman Uppal, Jean-Charles Rochet and Ming Yang.

"The sum of every series is the value of an expression which is defined by the process from which that series arises." Leonhard Euler, cited by G. C. Hardy (1973)

A debt crisis is a situation in which the financial market feels that the public debt is "out of control," and does not agree to roll it over. It is an unstable situation. Using a simple overlapping-generations (OLG) production economy, we show the existence, for a range of parameter values, of *two* steady states or "balanced growth paths." One is stable and the other unstable. Which steady state the economy goes to depends on initial conditions and the resulting transition paths. There is one saddle path emanating from a unique initial level of government debt that leads to the unstable steady state with a constant policy regarding taxation and expenditures. We call "debt capacity" the threshold level at the initial point in time. When debt starts above debt capacity, a crisis situation obtains. *The unstable steady state is the foundation of our notions of debt capacity and debt sustainability.* The other steady state is stable; for a given initial capital stock, many paths that start below debt capacity lead to it. In other words, we elucidate the policy meaning of the unstable steady state while few macroeconomic models incorporate the presence of an unstable steady state and, in practically all financial-equilibrium models, assumptions are made that bring about stability. In our model, the unstable steady state plays centerstage.

In our model, for some range of parameter values, the government budget can be perpetually in deficit without a crisis occurring. In the "high-income" countries of today, the real interest rate is below the growth rate while it is likely that their future primary budgets will remain in deficit.[1] Yet the prices of government bonds in the financial market remain finite and positive. We show that, in the presence of a primary budget perpetually in deficit, the endogenously determined equilibrium rate of interest can forever remain lower than the rate of growth, and that the debt per capita in an OLG model can be positive and finite at all times so that it can truly be rolled over eternally. This seems to correspond to the current situation.

Over the last seventy-five years, the debt of high-income nations has mostly increased, while prospects for government surpluses are dimming.[2] Specifically, the 2019 Long-Term Budget Outlook by the U.S. Congressional Budget Office predicts steady deficits, if not rising ones, for thirty years. In the same vein, the slowdown or even decline in population growth (Jones (2020)) as well the reduction in technological progress mirrored in reduced GDP growth rates would seem to make it unlikely that successful debt reduction is possible in the foreseeable future.[3]

Working in a similar framework, Blanchard (2019) suggests that, in the current situation of low interest rates, an increase of government debt may be feasible and beneficial.[4] He writes: "If the interest rate paid by the government is less than the growth rate, then the intertemporal budget constraint facing the government no longer binds. What the government can and should do in this case is definitely worth exploring."

One important aspect that Blanchard did not delve into is how deeply the government can be in debt. We show that, even when the financial market agrees to the debt rollover sequence that supports a positive value of the debt, the amount of debt that can be placed in the market is not infinite; there is still an upper bound on it, which is the debt capacity. For a level of debt above capacity, the debt is unsustainable (unless the government promises to change its policy and the promise is believed). Correspondingly, government expenditures are limited in size.

We think that this observation indicates that initial conditions and transition paths should be studied. Others also point in that direction. In a recent talk, Sims (2020) refers to debt issuance when the rate of

[1]While there is some uncertainty about whether riskless interest rates are below the growth rate, Blanchard (2019) contains strong and robust evidence in favor of the view that interest rates are below growth. For instance, in the United States since 1950 even the nominal 10-year rate with an average of 5.6 percent is below the nominal GDP growth with an average of 6.3 percent.

[2]Historically, sovereigns have borrowed to finance wars. Only in the 19th century governments started to systemically borrow to build ports, railways, roads, schools and universities. In their history of debt, Eichengreen, El-Ganainy, Esteves, and Mitchener (2019) mention only three successful debt reduction episodes: Great Britain after the Napoleonic Wars, the United States in the last third of the 19th century, and France in the decades leading up to 1913.

[3]Only a few countries seem to defy this trend. With its "debt brake" constitutional amendment, Germany is actively doing that: "The federal and state budgets shall in general be balanced without proceeds from borrowing. The federal and state governments can provide for rules to take into account the effects of deviations from normal cyclical developments, as well as a derogation for natural disasters or exceptional emergency situations that are beyond the control of the state and significantly affect the state's financial situation. A corresponding repayment plan must be provided for any derogation." *Constitution of the Federal Republic of Germany*, Article 109, Section 3.

[4]See also Furman and Summers (2020). Even after the rise in nominal interest rates that occurred in 2022-2023, high inflation causes these to remain smaller than the growth rate of nominal GDP.

interest is below the rate of growth as "zero fiscal cost debt." Yet, he is careful to state: "When the real interest rate on debt is below the growth rate of the economy, the government can issue and roll over debt forever without backing it by new taxes and still see the debt-to-GDP ratio shrink over time. But this only applies when the government makes a one-time increase in debt unaccompanied by increased taxation. This is not the situation we're in, even when it concerns pandemic relief spending. Given past, steady spending increases without tax increases, we unfortunately cannot view pandemic relief spending as a single, wartime increase in debt. Ultimately, these steady increases will affect the interest rate on debt and require dynamic solutions." In other words, the zero fiscal cost of today may foreshadow a much larger one later. The true fiscal cost of excessive government debt issuance cannot be assessed just from the current rate of interest or any current macroeconomic variable, or on the basis of an exogenous future path of the rate of interest. Rather, it should be assessed in a dynamic context reflecting anticipated deficits and growth of population (plus labor augmenting progress), going forward. For that, one must determine the future path of the rate of interest and one needs a dynamic model of growth and capital accumulation, with the interest rate being affected by the government primary deficit/surplus directly or indirectly. Our model articulates that link.

We study debt sustainability and capacity with a primary government deficit that varies with the growth of the economy. Extant models of the same kind assume a fixed path for the deficit. For purposes of illustration, we fix the relation that ties the budget to the economy by assuming that it arises from a social-security scheme; it is then driven by the fundamentals of the economy, namely, the terms of the scheme, population growth, preferences and production technology. We justify that form of the deficit on the grounds that it is welfare improving. Our analysis, however, is not limited to that illustration. We contribute fiscal and monetary policy experiments illuminating the importance of financial-market trust in the government and the belief that public debt is not out of control. We also consider the role of the government in stimulating the growth rate through innovation.

Since humans die while governments live for many generations, it seems natural to accomplish that task by means of an OLG model with finite lives of households. Its initiator, Samuelson (1958), had already indicated that the "social contrivance of money" was welfare improving because it bridged the non coincidence of needs of different generations. Wallace (1980) mused that "neither [Samuelson] nor most economists seem to take it seriously as a model of fiat money."[5] The reason was that money is needed obviously as an every-day device, not one used across generations as a form of social security.[6] But an application of the very same argument to government debt (alongside money or without it) seems vividly relevant and realistic. Along that line, Diamond (1965), in a production-economy model, sets the debt per capita to be equal to a constant and introduces the lump-sum taxes needed to pay for the cost of financing the budget. The stable steady state that he obtains is not "efficient" (i.e., welfare maximizing) for the well-known reason that each generation, in order to finance their retirement, saves in excess of what they would if the welfare of all generations were optimized: capital is being overaccumulated relative to the Golden Rule.

Tirole (1985), in a Diamond economy, but one with zero deficit and zero government debt, points out that the efficient allocation is another steady state, but an unstable one. He shows that the introduction in the exact right amount (at the initial point in time) of a financial security that makes no payments allows the economy to follow a saddle path that will reach the Golden Rule allocation. He calls that security a rational, intergenerational "bubble," the value per capita of which is finite. It has positive value only because the next generation will be willing to buy it.

We consider instead a preexisting outstanding debt and we allow further issuance dynamically and *endogenously* to finance additional government deficit each period. The social-security scheme of our paper does not bring about full efficiency because we keep it in a situation of deficit; as a result, a bit of overaccumulation is still present, which keeps the rate of interest below the Golden Rule. Our government debt plays a role similar to Tirole's bubble and our mathematics show some resemblance with his. We must say, therefore, that our government debt "is" or "contains" a bubble.[7]

[5] Contributions pertaining to OLG as a model of fiat money include: Shell (1971), Wallace (1980), and Kocherlakota (1998) among many others.

[6] A variation on the same reasoning appears in Kiyotaki and Wright (1989), which is the foundation of the so-called "New Monetarism" school. Instead of generations overlapping partially, traders with different needs meet randomly and, for that reason, benefit from a universal medium of exchange.

[7] Here we have in mind the following definition of the word "bubble." A bubble is an asset with a cash flow stream and price sequence that violate a transversality condition at infinity, in the sense that the remainder in Equation (13) below does not go to zero when T goes to infinity.

It seems that Chalk (2000) was the first to point out that, in an OLG model, there is an upper bound on the size of the debt. But he assumes that the primary deficit per capita (arising from a wasteful expenditure) is constant. Further, in his variant of Diamond's model, agents work during all periods and, therefore, there is no role for social security. A comprehensive and rigorous treatise on dynamics and policies in OLG models is available in De la Croix and Michel (2002). More recently, Cochrane (2021) presents debt arithmetics that point to the crucial role of the size of the debt.

As a first investigation, we present a deterministic version of the model. Although risk is no doubt important and would offer a way to differentiate the rate of return on public debt from the rate of return on capital, we do that because of the crucial role played here by the unstable steady state, a role that is not directly generalized to a stochastic setting.[8] We deliberately study the deterministic equilibrium so that the reader will see the unstable steady state.[9] We show that, within debt capacity, agents with finite lives value securities in a myopic way (limited by the horizon of their lives) but the level of debt capacity itself, or equivalently the question of debt sustainability, relies on a long-range calculation involving all future generations and their willingness to roll over the debt. A stochastic setting would raise the as yet unresolved question of the pricing of the risk of the inability to roll over.[10] See the discussion in Section 3.2 below. Further, the deterministic setting allows us to cleanly see the impact of policy changes along paths that started above capacity and the transition to a new steady state.

More research is needed in the direction of a generalization to a stochastic setting. In our analysis, the unstable steady state and the path to it are the elephants in the room; it is unlikely that their role will fade away in the stochastic case. In an OLG model with riskless population growth but risky investments, Hellwig (2021) confirms that the comparison between the riskless interest rate and the rate of population growth is the correct test of constrained Pareto efficiency but does not address the issue of largest possible amount of debt. Abel and Panageas (2022), in an OLG model in which the depreciation rate of capital alone is stochastic, do produce, like we do, a maximum amount of public debt that can be sustained along a stable balanced growth path.

Several economic frameworks, other than OLG, allow the government to play a role similar to the one it plays in our model. They are usually couched in terms of one or several rational bubbles, which modify the equilibrium and allows the government to mine it and increase its ability to finance deficits.[11] Such settings include constraints on trading that can also generate bubbles even when agents are infinitely lived. Excellent surveys of that approach are Miao (2014) and Martin and Ventura (2018). Kaas (2016), for instance, considers an economy with credit market frictions and heterogeneous firms; like we do, he finds a stable and an unstable steady state (with a bubble). Brunnermeier, Merkel, and Sannikov (2022), Reis (2021), Gersbach, Rochet, and von Thadden (2022) and Krueger and Uhlig (2022) consider an incomplete-market stochastic model featuring infinitely-lived log-utility agents. In Brunnermeier, Merkel, and Sannikov (2022), the capital owned by each agent is exposed to counter-cyclical idiosyncratic risk because a "skin-in-the game" constraint requires the agent to hold more than a minimum amount of his capital, while the rest is tradable. Each agent is equipped with a strictly positive personal stochastic discount factor and applies a transversality condition written with it. Nonetheless, because of market incompleteness, there can be enough of a precautionary saving motive to bring the rate of interest to be below the rate of growth. In that case, an aggregate bubble can be present in the government debt.[12]

For our purposes, empirical work on the presence of bubbles is very informative. Jiang, Nieuwerburgh, and Xiaolan (2021) apply a Campbell and Shiller (1988)-style decomposition to the market value of the U.S.

[8] In all stochastic models that we know either all agents have the same finite lives or, if they are infinitely-lived, a transversality condition at infinity is imposed from the start.

[9] This is a humorous parody of Pablo Picasso's statement: "I deliberately painted this crooked nose so that you are forced to see a nose," which he made about the *Woman in a Green Hat* painting, 1947 (Albertina museum, Vienna).

[10] So far, only the risk of future revision of policy has been incorporated by Croce, Kung, Nguyen, and Schmid (2012a), Croce, Kung, Nguyen, and Schmid (2012b), Pástor and Veronesi (2012), Kelly and Veronesi (2016), Croce, Raymond, and Schmid (2019), Corhay, Kind, Kung, and Morales (2021).

[11] Santos and Woodford (1997) give conditions under which no bubble is possible in intertemporal competitive-equilibrium settings.

[12] Settings of that kind are sometimes combined with a neo-keynesian macroeconomic model as in Elenev, Landvoigt, Shulz, and Nieuwerburgh (2021) and references therein. The purpose is to capture the effect of central-bank intervention policy. Or they can be combined with the Fiscal Theory of the Price Level, which accounts for the fact that government debt is mostly denominated in nominal units, a matter we consider in Section 5.1 below. See Bassetto and Cui (2018), Farmer and Zabczyk (2020) and Reis (2021).

federal government's current debt divided by the U.S. gross domestic product. The variance of that ratio must be accounted for by news about future real returns on debt, news about future surplus/output ratios or a third term reflecting news about future debt/output ratios. They find that the last term accounts for 100% of the variance. This result must be interpreted as evidence of the presence of a bubble, or violation of a form of transversality condition at infinity.[13]

Using the OLG deterministic model, we explore the existence of steady states, to which one converges or does not converge given initial conditions. Then we extend the model and its concept of debt capacity to two policy-relevant settings. First, the ability to run a monetary policy may increase the debt capacity or fiscal space. To investigate this issue, we allow the government consolidated with the central bank, in addition to collecting taxes and paying benefits, either to follow a policy centered on monetary balances or to intervene in the money market following a Taylor rule, thus controlling the nominal rate of interest. We find that monetary intervention with a Taylor coefficient greater than 1 increases total financing capacity (putting together bonds and money), in comparison with the case of no intervention and the case of a coefficient lower than 1.

Second, as debt capacity depends on growth, one might hope that endogenous productivity increases would raise the limit. We, therefore, ask whether a government can increase its debt capacity by subsidizing innovation, which ultimately raises productivity and growth. To answer this question, we adjust our framework along the lines of a revised Romerian approach but let the government finance R&D in addition to paying for social security. Overall, our numerical illustrations under these varied scenarios show that public R&D spending does not lift debt capacity miraculously.[14]

In policy experiments, we use the notion of debt capacity to explore the government responses (such as raising taxes) that are needed in case debt, for whatever reason, exceeds it. These capture the complete fiscal cost of exceeding debt capacity today. By the logic of debt accumulation, we conclude that the later the policy response, the larger it has to be. We also explain how debt could come to exceed debt capacity. For that, we imagine an unexpected population-growth declines. We again calculate the needed policy response. Our model can arguably be interpreted as saying that it is better to implement a policy that reduces the debt automatically during normal times, as a way to aim towards the stable steady state.

That part of our paper bears some relation to the vast literature that developed tax and social-security policy scenarios in an OLG context.[15] The applied models of that literature are much more detailed and incorporate more features than we have here, going as far as to imbed tax calculators corresponding to the tax law of a specific year (see Moore and Pecoraro (2018)), or to include a population of 55 generations (Diamond and Zodrow (2006)). Their purpose is advisory; it is to forecast the effects of specific aspects of tax reforms: general lowering of taxes, or replacement of one form of tax by another. Many of these papers take the rate of interest as exogenous. They also take the amount of debt of a base year as given, without any emphasis on the largest viable amount of debt. Some address the issue of the financing of a tax reform (Diamond (2005), Diamond and Viard (2008)) but all of them consider scenarios in which, at the tail end of the tax reform, budget balance will be restored, so that the so-called "intertemporal budget constraint" of the government binds, with a value of debt equal to present discounted value of future surpluses over a finite horizon. In other words, a terminal condition of return to a stable steady state is imposed explicitly or implicitly. The lesson from our paper regarding this advisory work is that the set of scenarios to be considered can and should be greatly expanded to allow for perpetual refinancing, and for a definition of debt sustainability and debt capacity that is grounded on the unstable steady state.

The balance of the paper is organized as follows. Section 1 presents a deterministic model with perpetual refinancing of a social-security driven government deficit, and determines under what conditions the paths of

[13]A recent study by Jiang, Lustig, Nieuwerburgh, and Xiaolan (2021) aims to explain, as we do, the valuation of government debt. It features a thorough empirical investigation of the stochastic process of government surplus, – carefully estimating separate processes for government revenues and government expenditures –, postulates an exponential affine stochastic discount factor for infinitely lived investors and derives the risk premium of government debt. The authors state that "In rational bubble models, the debt/GDP ratio declines over time." In our model, the debt/GDP ratio can rise either temporarily or permanently.

[14]Compare this to the statement by Prime Minister Draghi and President Macron in the *Financial Times*, December 23, 2021: "We need to have more room for manoeuvre and enough key spending for the future and to ensure our sovereignty. Debt raised to finance such investments, which undeniably benefit the welfare of future generations and long-term growth, should be favoured by the fiscal rules, given that public spending of this sort actually contributes to debt sustainability over the long run."

[15]That literature issued from Summers (1981) and Auerbach and Kotlikoff (1987). For a thorough survey, see Zodrow and Diamond (2013).

the economy exist and are unique for given initial conditions. Section 2 studies the paths of the economy and investigates the existence of steady states. Our definition of debt capacity is in Section 3. Section 4 contains policy experiments and describes the consequences of a decline in the population growth rate. In Section 5, we turn to two extensions of the model: first, we examine the implications for inflation of a high level of nominal government debt and consider the role of the central bank; second, we envision the possibility that the government may sustain its debt by subsidizing R&D, in addition to supporting a social-security scheme. The final section contains the conclusion.

1 A deterministic model of perpetual refinancing

1.1 The components of the system

We build an OLG model with population growth and physical capital accumulation with a single good, similar to De la Croix and Michel (2002). The economy comprises a production sector, a household sector and a government sector. We assume in this section that the initial amount of debt, inherited from history, is contractually denominated as a real amount of the good. The alternative of nominal denomination is explored in Section 5.1 below.

The production function is

$$Y_t = F\left(K_t, L_t\right)$$

where K_t and L_t are the inputs of capital and labor. The function F is continuous, twice differentiable, homogeneous of degree 1 in its two arguments, strictly increasing and strictly concave in each. To exploit the homogeneity, we write

$$k_t \triangleq \frac{K_t}{L_t}; f\left(k_t\right) \triangleq F\left(k_t, 1\right)$$

We make the following assumptions about the production function:

Assumption 1. *For all $k \geq 0$, $f(k) > 0, f'(k) > 0, f''(k) < 0$.*

At $t \neq 0$, the capital stock K_t is set aside at time $t-1$ and chosen by the generation born at time $t-1$. Y_t is interpreted as output plus the capital recovered after depreciation.

The households/investors: Like in Diamond (1965), we introduce the following notation and assumptions: c_t^t is the consumption at date t of one household of the generation born at date t while c_{t+1}^t is the consumption at date $t+1$ of the generation born at date t. L_t is the exogenous number of individuals in the generation born at time t. Their lives are summarized with two periods and they work at the first point in time only; their supply of labor is inelastic. L_t grows at the constant rate n per period. The number n stands for population growth but possibly also for labor-augmenting technical progress. It is a catch-all for all forms of exogenous perpetual growth. We examine technical progress explicitly in Section 5.2.

Generations are born with an endowment of only one kind: their labor force. They collect a wage bill $w_t L_t$. At time 0, there are only "old" people born notionally at time -1 with arbitrary consumption c_0^{-1}.

Utility is additive and the two-points in time utility functions are one and the same with discount factor β:

$$U\left(c_t^t, c_{t+1}^t\right) = u\left(c_t^t\right) + \beta u\left(c_{t+1}^t\right); c_t^t > 0; c_{t+1}^t > 0; t \geq 0$$

We make the following assumption about the utility function:

Assumption 2. *For all $c > 0$, $u'(c) > 0, u''(c) < 0$, $\lim_{c \to 0} u'(c) = +\infty$.*

The financial market: Two assets are traded in the financial market. One is a one-period bond, which is a claim on the government and the other is the direct ownership of the capital that serves as input into the production system, and which can be rented to production facilities. In this deterministic world, the young households are indifferent between physical capital and government debt, so that we let them choose not each one separately but their sum which is called "savings" s_t. In total, they save an amount $s_t L_t$ at time t. This is a stock of savings.

The two assets being perfect substitutes in demand, they bring the same rate of return. The one-period rate of return or rate of interest quoted at time t is called r_{t+1}.

Taxation and spending: Taxation is in the form of a contribution to the social-security system. The time-t young make a total social-security contribution of $L_t \tau w_t$, where τ is the social-security tax rate and w_t is the wage rate.

Government spending is in the form of social-security defined benefits paid to the old households on the basis of the wages they were earning when young. Specifically, at time t the old receive a total social-security benefit of $L_{t-1}\theta w_{t-1}$, where θ is the social-security benefit ratio. Throughout, we consider the case in which the primary budget deficit is structural:[16]

Assumption 3. $0 \leq \tau < 1; 0 \leq \theta; \tau < \frac{\theta}{1+n}$.

In Appendix 2.2, we show that social security is a welfare-improving form of spending.[17] Our formulation differs from De la Croix and Michel (2002) and Chalk (2000) who assume a constant deficit per capita.

With this notation, the simultaneous flow budget constraints at time t are as follows:
for the young household,

$$c_t^t + s_t = (1-\tau)\, w_t \tag{1}$$

for the old household,

$$c_t^{t-1} = s_{t-1} \times (1+r_t) + \theta \times w_{t-1} \tag{2}$$

for the government,

$$-G_{t+1} + \theta \times w_{t-1} \times L_{t-1} = \tau \times w_t \times L_t - (1+r_t)\, G_t \tag{3}$$

where G_t is the total debt with which the government enters time t and G_{t+1} is the debt with which it exits time t. In per capita notation, this is:

$$-(1+n)\, g_{t+1} + \theta \times w_{t-1} \frac{1}{1+n} = \tau w_t - (1+r_t)\, g_t \tag{4}$$

where $g_t \triangleq G_t / L_t$.

Market clearing: By using the same notation L_t for the size of the population, which is the labor supply, and for the input to the production function, we have presumed that the labor market clears. The market for goods clears

$$L_t \times c_t^t + L_{t-1} \times c_t^{t-1} + K_{t+1} = F(K_t, L_t)$$

or in per capita notation:

$$c_t^t + \frac{1}{1+n} c_t^{t-1} + (1+n)\, k_{t+1} = f(k_t) \tag{5}$$

The financial market clears:

$$s_t \times L_t \triangleq K_{t+1} + G_{t+1}$$

or on a per capita basis:

$$s_t \triangleq (1+n)(k_{t+1} + g_{t+1}) \tag{6}$$

Naturally, by Walras Law, one of the market-clearing equations is redundant.

For each generation, it would be suboptimal to finish its life with total savings of capital and government-debt holdings greater than zero. And default is not allowed, so that their total stock of savings at the end of their life is set exactly at zero. At the end of its two-period life each household consumes its entire wealth, including the value of the capital stock, which is part of its savings and which it sells to the young, leaving nothing behind. These are the only terminal conditions of optimality that are present in the model; they have already been coded into the budget constraints above.

We define the savings function

Definition 1.

$$\mathfrak{s}(w, r) \triangleq \arg\max_s \left[u\left((1-\tau)\, w - s\right) + \beta u\left(s \times (1+r) + \theta \times w\right)\right] \tag{7}$$

[16] In Section 4, however, we consider changes of policy made necessary in case debt is unsustainable.
[17] See the policy recommendation of L. Summers, *Washington Post*, Jan 7, 2020.

Under the assumptions on utility $\mathfrak{s}(w,r)$ is uniquely defined and satisfies

$$(1-\tau)\,w-s>0 \tag{8}$$

and

$$s\times(1+r)+\theta\times w>0 \tag{9}$$

as long as this feasibility restriction is satisfied:

$$(1-\tau)\,w+\frac{\theta\times w}{1+r}>0$$

The saving function satisfies several properties listed in Appendix B.

We define the functions giving the rate of interest and the wage rate that reflect the first-order conditions of the firm:

Definition 2. [18]

$$\mathfrak{r}(k)\triangleq f'(k)-1;\;\; \mathfrak{w}(k)\triangleq f(k)-kf'(k)$$

Let the deficit function be:

Definition 3.

$$\mathfrak{d}(k_{t-1},k_t)\triangleq\frac{\theta}{1+n}\mathfrak{w}(k_{t-1})-\tau\times\mathfrak{w}(k_t) \tag{10}$$

Proposition 1. *The evolution of the economy is dictated by a first-order system of difference equations giving (k_{t+1},g_{t+2}) from (k_t,g_{t+1}):*

$$\left\{\begin{array}{c}\mathfrak{s}(\mathfrak{w}(k_t),\mathfrak{r}(k_{t+1}))=(1+n)(k_{t+1}+g_{t+1})\\(1+n)\,g_{t+2}=(1+\mathfrak{r}(k_{t+1}))\,g_{t+1}+\mathfrak{d}(k_t,k_{t+1})\end{array}\right. \tag{11}$$

*Initial conditions are set by the time-*0 *capital stock per capita k_0 and the time-*1 *debt g_1.*[19]

The first equation of the system is the market-clearing equation (6) and the second one is the government's debt accumulation equation (4) shifted forward by one period.

1.2 Existence and uniqueness of equilibrium paths for given initial conditions

We now establish conditions under which, given an initial point, the system (11) has a solution in the form of a growth path in the (k_t,g_{t+1}) space, and conditions under which said path is well-defined, i.e., unique.

Define a function that will serve to write market clearing in the form $\Delta(k_{t+1})=0$ for $g=g_{t+1}$ and $w=\mathfrak{w}(k_t)$:

Definition 4.

$$\Delta(k)\triangleq(1+n)(k+g)-\mathfrak{s}(w,\mathfrak{r}(k))$$

The equilibrium exists if, for any given value of g and w the equation $\Delta(k)=0$ admits a solution k.

Theorem 1. *If, in addition to the assumptions on the production function and the utility function, we assume that $\lim_{k\to0,k>0}\mathfrak{s}(w,\mathfrak{r}(k))>0$ and that $(1+n)\,g<\lim_{k\to0}\mathfrak{s}(w,\mathfrak{r}(k))$ for all $w>0$, the function $\Delta(k)$ has at least one root k for all g and $w>0$.*

Proof. See Appendix C. ■

The equilibrium is unique if and only if the equation $\Delta(k)=0$ has a unique solution k, which it does if $\Delta'(k)>0$ for all k. In that way, System (11) will give a unique value for k_{t+1}, which it does also for g_{t+2}, which is explicit.

Define the elasticity of intertemporal substitution $\mathfrak{e}(s,r)$:

[18] As mentioned above, Y_t is interpreted as output plus the capital recovered after depreciation. In the alternative interpretation in which Y_t was output pure and simple, we would be assuming a depreciation rate equal to 1. We note that an annual depreciation rate of 0.1 would compound, over the length of a generation (say 25-years), to a depreciation rate of $1-(1-0.1)^{25}=0.93$ and that such a depreciation rate would not materially change our numerical results.

[19] If one regarded g_0 as the initial debt per capita, one would have to specify in an ad hoc fashion the amount of benefits paid to the old generation at time 0.

Definition 5.
$$\frac{1}{\mathfrak{e}(s,r)} \triangleq \frac{-u''(s\times(1+r)+\theta\times w)\times s}{u'(s\times(1+r)+\theta\times w)}\times(1+r) \tag{12}$$

Assumption 4. *$\mathfrak{e}(s,r) > 1$ for all k and w that satisfy (8) and (9).*

Theorem 2. *Under Assumption 4, $s'_r(w,r) > 0$ and $\Delta'(k) > 0$.*

Proof. Lemma 7 in Appendix B shows that $\mathfrak{s}'_r > 0$. Further,

$$\Delta'(k) = 1 + n - \mathfrak{s}'_r(w, \mathfrak{r}(k)) \times f''(k) > 0$$

■

Uniqueness follows.

1.3 Rolling over

When a growth path exists, we want to know the value of the debt along that path. Incrementing the government budget Equation (4), one gets an expression for the value of debt at time 1 (writing d_t for $\mathfrak{d}(k_{t-1}, k_t)$ and r_u for $\mathfrak{r}(k_u)$):

$$g_1 = \underbrace{\frac{1}{1+n}\sum_{t=1}^{T-1}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}}}_{\text{Partial sum}} + \underbrace{\frac{g_T}{\prod_{u=1}^{T-1}\frac{1+r_u}{1+n}}}_{\text{Remainder}};\forall T > 1 \tag{13}$$

The proof is in Appendix D.

Equation (13) gives a decomposition of the time-1 value of debt into two components. The partial sum is the present discounted value of future surpluses up to time T. The remainder component clearly captures the ability further to roll over the debt at price g_T.[20] When $r_t < n$ and $d_t > 0$, the first component tends to minus infinity as $T \to +\infty$. So, if we summed the series of the present values of current and future surpluses by the technique, familiar from undergraduate Calculus, of taking the limit of partial sums, we could not write that it is equal to g_1, which may very well be finite, without some extreme and offsetting assumptions on the remainder.[21]

We show now, by means of a stylized example, that the limit of partial sums is *not the financially relevant method of summation.* Our reasoning will be quite familiar to macroeconomists (see, for instance, Blanchard (2022), page 53), and is Accounting pure and simple. For simplicity, let the rate of interest be constant at r and suppose that the government exits time 0 with a debt in the amount of $G_1 = 1$ so that we know from the start that the value of that debt is finite. From there, following Leonhard Euler's precept quoted as the epigraph of our paper, we move forward and implement the "process from which that series arises."[22] At time 1, the government must repay the amount $1 + r$, which it can do by issuing new debt in an amount $G_2 = 1 + n$ and also finance a positive *deficit* equal to $D_1 = (1 + n) - (1 + r) = n - r$. At time 2, the government must repay the amount $(1 + n) \times (1 + r)$, which it can do by issuing new debt in an amount $G_3 = (1 + n)^2$ and also finance a positive deficit equal to $D_2 = (1 + n)^2 - (1 + n)(1 + r) = (n - r) \times (1 + n)$ and so on.[23] In other words, the government budget constraint is:

$$-G_{t+1} + D_t = -(1+r)\times G_t; G_1 = 1 \tag{14}$$

with

$$D_{t+1} = (1+n)\times D_t; D_1 = n - r$$

[20]If we assumed $\lim_{T\to\infty} g_T / \prod_{u=1}^{T-1}(1+r_u)/(1+n) \to 0$, we would be imposing a so-called transversality condition at infinity.

[21]The remainder would have to tend to plus infinity.

[22]We also take into account a remark of Jean-Charles Rochet to whom we are grateful.

[23]Charles Ponzi, of course, did exactly the same thing except that he could not keep it up (the growth rate could not be kept constant), whereas the long-lived government can, generation after generation, except in the case we consider below in Section 3.2.

which is analogous to but much simpler than Equation (3) above.

Since the debt remains riskless, the financial market agrees to that rollover sequence and values the debt at the present value of the stream of per capita surpluses discounted at the rate r:

$$1 = (r - n) \sum_{t=1}^{\infty} \frac{(1+n)^{t-1}}{(1+r)^{t}} \tag{15}$$

Now, if we tried to sum the series on the right-hand side of this equation by the technique of partial sums, we would easily find that the series is divergent when $r < n$. So, we would have an infinite number on the right-hand side and a finite one on the left-hand side. In this context, partial sums are obviously not the right way to go.

Over the last few centuries, mathematicians worked on series that are divergent in the meaning of partial sums. They have proposed other definitions of the sum of a series based on three axioms: regularity,[24] linearity and translativity.[25] Implementing their idea, write the right-hand side of Equation (15) as $x_1 \times (r - n) / (1 + n)$ where x_1 is

$$x_1 = \sum_{t=1}^{\infty} \frac{(1+n)^{t}}{(1+r)^{t}}$$

and is now calculated. By the linearity and translativity axioms,

$$x_1 = \frac{1+n}{1+r} + \frac{1+n}{1+r} x_1$$

Therefore (unless $r = n$),

$$x_1 = \frac{\frac{1+n}{1+r}}{1 - \frac{1+n}{1+r}} = \frac{1+n}{r-n}$$

and

$$G_1 = \frac{r-n}{1+n} \frac{1+n}{r-n} = 1$$

which is the right answer.

Therefore, Equation (15), which was paradoxical at first sight in the case $r < n$, is true – under an alternative definition of summation but, more important for us, in a Finance relevant sense –, for all real values of $r \neq n$.[26] It is remarkable that the right-hand of Equation (15) is an infinite series of all-negative numbers and yet the sum is a positive number.

By extension, that sort of sum is also *the one delivered by the forward solution* of our difference-equation system in the broader context of our dynamic model. The step-by-step forward solution generates the series in the way Euler suggested. By way of illustration, solve analogous Equation (14) forward, and get:

$$G_t = (1+n)^{t-1}$$

with

$$D_t = (1+n)^{t-1} \times (n - r)$$

which, with $r < n$, is a growth path with positive deficit *and* positive debt. The same property obtains when solving our nonlinear system forward.

1.4 The case of CES production cum isoelastic utility

For the sake of concreteness, we apply the conditions of existence and uniqueness of the equilibrium paths to the case of CES production:[27]

$$f(k) = \begin{cases} A \times [\alpha k^{\rho} + 1 - \alpha]^{\frac{1}{\rho}} & \text{if } \rho \neq 0 \\ A k^{\alpha} \text{ if } \rho = 0 \end{cases} ; \rho < 1; A > 0; 0 < \alpha < 1 \tag{16}$$

[24] That is, the sum must agree with the limit of the partial sums in case the latter exists.

[25] See: https://en.wikipedia.org/wiki/Divergent_series. The most famous textbook on the topic is Hardy (1973).

[26] In other words, for this particular cash flow stream, the familiar Gordon formula is valid even when the rate of growth is larger than the rate of discount.

[27] As is well-known, the functional form can be taken to the limit $\rho \to 0$ where it becomes the Cobb-Douglas production function. If $\rho \leq 0$, $f(0) = 0$. If $\rho > 0$, $f(0) > 0$.

together with isoelastic utility:[28]

$$u(c)=\left\{\begin{array}{c} c^{1-\zeta}/(1-\zeta)\ \text{if}\ \zeta\neq 1\\ \log c\ \text{if}\ \zeta=1\end{array}\right.\quad ;\zeta>0$$

for which the elasticity of technical substitution (ETS) between capital and labor, $\eta = 1/(1-\rho)$, and the elasticity of intertemporal substition (EIS), $1/\zeta$, are explicit parameters.

Existence of equilibrium:

$$\begin{aligned}
\mathfrak{s}(w,r) &= \frac{(1-\tau)\beta^{\frac{1}{\zeta}}(1+r)^{\frac{1}{\zeta}-1}-\frac{\theta}{1+r}}{1+\beta^{\frac{1}{\zeta}}(1+r)^{\frac{1}{\zeta}-1}}w\\
\lim_{k\to 0,k>0} f'(k) &= \left\{\begin{array}{ll} A\alpha^{\frac{1}{\rho}} & \text{for}\quad \rho<0\\ +\infty & \text{for}\quad 0\leq\rho<1\end{array}\right. \quad ;\quad \lim_{r\to+\infty}\mathfrak{s}(w,r)=\left\{\begin{array}{c}(1-\tau)w\ \text{if}\ \zeta<1\\ (1-\tau)w\frac{\beta}{1+\beta}\ \text{if}\ \zeta=1\\ 0\ \text{if}\ \zeta>1\end{array}\right.\\
\lim_{k\to 0,k>0}\mathfrak{s}(w,\mathfrak{r}(k)) &= w\times\left\{\begin{array}{ll} \frac{1}{A\alpha^{\frac{1}{\rho}}}\frac{(1-\tau)\beta^{\frac{1}{\zeta}}\left(A\alpha^{\frac{1}{\rho}}\right)^{\frac{1}{\zeta}}-\theta}{1+\beta^{\frac{1}{\zeta}}\left(A\alpha^{\frac{1}{\rho}}\right)^{\frac{1}{\zeta}-1}} & \text{for}\quad\rho<0\\ \left.\begin{array}{c}(1-\tau)>0\ \text{if}\ \zeta<1\\ (1-\tau)\frac{\beta}{1+\beta}>0\ \text{if}\ \zeta=1\\ 0\ \text{if}\ \zeta>1\end{array}\right\} & \text{for}\quad 0\leq\rho<1\end{array}\right.
\end{aligned}$$

Given that explicit form for $\lim_{k\to 0}\mathfrak{s}(w,\mathfrak{r}(k))$, Theorem 1 tells us that there exists an equilibrium when $\rho<0$ and $\theta<(1-\tau)\beta^{1/\zeta}\left(A\alpha^{1/\rho}\right)^{1/\zeta}$ (benefits are limited in size), and also when $0\leq\rho<1$ and $\zeta\leq 1$. In both cases, however, Theorem 1 places an upper bound on debt for the equilibrium to exist: $(1+n)g<\lim_{k\to 0}\mathfrak{s}(w,\mathfrak{r}(k))$.

Uniqueness of equilibrium:

$$\mathfrak{e}(s,r)=\frac{1}{\zeta}\times\frac{s\times(1+r)+\theta\times w}{s\times(1+r)}$$

The sufficient condition $\mathfrak{e}(s,r)>1$ in Theorem 2 certainly holds if $\zeta\leq 1$.

Numerical examples:

Example 1. *Consider the special case of logarithmic utility (isoelastic with $\zeta\to 1$) and Cobb-Douglas production function (CES with $\rho\to 0$). The savings function is especially simple:*

$$\mathfrak{s}(w,r)=w\frac{\beta(1-\tau)-\frac{\theta}{1+r}}{\beta+1}$$

and

$$f(k)=A\times k^{\alpha};\ \mathfrak{w}(k)=A\times(1-\alpha)k^{\alpha};\ \mathfrak{r}(k)=A\times\alpha k^{\alpha-1}-1 \tag{17}$$

Suppose that a period of the model is equal to twenty-five years and that: $A=1$, $n=(1+0.02)^{25}-1$ (2%/year), $\alpha=0.2$, $\beta=0.99^{25}$ (0.99/year), $\theta=0.165$, $\tau=0.1$. Because $\rho=0$ and $\zeta=1$, Theorems 1 and 2 tells us that the equilibrium exists and is unique irrespective of θ, although an upper bound must be placed on debt per capita:

$$g<\frac{1-\tau}{1+n}\frac{\beta}{1+\beta}=\frac{1-0.1}{(1+0.02)^{25}}\frac{0.99^{25}}{1+0.99^{25}}=0.24$$

which will be much above debt capacity (see Definition 6 and numerical examples below).

[28] As is well-known, the functional form can be taken to the limit $\zeta\to 1$ where it becomes the logarithmic utility function.

Example 2. *Consider logarithmic utility ($\zeta = 1$), CES production (16) with $\rho = -1$ (ETS equal to 1/2), and other parameters as in the previous example. For existence, because $\rho < 0$, an upper bound must be verified on benefits:*

$$\theta < (1-\tau)\,\beta\left(A\alpha^{1/\rho}\right) = (1-0.1)\,0.99^{25}\left(0.2^{-1}\right) = 3.5$$

and on debt:

$$g < \frac{1-\tau}{1+n}\frac{1}{A\alpha^{-1}}\frac{\beta\left(A\alpha^{-1}\right)-\theta}{1+\beta\left(A\alpha^{-1}\right)^{-1}} = \frac{1-0.1}{(1+0.02)^{25}}\frac{1}{0.2^{-1}}\frac{0.99^{25}\left(0.2^{-1}\right)-0.165}{1+0.99^{25}\left(0.2^{-1}\right)^{-1}}0.354 = 0.125$$

2 Existence and stability of steady states

To examine the existence of steady states, we obtain existence conditions for the benchmark case of zero deficit. Then, for a deficit small enough, we examine the way in which the steady states are displaced relative to that benchmark. To accomplish that task, we need to define some functions reflecting the behavior of the economy along equilibrium paths.

2.1 The paths

Observe the following property of the function Δ of Definition 4:[29]

$$\frac{\partial}{\partial w}\Delta(k) = -\mathfrak{s}'_w\left(w, \mathfrak{r}(k)\right) < 0$$

Observe also that $w = \mathfrak{w}(k_t)$ only depends on the current value, k_t, in an increasing manner. Therefore, a higher value of k_t, which implies a higher value of w_t, shifts down the upward sloping $\Delta(k)$ curve ($\partial\Delta(k)/\partial k_t < 0$). A higher value k_{t+1} of the solution k of the equation $\Delta(k) = 0$ is induced. Hence, if debt were constant – which it cannot be under Assumption 3 –, it would be true that $dk_{t+1}/dk_t > 0$ for all t and the sequence $\{k_t\}$ would be monotone, and it would be true that the sequence $\{k_t\}$ is either increasing at all times or decreasing at all times. When debt moves, the following statement concerning the comparison of k_{t+1} with k_t can be made anyway:

Proposition 2. *Assume $\Delta'(k) > 0$ for a wide enough neighborhood of a value of k for which $\Delta(k) = 0$. If $\Delta(k_t) > 0$ with $g = g_{t+1}$, then $k_{t+1} < k_t$, and if $\Delta(k_t) < 0$ with $g = g_{t+1}$, then $k_{t+1} > k_t$.*

To formalize this proposition, define the function $\mathfrak{D}$ that is equal to the function $\Delta(k)$ with g_{t+1} substituted for g, and $\mathfrak{w}(k_t)$ for w:

$$\begin{aligned}\mathfrak{D}(k; k_t, k_{t-1}) \quad &\triangleq \quad (1+n)\,k - \mathfrak{s}\left(\mathfrak{w}(k_t), \mathfrak{r}(k)\right) \\ &\qquad \underbrace{+\frac{1+\mathfrak{r}(k_t)}{1+n}\underbrace{\left[\mathfrak{s}\left(\mathfrak{w}(k_{t-1}), \mathfrak{r}(k_t)\right) - (1+n)\,k_t\right]}_{=(1+n)g_t} + \mathfrak{d}(k_{t-1}, k_t)}_{=(1+n)g_{t+1}}\end{aligned}$$

k_{t+1} is the solution for k of the equation $\mathfrak{D}(k; k_t, k_{t-1}) = 0$. We have studied the existence and uniqueness of the solution in Section 1. Another way of stating the first-order difference-equation system (11) is the second-order difference equation

$$\mathfrak{D}(k_{t+1}; k_t, k_{t-1}) = 0$$

which provides k_{t+1} from k_t and k_{t-1}.

Proposition 2 implies that the locus of points such that $\mathfrak{D}(k_t; k_t, k_{t-1}) = 0$ delimits two sets of zones in the (k_{t-1}, k_t) plane: one in which $\mathfrak{D}(k_t; k_t, k_{t-1}) > 0$ and $k_{t+1} < k_t$ and one in which $\mathfrak{D}(k_t; k_t, k_{t-1}) < 0$ and $k_{t+1} > k_t$. The following shows that each set of zones is connected, or that there are literally only two zones.

[29] See Lemma 8 in Appendix B.

Lemma 1. *Under the assumption of Lemma 8 (in Appendix B),* $\mathfrak{D}(k; k_t, k_{t-1})$ *is strictly increasing with respect to* k_{t-1}.

Proof.

$$\frac{\partial \mathfrak{D}}{\partial k_{t-1}}(k_t; k_t, k_{t-1}) = \frac{1+\mathfrak{r}(k_t)}{1+n}\mathfrak{s}'_w(\mathfrak{w}(k_{t-1}), \mathfrak{r}(k_t))\,\mathfrak{w}'(k_{t-1}) + \frac{\theta}{1+n}\mathfrak{w}'(k_{t-1}) > 0$$

since $\mathfrak{s}'_w(\mathfrak{w}(k_{t-1}), \mathfrak{r}(k_t)) > 0$ under the assumption of Lemma 8. ■

Hence, there exists a function $\Psi(k_t)$ defined by

$$\begin{aligned}
\mathfrak{D}(k_t; k_t, k_{t-1}) &= 0 \Leftrightarrow k_{t-1} = \Psi(k_t) \\
\mathfrak{D}(k_t; k_t, k_{t-1}) &> 0 \text{ (and } k_{t+1} < k_t) \Leftrightarrow k_{t-1} > \Psi(k_t) \\
\mathfrak{D}(k_t; k_t, k_{t-1}) &< 0 \text{ (and } k_{t+1} > k_t) \Leftrightarrow k_{t-1} < \Psi(k_t)
\end{aligned}$$

The function $\Psi(k_t)$ allows us to draw a phase diagram, an example of which is to be found in Figure 1.

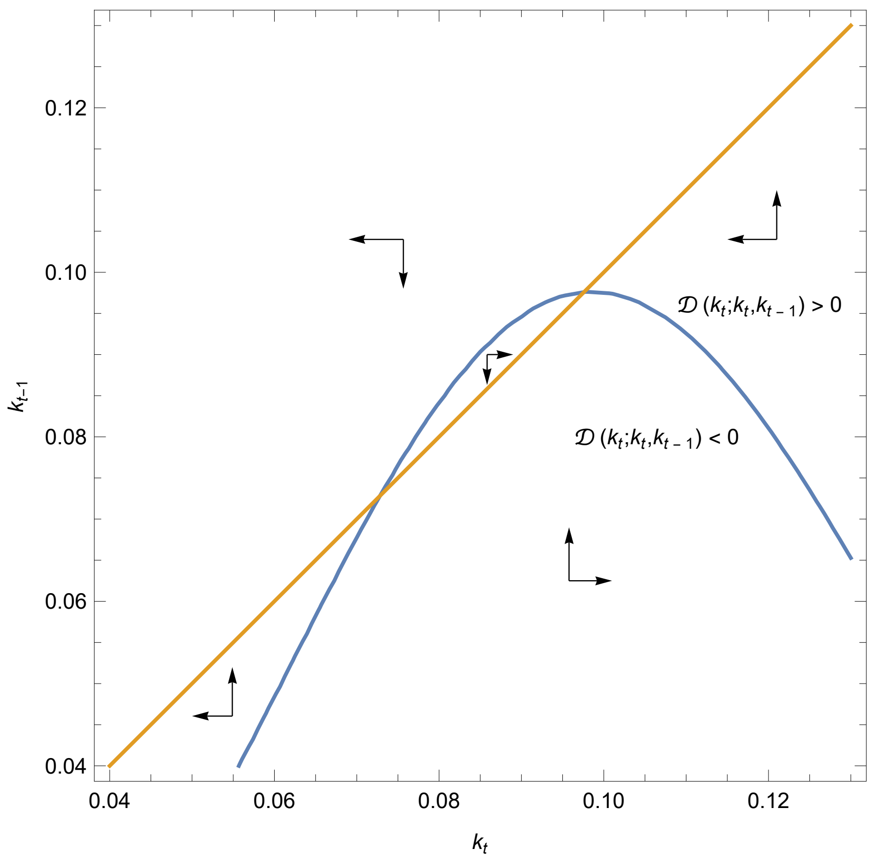

Figure 1: **Phase diagram or the directions of evolution of the pair (k_t, k_{t-1}) of the capital stock per capita.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $n = (1+0.02)^{25} - 1$ (2%/year), $\alpha = 0.2$, $\beta = 0.99^{25}$ (0.99/year), $\theta = 0.165$, $\tau = 0.1$. The blue schedule is the graph of the function $\Psi(k_t)$. An arrow towards the right means $k_{t+1} > k_t$. An arrow towards the top means $k_t > k_{t-1}$.

2.2 Steady states

A steady state is a value of k_t at which $\mathfrak{D}(k_t; k_t, k_t) = 0$. We now show, as an immediate consequence of the definition, that the value (13), summed the proper way, as in Equation (15), can very well be finite. Indeed, that is true at a steady state, as it would be in a simpler model with constant deficit. Call g a steady-state per-capita value of government debt; as per Equation (4):

$$g = \frac{-\mathfrak{d}(k,k)}{\mathfrak{r}(k) - n} \tag{18}$$

where the formula is valid if $n \neq \mathfrak{r}(k)$. When there exists a steady state k of the capital stock per capita, then there exists a steady state value g of government debt per capita. If households hold a positively-valued amount of debt $g > 0$, and there is a deficit $\mathfrak{d}(k,k) > 0$, it must be that $\mathfrak{r}(k) < n$.

Deficits in the steady states – stable or unstable – imply that $\mathfrak{r}(k) < n$ and vice versa, a configuration that best reproduces current economic conditions. To the opposite, when the steady-state social-security scheme produces surpluses, $\mathfrak{r}(k) > n$. In a steady state, it may very well happen that the government's social-security budget per capita is permanently in deficit ($\mathfrak{d}(k,k) > 0$), while, if $\mathfrak{r}(k) < n$, government debt still has positive market value. That, in a nutshell, is the point made by Blanchard (2019), in a simpler setting of exogenous deficits and interest rates.

In the (k_t, k_{t-1}) plane, any steady state is located at the intersection of the two loci $\mathfrak{D}(k_t; k_t, k_{t-1}) = 0$ (or, equivalently, $k_{t-1} = \Psi(k_t)$) and $k_{t-1} = k_t$.

Define the function $\Phi_0(k_t)$ as follows:

$$\mathfrak{d}(k_{t-1}, k_t) = 0 \Leftrightarrow k_{t-1} = \Phi_0(k_t)$$

which is the line of zero deficit in the (k_t, k_{t-1}) plane.

Lemma 2. *Under deficit Assumption 3, the deficit is positive at $k_t = k_{t-1}$.*

The proof is obvious. In other words, the line of zero deficit always lies below the line $k_{t-1} = k_t$.

Consider the function $\mathfrak{D}_0$ without the deficit term:

$$\begin{aligned}\mathfrak{D}_0(k; k_t, k_{t-1}) \quad &\triangleq \quad (1+n)k - \mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k)) \\ &+\frac{1+\mathfrak{r}(k_t)}{1+n}\left[\mathfrak{s}(\mathfrak{w}(k_{t-1}), \mathfrak{r}(k_t)) - (1+n)k_t\right]\end{aligned}$$

so that, at $k_t = k_{t-1}$,

$$\mathfrak{D}_0(k_t; k_t, k_t) = \left[(1+n)k_t - \mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))\right]\frac{n - \mathfrak{r}(k_t)}{1+n}$$

There would be evidently two (sets of) solutions k_t to the equation $\mathfrak{D}_0(k_t; k_t, k_t) = 0$. These would be

- the value of k, which we can call k_{GR} (the Golden Rule point) that solves $\mathfrak{r}(k_t) = n$, which exists provided $\lim_{k_t \to +\infty} f'(k_t) \leq 1 + n \leq \lim_{k_t \to 0} f'(k_t)$, and is unique since $f'(k_t)$ is monotonic.
- and the value of k, which we can call k_D, (the "Diamond" point) that solves an equation that requires zero steady-state debt:[30]

$$\frac{\mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))}{k_t} = 1 + n$$

k_D, in case it exists, is unique if $\mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))/k_t$ is strictly decreasing throughout:

Assumption 5.

$$\mathfrak{s}'_w(\mathfrak{w}(k_t), \mathfrak{r}(k_t))\,\mathfrak{w}'(k_t) + \mathfrak{s}'_r(\mathfrak{w}(k_t), \mathfrak{r}(k_t))\,\mathfrak{r}'(k_t) < \frac{\mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))}{k_t}$$

If that condition holds, k_D exists and is *strictly positive* if and only if the function $\mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))/k_t$ satisfies[31]

$$\lim_{k_t \to 0} \frac{\mathfrak{s}(\mathfrak{w}(k_t), \mathfrak{r}(k_t))}{k_t} > 1 + n \tag{19}$$

[30] This is the zero-debt steady state but with a savings function that contains τ and θ.

[31] It satisfies $\lim_{k \to +\infty} \mathfrak{s}(\mathfrak{w}(k), \mathfrak{r}(k))/k = 0$ (see Equation (37) in Appendix B).

Remark 1. *For some parameter configurations, $k_t = 0$ is a mathematically and economically valid solution. Or it can be that there is no strictly positive k_D.*

In general, the two solutions above do not coincide. Therefore, there exists between them an open interval $\mathcal{E}$.

Define the function $\Psi_0(k_t)$ as follows:

$$\begin{aligned}\mathfrak{D}_0(k_t; k_t, k_{t-1}) &= 0 \Leftrightarrow k_{t-1} = \Psi_0(k_t) \\ \mathfrak{D}_0(k_t; k_t, k_{t-1}) &> 0 \Leftrightarrow k_{t-1} > \Psi_0(k_t) \\ \mathfrak{D}_0(k_t; k_t, k_{t-1}) &< 0 \Leftrightarrow k_{t-1} < \Psi_0(k_t)\end{aligned}$$

Lemma 3. *Over the interval $\mathcal{E}$, $\Psi_0(k_t) > k_t$.*

Proof. See Appendix F ■

Next, we compare $\Psi(k_t)$ to $\Psi_0(k_t)$.

Lemma 4. *When the deficit is positive,*

$$\Psi(k_t) < \Psi_0(k_t)$$

The opposite is true when the deficit is negative.

Proof. By definition

$$\mathfrak{D}(k; k_t, k_{t-1}) = \mathfrak{D}_0(k; k_t, k_{t-1}) + \mathfrak{d}(k_{t-1}, k_t)$$

Substitute in $k_{t-1} = \Psi_0(k_t)$ and $k = k_t$

$$\mathfrak{D}(k_t; k_t, \Psi_0(k_t)) = \mathfrak{D}_0(k_t; k_t, \Psi_0(k_t)) + \mathfrak{d}(\Psi_0(k_t), k_t) = \mathfrak{d}(\Psi_0(k_t), k_t) > 0$$

Since $\mathfrak{D}(k_t; k_t, \Psi(k_t)) = 0$ and the function $\mathfrak{D}$ is strictly increasing in its third argument (see Lemma 1), as long as the deficit is positive, the statement is true. ■

Geometrically speaking, under the assumption that k_D exists, a necessary and sufficient condition for the existence of steady state(s) is that there exist values of k for which

$$\mathfrak{D}(k; k, k) = \underbrace{[(1+n)k - \mathfrak{s}(\mathfrak{w}(k), \mathfrak{r}(k))] \frac{n - \mathfrak{r}(k)}{1+n}}_{<0} + \underbrace{\mathfrak{d}(k, k)}_{>0} \leq 0 \tag{20}$$

or, equivalently, for which the value of the deficit falls below a ceiling:

$$0 < \mathfrak{d}(k, k) \leq -\mathfrak{D}_0(k; k, k)$$

We can get a more explicit characterization if we restrict the deficit as follows:

$$0 < \frac{\theta}{1+n} - \tau < \varepsilon$$

For ε small enough, by continuity, there exist solutions k_t to the equation $\mathfrak{D}(k_t; k_t, k_t) = 0$. Two of them lie in a neighborhood of the two solutions k_{GR} and k_D of the equation $\mathfrak{D}_0(k_t; k_t, k_t) = 0$.[32]

Theorem 3. *When $0 < \theta/(1+n) - \tau < \varepsilon$, for ε small enough, the solutions to the equation $\mathfrak{D}(k_t; k_t, k_t) = 0$ that lie in a neighborhood of the two solutions k_{GR} and k_D of the equation $\mathfrak{D}_0(k_t; k_t, k_t) = 0$, exist and are within the open interval $\mathcal{E}$.*

Proof. See Appendix F ■

The result implies that the two steady states are always on the same side of the value of k for which $\mathfrak{r}(k) = n$.

[32] $\Psi(k_t)$ and $\Psi_0(k_t)$ are expected to be concave functions. If they were not, the equation $\Psi(k_t) = k_t$, could have more solutions than does $\Psi_0(k_t) = k_t$.

2.3 Stability

Linearize the difference equation $\mathfrak{D}(k_{t+1}; k_t, k_{t-1}) = 0$ around a steady state generically called SS (near k_{GR} or near k_D):[33]

$$\begin{bmatrix} k_{t+1} - k_{SS} \\ k_t - k_{SS} \end{bmatrix} = \begin{bmatrix} \mathcal{T} & -\mathcal{D} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} k_t - k_{SS} \\ k_{t-1} - k_{SS} \end{bmatrix}$$

where

$$\mathcal{T} = \left.\frac{dk_{t+1}}{dk_t}\right|_{\mathfrak{D}=0} = -\frac{\frac{\partial \mathfrak{D}}{\partial k_t}}{\frac{\partial \mathfrak{D}}{\partial k_{t+1}}}; \mathcal{D} = -\left.\frac{dk_{t+1}}{dk_{t-1}}\right|_{\mathfrak{D}=0} = \frac{\frac{\partial \mathfrak{D}}{\partial k_{t-1}}}{\frac{\partial \mathfrak{D}}{\partial k_{t+1}}}$$

The characteristic polynomial is $P(\lambda) \triangleq \lambda^2 - \mathcal{T}\lambda + \mathcal{D} = 0$. The product of eigenvalues is

$$\lambda_1 \lambda_2 = \mathcal{D} = -\left.\frac{dk_{t+1}}{dk_{t-1}}\right|_{\mathfrak{D}=0} = \frac{\underbrace{\frac{1+\mathfrak{r}}{1+n}\mathfrak{s}'_w \mathfrak{w}'}_{>0} + \underbrace{\frac{\theta}{1+n}\mathfrak{w}'}_{>0}}{\underbrace{1 + n - \mathfrak{s}'_r f''}_{>0}} > 0$$

so that the two eigenvalues are of the same sign (the sign of the denominator being justified by the uniqueness of equilibrium path, as examined above).[34] The sum of eigenvalues is

$$\lambda_1 + \lambda_2 = \mathcal{T}$$

Assume that the function $\Psi(k_t)$ is differentiable:

Lemma 5.

$$\frac{\partial \Psi}{\partial k_t} > 1 \Leftrightarrow 1 + \mathcal{D} < \mathcal{T}$$

Proof.

$$\frac{\partial \Psi}{\partial k_t} = -\frac{\frac{\partial \mathfrak{D}}{\partial k_{t+1}} + \frac{\partial \mathfrak{D}}{\partial k_t}}{\frac{\partial \mathfrak{D}}{\partial k_{t-1}}} = -\frac{\frac{\partial \mathfrak{D}}{\partial k_{t+1}}}{\frac{\partial \mathfrak{D}}{\partial k_{t-1}}} - \frac{\frac{\partial \mathfrak{D}}{\partial k_t}}{\frac{\partial \mathfrak{D}}{\partial k_{t-1}}} = -\frac{1}{\mathcal{D}} + \frac{\mathcal{T}}{\mathcal{D}}$$

Since $\mathcal{D} > 0$, the result follows. ∎

The results of Section 2.2 imply the following:

- In the case $k_{GR} < k_D$, at the steady state near and above k_{GR}, $\partial\Psi/\partial k_t > 1$, whereas at the steady state near and below k_D, $\partial\Psi/\partial k_t < 1$.
- In the case $k_D < k_{GR}$, at the steady state near and above k_D, $\partial\Psi/\partial k_t > 1$, whereas at the steady state near and below k_{GR}, $\partial\Psi/\partial k_t < 1$

Hence:

- In the case $k_{GR} < k_D$, the steady state near and above k_{GR} is a saddle (i.e., an unstable steady state) because $1 + \mathcal{D} < \mathcal{T}$ whereas the steady state near and below k_D is either a sink (i.e., a stable steady state, in case $\mathcal{D} < 1$) or a source, because $1 + \mathcal{D} > \mathcal{T}$.
- In the case $k_D < k_{GR}$, the steady state near and above k_D is a saddle because $1 + \mathcal{D} < \mathcal{T}$, whereas the steady state near and below k_{GR} is either a sink (if $\mathcal{D} < 1$) or a source, because $1 + \mathcal{D} > \mathcal{T}$.

In Section 3.2 below, we suggest that outside the area of attraction of the stable steady state, k reaches 0 in finite time, which means that the income of the young no longer allows them to buy the savings of the old.

[33] This derivation generalizes De la Croix and Michel (2002), Section 4.3.1, pages 196ff, Propositions 4.7 and 4.8; and Section 4.3.3, pages 206ff, Proposition 4.11; and Appendix 3, Section 4, page 322.

[34] We do not know a general condition that would guarantee that $\mathcal{D} < 1$.

2.4 The case of CES production cum isoelastic utility

Here again, the form of the production and utility function allow us to make an explicit statement about the existence of the Diamond solution. Also, in the general case, we are not able to state whether $k_{GR} < k_D$ or the opposite is true. But, with these production and utility function, we can.

Existence of k_D: a sufficient condition is stated in Equation (19). To implement it, we need (see derivation in Appendix E):

$$\lim_{k\to 0, k>0} \frac{\mathfrak{s}\left(\mathfrak{w}\left(k_t\right), \mathfrak{r}\left(k_t\right)\right)}{k_t} = \begin{cases} \dfrac{(1-\tau)\beta^{\frac{1}{\zeta}}\left(A\alpha^{\frac{1}{\rho}}\right)^{\frac{1}{\zeta}}-\theta}{1+\beta^{\frac{1}{\zeta}}\left(A\alpha^{\frac{1}{\rho}}\right)^{\frac{1}{\zeta}-1}} & \text{for} \quad \rho<0 \\ \left.\begin{array}{c} +\infty \text{ if } \zeta \leq 1 \\ \frac{1}{\alpha}\left(1-\tau\right) \text{ if } \zeta > 1 \end{array}\right\} & \text{for} \quad \rho = 0 \\ +\infty & \text{for} \quad 0<\rho<1 \end{cases}$$

When $\rho = 0$ and $\zeta \leq 1$ and when $0 < \rho < 1$, the existence of the Diamond solution of the equation $k = \Psi_0\left(k\right)$ is guaranteed. In other cases, a sufficient condition for existence is obtained by placing restrictions on the values of the parameters as per Equation (19).

Relative position of the two steady states, assuming they exist: the relative position determines whether the steady states will occur in a regime of overaccumulation ($r < n$) or underaccumulation.

At $\mathfrak{r}\left(k_t\right) = n$, is debt positive? If the answer is yes, $k_{GR} < k_D$. Otherwise, the opposite is true. $\mathfrak{r}\left(k_t\right) = n$ means:

$$\left(\frac{\left(\frac{1+n}{\alpha\times A}\right)^{\frac{\rho}{1-\rho}}-\alpha}{1-\alpha}\right)^{-\frac{1}{\rho}} = k; \mathfrak{w}\left(k\right) = A\times\left(1-\alpha\right)^{\frac{1}{\rho}} \frac{\frac{1+n}{\alpha\times A}}{\left[\left(\frac{1+n}{\alpha\times A}\right)^{\frac{\rho}{1-\rho}}-\alpha\right]^{\frac{1-\rho}{\rho}}}$$

At k_t such that $\mathfrak{r}\left(k_t\right) = n$, debt is positive (which is $\mathfrak{s}\left(\mathfrak{w}\left(k_t\right), \mathfrak{r}\left(k_t\right)\right)/k_t > 1+n$) if and only if

$$\frac{1}{\alpha}\left[\left(\frac{1+n}{\alpha\times A}\right)^{\frac{\rho}{1-\rho}}-\alpha\right]\frac{1-\tau-\theta\beta^{-\frac{1}{\zeta}}\left(1+n\right)^{-\frac{1}{\zeta}}}{1+\beta^{-\frac{1}{\zeta}}\left(1+n\right)^{-\frac{1}{\zeta}+1}} > 1 \tag{21}$$

Numerical examples:

Example 3. *(Example 1 continued) Because the left-hand side of Condition (21) is equal to* 1.3488*, we can tell that* $k_{GR} < k_D$*. Actually, in this special case of logarithmic utility and Cobb-Douglas production function, there exists an analytical solution for the steady states.*[35] *The equation has two positive real roots:*
Root 1: $k = 0.09767$, $r = 1.0111\%/\text{year}$ *(near* k_D*). The eigenvalues are* $\{0.77, 0.24\}$ *and* $\mathcal{D} = 0.19$*.* $1+\mathcal{D} = 1.19$ *is larger than* $\mathcal{T} = 1.01$*. The steady state is a sink.*
Root 2: $k = 0.07264$, $r = 1.9725\%/\text{year}$ *(near* $\mathfrak{r}\left(k_{GR}\right) = n$ *(2%/year)). The eigenvalues are* $\{1.287, 0.227\}$ *and* $\mathcal{D} = 0.29 < 1$*.* $1+\mathcal{D} = 1.29$ *is smaller than* $\mathcal{T} = 1.51$*. The steady state is a saddle.*

Example 4. *(Example 2 continued) The steady-state equation admits only one strictly positive real root, which is near* k_{GR}*:*[36]
$k = 0.186314$, $r = 2.002\%/\text{year}$ *(r is near but above* $n = 2\%/\text{year}$*),* $\mathcal{T} = 0.74379$*,* $\mathcal{D} = 0.361264$ *while the eigenvalues are complex.* $1+\mathcal{D} > \mathcal{T}$ *and* $\mathcal{D} < 1$*. The steady state is a sink.*
Necessary condition (19) for the existence of $k_D > 0$ *is satisfied (*$1.876 > 1.6406$*) in this example. The non existence of* k_D *that we observe implies that Assumption (5) is not satisfied.*

3 Debt capacity

In this section, we continue to assume that the initial amount of debt, set by history or reset by fiat, is contractually denominated as a real amount of the good and we turn to the important matter of global convergence or divergence of paths.

[35] It is a bit simpler to rewrite the equation system with $x \triangleq 1+r$ as an unknown, to discover that the equation is cubic in x. There exist explicit but fairly cumbersome formulae for the roots of a cubic equation. Here, we just give the numerical result.

[36] The other solution is $k = 0$, which illustrates Remark 1 but at which output is equal to zero since $\rho < 0$.

3.1 Debt capacity defined

Consider a parametrization in which a stable steady state produces a deficit period by period financed by a never ending issuance of debt while the real rate of interest is lower than the real rate of growth of the economy. The economy might be (already) at the stable steady state or on its way to it.

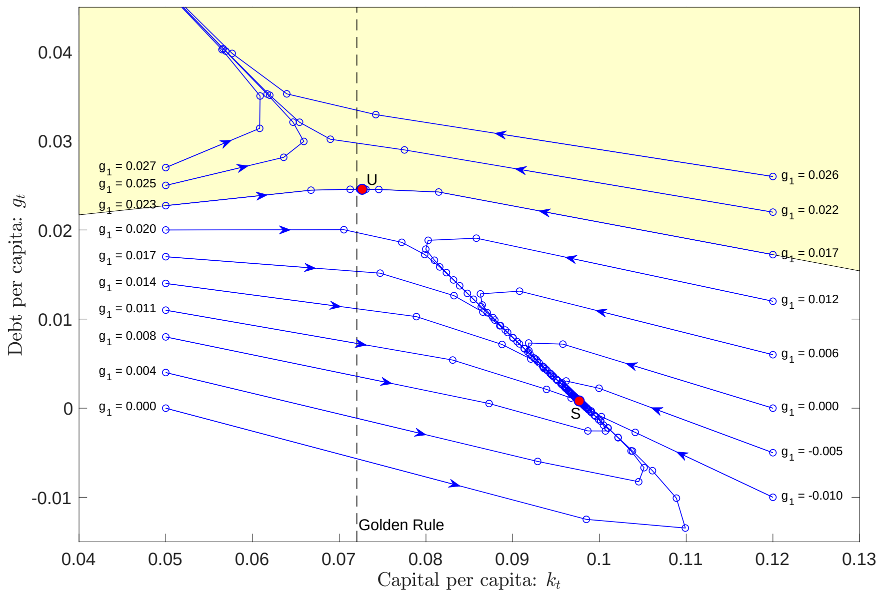

Figure 2: **The paths of the debt per capita and the capital stock per capita for two initial values k_1 and several initial values g_1.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $A = 1$, $n = (1 + 0.02)^{25} - 1$ (2%/year), $\alpha = 0.2$, $\beta = 0.99^{25}$ (0.99/year), $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U".

Example 5. *Example 1 continued: the joint dynamics of the capital stock and the debt are illustrated in the diagram of paths, Figure 2.*

Figure 2 illustrates the fact that many paths with many levels of deficit-to-GDP (or debt to GDP) ratios all lead to the same stable steady state.

Definition 6. *Debt capacity, for a given level of k_0, is the highest level of g_1 such that convergence occurs* without any change of policy parameters (θ, τ).[37]

Debt capacity is also the level of debt today that would lead to the unstable steady state along a saddle path. Provided debt starts from any level strictly within capacity, it converges to the stable steady state. An economy can start above debt capacity, on a seemingly explosive path with a rising rate of interest, if it can be anticipated that the government will, at some point, increase the tax rate, τ, decrease the social-security benefit ratio, θ, or both, in order to help return to a sustainable steady state. In Section 4 below, we examine such policy responses. We stress that our definition is not based on the existence or inexistence of a locally stable steady state. *Instead, it is based on the existence of a locally unstable steady state.* The non linearity in the model is such that the economy does not converge to a steady state when debt starts above capacity.

To make more concrete the threshold between convergence and divergence, we display in Figure 3, for initial conditions (k_1, g_1), the contours – in the spaces of the debt ratio versus the real interest rate and of the deficit ratio versus the real interest rate – that separate the area of divergence (upper, shaded area) from the area of convergence. In the unstable steady state, the rate of interest is just below the Golden Rule rate n.

[37] Given k_0 and g_1, k_1 follows by virtue of difference-equations (11), and vice versa. Hence, we can also measure debt capacity with the pair (k_1, g_1).

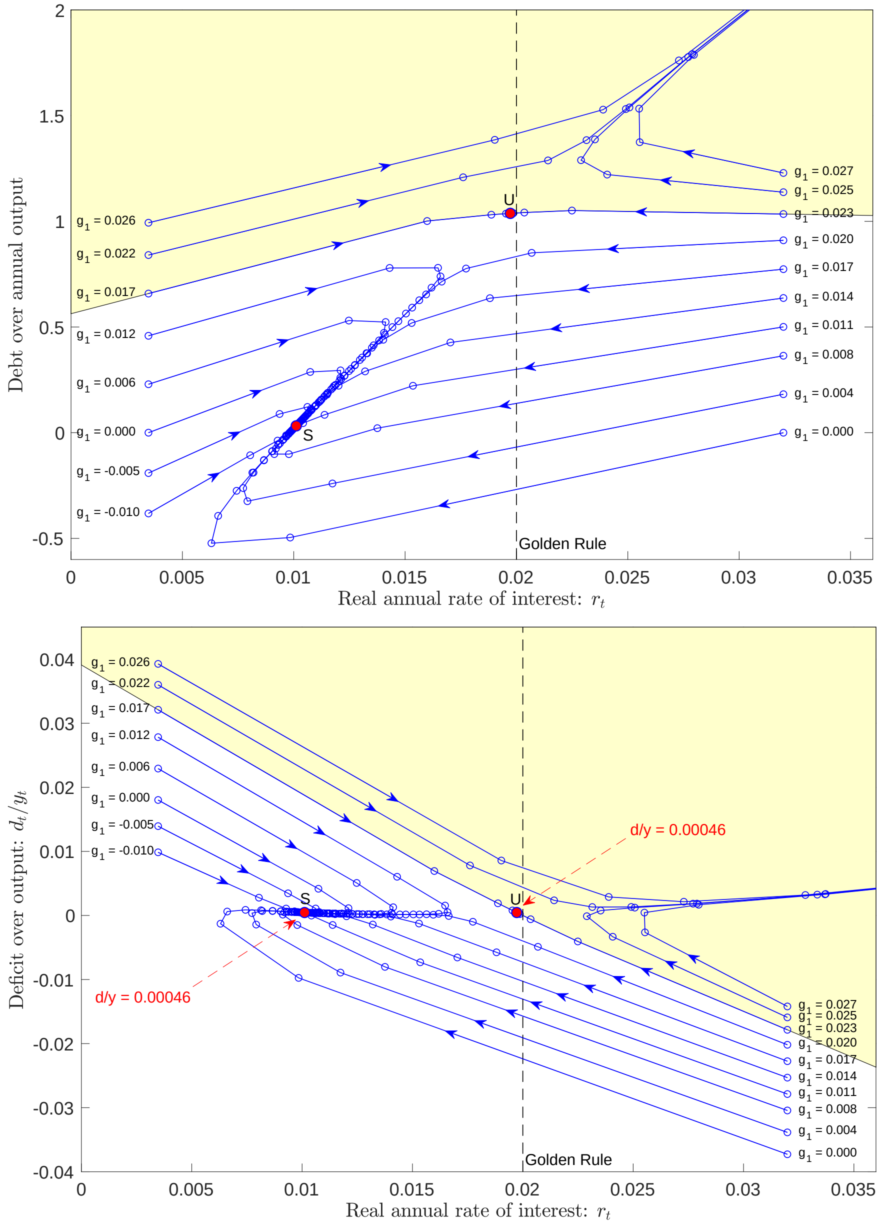

Figure 3: **The paths of the debt ratio and the real interest rate and of the deficit ratio and the real interest rate for two initial values k_1 and several initial values g_1.** Illustration with log utility and Cobb-Douglas production function. Parameter values are: $A = 1$, $n = (1+0.02)^{25} - 1$ (2%/year), $\alpha = 0.2$, $\beta = 0.99^{25}$ (0.99/year), $\theta = 0.165$, $\tau = 0.1$. The stable steady state is marked "S" and the unstable one "U". g is debt per capita; y is output per capita over 25 years and d is deficit over 25 years. Annual output is $y/25$.

3.2 Short-lived but farsighted households

In Section 1.3, we argued that the solution to our system of difference equations should be calculated forward, using as an argument the mathematical theory of "Divergent Series."

Such was the relevant solution because the debt could be rolled over forever. When the old generation sells the debt to the young generation, the young are willing to buy it because they know that the generation after them will buy it and so on and so forth. Obviously, for this reasoning to be valid there should be no "end of time," and no terminal condition.[38] In this case, only initial conditions are to be specified. This is a case of rational myopia, in which future events, beyond one period, are not included in the marginal calculation of economic agents.

The reasoning fails, as it does in a Ponzi scheme, if there exists an end of time. Despite the myopia of each individual generation, there is an upper bound on the level of debt for the following reason.

If the path of debt is explosive, there occurs at some point in time a situation in which the debt has become too large to be purchased by the young generation. To stay within their budget constraint, the young would attempt to build a negative capital stock, which is not allowed on physical grounds. Since this would be known to the last young generation that lived just before this happened, the debt could not be sold to them.[39] It then could not be sold to the previous generation and so on; hence, any equilibrium with explosive debt would unravel. This means that rational explosive paths cannot even begin: even at the initial date, the debt cannot be sold if its face value is larger debt capacity. Households are farsighted in the sense that they are able to anticipate a refinancing difficulty in the distant future.

When convergent, the paths of debt are calculated forward. However, the elimination of divergent paths with the attendant calculation of debt capacity, is a forward looking operation. They are, therefore, calculated backward in a rationally expected way, as though households were infinitely lived. That is, indeed, true for the saddle path that leads to the unstable steady state.[40]

When the time-0 face value of debt is above debt capacity, one can call it "explosive." But it is more properly called "unsustainable" or, at most, "potentially explosive." A "debt crisis" occurs. Unless a change of policy can be expected, we must then imagine a renegotiation taking place between the old generation and the young one outside the financial market. The old generation has little choice but to accept a write down of the debt to the level of debt capacity.

3.3 Example of CES production cum isoelastic utility

Example 6. *Example 1 continued: as mentioned earlier, we include in the top plot in Figure 3 a diagram of paths in the plane of the ratio of debt g to annual output ($y/25$) versus the real rate of interest r. The diagram shows that, for the higher of the two initial capital stocks displayed, a g_1 of* 0.0172 *(which is at about* 65.83% *of annual output), and for the lower one a g_1 of* 0.0227 *(which is at* 103.46% *of annual output), lead to the unstable steady state. At the unstable steady state itself, the debt is at* 103.76% *of annual output. At the stable steady state, the debt is at* 3.23% *of annual output. We also display in the bottom plot in Figure 3 the paths in the plane of the ratio of budget deficit to output d/y versus the real rate of interest r. The ratio d/y can be viewed as a Maastricht criterion. We see that the deficit ratio is equal to* 0.0458% *in both steady states. Indeed, in the case of the Cobb-Douglas production function, the labor share $\mathfrak{w}(k)/f(k)$ is equal to $1-\alpha$, independent of k, so that the steady-state social-security deficit over output ratio is*

$$(1-\alpha)\left(-\tau+\frac{\theta}{1+n}\right) \tag{22}$$

in both steady states.

4 Fiscal policy experiments and the role of population growth

We now make use of the concept of debt capacity to run policy experiments. If future policies that are stabilizing are anticipated, correctly or incorrectly so, debt may start above capacity on a seemingly explosive

[38]Terminal condition such as transversality conditions at infinitty apply as necessary conditions of optimality when an agent *with an infinite lifetime* maximizes his lifetime utility. There is no such agent in this economy.

[39]We have not found any scenario in which the debt could be sold for a market value equal to a fraction of its face value.

[40]By the same token, when no steady state exists in our model, there can be no debt at all, as all paths are explosive.

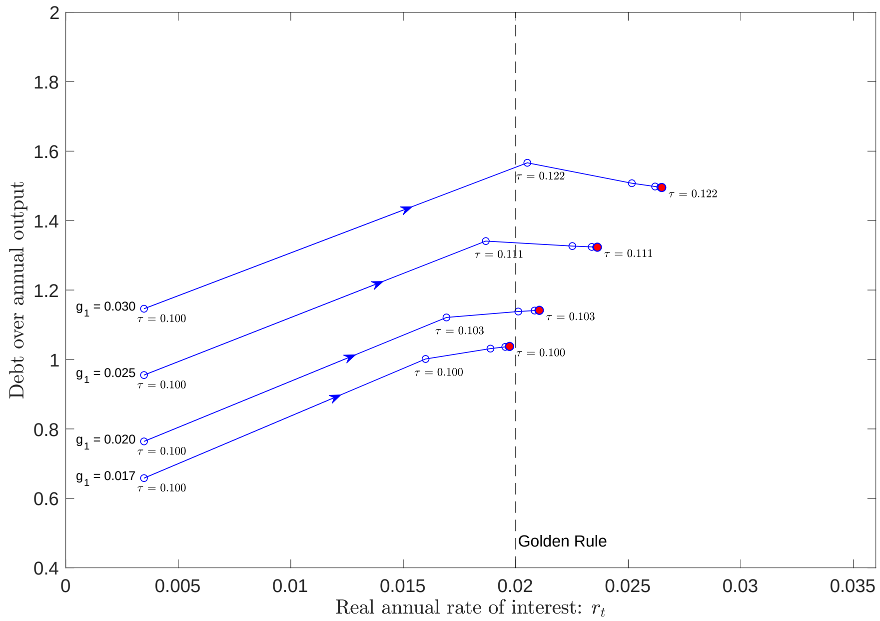

Figure 4: **The paths of the debt ratio and the rate of interest when debt starts above capacity and a policy response takes place with a delay of one period ($M = 1$).** Parameter values are: $A = 1$, $n = (1 + 0.02)^{25} - 1$ (2%/year)), $\alpha = 0.2$, $\beta = 0.99^{25}$ (0.99/year), $\theta = 0.165$, $\tau = 0.1$. Initial capital per capita is $k_1 = 0.12$ with $r_1 = 0.0035$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capacity saddle path ($g_1 = 0.0172$, which is a debt over annual output ratio equal to 0.6583). The policy responses are indicated in the column labelled "τ" in Table 1.

	Initial debt per capita	Initial debt / annual output	τ	Steady-state r/year	Steady-state debt /annual output	Steady-state deficit /annual output
$M = 0$	0.0300	1.1461	0.1145	0.0247	1.3920	−0.2787
	0.0250	0.9551	0.1077	0.0227	1.2633	−0.1432
	0.0200	0.7641	0.1024	0.0208	1.1217	−0.0359
	0.0172	0.6583	0.1000	0.0197	1.0376	0.0115
$M = 1$	0.0300	1.1461	0.1216	0.0265	1.4953	−0.4213
	0.0250	0.9551	0.1106	0.0236	1.3234	−0.2014
	0.0200	0.7641	0.1030	0.0210	1.1415	−0.0487
	0.0172	0.6583	0.1000	0.0197	1.0376	0.0115
$M = 2$	0.0300	1.1461	0.1411	0.0305	1.6893	−0.8103
	0.0250	0.9551	0.1171	0.0254	1.4318	−0.3296
	0.0200	0.7641	0.1042	0.0215	1.1748	−0.0717
	0.0172	0.6583	0.1000	0.0197	1.0376	0.0115

Table 1: **Tax rate, interest rate, and deficit responses to over-capacity initial debt.** Parameter values are: $A = 1$, $n = (1 + 0.02)^{25} - 1$ (2%/year), $\alpha = 0.2$, $\beta = 0.99^{25}$ (0.99/year), $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k_1 = 0.12$ with $r_1 = 0.0035$, which is a high amount of capital and a low rate of interest in the context of our model illustration. Debt starts on or above the debt-capapcity saddle path. The policy response can take place at the initial point, $M = 0$, or with a delay, $M = \{1, 2\}$.

path. The stabilizing responses that are needed represent *the true fiscal cost of exceeding debt capacity.* We illustrate such scenarios in Figure 4 for the case of high initial capital stock. As before, the debt capacity per capita at the initial point in time is $g_1 = 0.0172$ (debt/annual output = 65.83%), which is on the saddle path leading to the steady state marked U. If the debt starts above that level (at $g_1 = 0.02$, for instance), it embarks on a seemingly explosive path that is rectified after one period of 25 years, by means of an increase in the wage-tax rate τ, in order to put the economy on another saddle path. Notice a very important effect of this rectification. The steady state that follows rectification features a rate of interest (2.08%/year) that is *above* the Golden-rule rate, equal to the rate of population growth $n = 2\%$/year. Correspondingly, the new steady-state primary budget is in surplus. The reason is that the initial steady state was close to n, leaving little room for extra debt. During the period of delay, deficits and the debt accumulate so that the delay in the response is in the direction of forcing the government thereafter to run a surplus by means of a higher tax increase or reduced expenditures (as happened in Italy in the years 2015-2019).

In Table 1, we calculate the needed quantitative responses. When the response is immediate ($M = 0$), an initial debt equal to 114.61% of annual output ($g_1 = 0.03$) requires a tax of 11.45%, which is above the 10% considered so far and leads to a steady-state rate of interest equal to 2.47%/year. When the response is delayed to the next generation ($M = 1$), the tax rate for the same initial debt must be raised to 12.16%, which leads to a much larger steady-state surplus and a larger steady-state interest equal to 2.65%/year. We see again that, as the government delays the policy response, the new tax rate required to prevent the government debt from exploding is sufficiently high to cause a switch to $r > n$ and a large primary surplus.

	τ	Steady-state r/year	Steady-state debt /annual output	Steady-state deficit/annual output
		Initial drop to 1%/year		
$M = 0$	0.1285	0.0099	0.9060	0.0030
$M = 1$	0.1346	0.0129	1.2302	−0.1192
$M = 2$	0.1540	0.0185	1.6947	−0.5060
		Initial drop to 1.5%/year then to 1%/year		
$M = 0$	0.1301	0.0109	1.0155	−0.0284
$M = 1$	0.1418	0.0154	1.4513	−0.2635
$M = 2$	0.1755	0.0229	1.9619	−0.9364

Table 2: **Tax rate, interest rate, debt and deficit responses to declining population growth.** Parameter values are: $A = 1$, $n = (1 + 0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\theta = 0.165$, $\tau = 0.1$. Initial capital per labor is $k_1 = 0.12$ with $r_1 = 0.0035$, which is a high amount of capital and a low rate of interest in the context of the model. Debt starts on the saddle path, which means at capacity (debt per capita $g_1 = 0.0172$, which is debt/annual output = 0.6583). At the initial point in time, or at the initial point and then again at the second point in time, the annual population growth rate drops to the levels indicated. The policy response can take place immediately, $M = 0$, or with a delay, $M = \{1, 2, 3\}$.

We also explain how debt could come to exceed debt capacity. Debt may become unsustainable, for instance, because of a drop in the growth rate of the population that is not expected by the financial market. For that reason, we turn to a major concern that one might have regarding the debt capacity of a government. In real life, population growth has been in decline in every single industrialized economy, with technological progress being a partial offset. In our model, when the population growth rate falls, the debt capacity shrinks and the economy may move to an exploding path, which, as we saw, would actually unravel. Here again, in case of explosion, the government could no longer sell its debt unless it increased taxes, or promised to do so, and thereafter embarked on a new saddle path.

To generalize the principle that the later the policy response, the larger it has to be, we develop hypotheses of lower population growth, starting debt exactly on the saddle path and varying the timing of the response. Specifically, we start with a high k_1 (0.12) and with a debt at capacity (debt over annual output g_1 equal to 65.83%). In the first hypothesis, the rate of population growth is 1%/year instead of 2%/year. The policy response can take place immediately ($M = 0$) or with a delay of several quarter centuries ($M = \{1, 2\}$). The top panel of Table 2 indicates the value to which the government must raise the contribution or tax rate τ in order to stay on a saddle path and avoid an unsustainable situation. It is clear from this panel that, if the government increases the tax rate with delay ($M > 0$), it must raise it more than otherwise: changing the tax rate in the same period as the decline in n requires an increase from 10% to 12.85% while two periods after the decline in the population growth rate an 15.4% tax is required.

In the second hypothesis (bottom panel of Table 2), the rate of population growth is 1.5%/year initially and drops 1%/year at the next generation. If this demographic evolution is anticipated by the government so that it acts right at the initial point in time, the tax needed for sustainability is only 13.01%, whereas, if it waits till the second drop in population growth, the tax must be as high as 14.18%. In both scenarios, the need arises in case of delay to generate a huge steady-state budget surplus.

What these policy experiments teach us is that the true fiscal cost of excessive government debt issuance should be assessed in a dynamic context reflecting anticipated deficits and population growth going forward. A switch to surpluses may be needed in the future, either through increased tax rates, reduced social-security benefits, or both, all being politically painful.

5 Extensions

We have shown so far how debt capacity can be defined on the basis of a very simple OLG model with growth. In this section, we want to enrich the model, give it more policy substance and open the way towards future implementation. Specifically, we consider two additional forms of intervention by the government. First, we aim to increase our model's degree of realism by introducing nominal considerations. For that, we allow the debt of the government (consolidated with the central bank) to be nominal, and for it to be bought and sold as a way to implement a form of monetary policy. Second, we ask whether a government can increase its debt capacity by subsidizing innovation, which subsequently raises productivity and growth.

5.1 Money and the role of the central bank

The prospect of the money vs. bond trades of the central bank, and of monetization of the debt, might modify the debt capacity of the government, because households are forced to hold money whereas they choose freely to hold bonds and to subscribe to new issues.

To investigate these issues, we now assume realistically that debt is contractually denominated as a nominal amount, this debt being the consolidated debt of the government and the central bank. The same is true, of course, for the amount of money in circulation.[41] We consider first the case in which the government intervenes in the money market, buying and selling bonds, as a way to fix the nominal rate of interest in accordance with a simple version of the Taylor rule, in addition to collecting taxes and paying benefits. As before, government debt is a one-period debt. Let the nominal rate of interest be i_t. We then compare the results with those obtained when the government does not intervene.

Let $M_{1,t}$ be the young households' total money demand in *real* units (total nominal money balances deflated by the price level P_t) and $m_{1,t} = M_{1,t}/L_t$ is the *per capita* money demand in real units. They need it because they must turn into cash their wage, which is paid to a bank account. Cash can be withdrawn by taking trips to the bank. Each trip costs a fixed real amount ν (a transaction technology coefficient). Old households do not demand money: the social-security benefits are paid directly in cash. The per capita cost of trips to the bank is refunded to the young households, as a real *lump-sum amount* coming from unspecified outside resources and equal to:

$$\zeta_{1,t} \triangleq (1-\tau) \times w_t \times \frac{\nu}{2 \times m_{1,t}}$$

The simultaneous budget constraints at time t are as follows:

- for the young household,

$$c_t^t + s_t + m_{1,t} = (1-\tau) \times w_t \times \left(1 - \frac{\nu}{2 \times m_{1,t}}\right) + \zeta_{1,t} \tag{23}$$

[41] This implies that, in this section, the real amounts of debt and money outstanding are not given contractually at the initial point in time. As a remedy, we assume that the initial price level is inherited from history.

- for the old household,[42]

$$c_t^{t-1} = s_{t-1} \times \frac{1+i_t}{1+\pi_t} + m_{1,t-1}\frac{1}{1+\pi_t} + \theta \times w_{t-1} \tag{24}$$

where $\pi_t \triangleq P_t/P_{t-1} - 1$ is the rate of inflation and i_t is the nominal rate of interest,

- for the government (consolidated with the central bank),

$$-G_{t+1} - M_{2,t} + \theta w_{t-1}L_{t-1} = \tau w_t L_t - (1+i_t)\frac{1}{1+\pi_t}G_t - \frac{1}{1+\pi_t}M_{2,t-1}$$

or,

$$-(1+n)g_{t+1} - m_{2,t} + \theta \times w_{t-1}\frac{1}{1+n} = \tau \times w_t - \frac{1+i_t}{1+\pi_t}g_t - \frac{1}{1+\pi_t}\frac{1}{1+n}m_{2,t-1} \tag{25}$$

where $M_{2,t}$ is the total money supply in *real* units (and $m_{2,t} = M_{2,t}/L_t$), G_t is the total debt in *real* units (total nominal debt deflated by P_t) with which the government enters time t and G_{t+1} is the debt in real units with which it exits time t (and $g_t \triangleq G_t/L_t$).

The need to procure money when young is an additional tax collected by the government. When households get old, money is dispensed with, which constitutes an additional benefit for them. The new terms in the government budget constraint net out to:[43]

$$m_{2,t} - \frac{1}{1+\pi_t}\frac{1}{1+n}m_{2,t-1} = \left(m_{2,t} - \frac{1}{1+n}m_{2,t-1}\right) + \frac{1}{1+n}m_{2,t-1} \times \left(1 - \frac{1}{1+\pi_t}\right)$$

where the first term on the right-hand side is money creation and the second term is the seigniorage collected on existing balances. Seigniorage is determined by the rate of inflation.

Define the savings and money-holding functions:[44]

$$\begin{aligned}\{\mathfrak{s}(w,i,\pi), \mathfrak{m}(w,i)\} \quad \triangleq \quad & \arg\max_{s,m} \left[u\left((1-\tau) \times w \times \left(1 - \frac{\nu}{2 \times m}\right) + \zeta_{1,t} - s - m\right)\right. \\ & \left. +\beta u\left(s \times \frac{1+i}{1+\pi} + m\frac{1}{1+\pi} + \theta \times w\right)\right]\end{aligned}$$

Market clearing is still as it was in Section 1, but $m_{1,t} = m_{2,t}$ (where $m_{1,t} = \mathfrak{m}(w_t, i_{t+1})$; see (26)) is an additional market-clearing equation. The initial conditions are set by the initial capital stock $k_0 = K_0/L_0$ and the amounts of *nominal* debt and money $G_1 \times P_1$ and $M_{2,0} \times P_1$ (these products are given numbers) with which the government enters time 1.

Monetary policy affects debt capacity through two channels. Implicitly, cash holding m affects capital accumulation. Explicitly, it affects debt capacity g/y through the deficit term, which contains seigniorage revenues. In steady state,

$$\frac{g}{y} = \underbrace{\frac{1}{n-r}}_{g/d}\frac{d}{w}\frac{w}{y}; \frac{d}{w} = \underbrace{\left(\frac{\theta}{1+n} - \tau\right)}_{d/w \text{ without money}} + \underbrace{\left(\frac{1}{(1+n)(1+\pi)} - 1\right)\frac{m}{w}}_{\text{money-balance effect}}$$

[42] Seigniorage is not refunded.

[43] We now call "deficit" the following amount

$$d_t = \theta \times w_{t-1}\frac{1}{1+n} - \tau \times w_t - m_{2,t} + \frac{1}{1+\pi_t}\frac{1}{1+n}m_{2,t-1}$$

which includes the income for the government that arises from money. It is not the primary deficit as usually defined as it includes proceeds from money issuance.

[44] The solution for money demand is explicit and takes the form of the familiar Allais-Baumol-Tobin square-root rule of inventory management:

$$\mathfrak{m}(w,i) = \sqrt{\frac{(1-\tau) \times w \times \left(\frac{\nu}{2}\right)}{1 - \frac{1}{1+i}}} \tag{26}$$

Monetary intervention mode: the behavior of the government is dictated by the following simple Taylor rule:

$$1+i_{t+1}=(1+\bar{\imath})\times\left(\frac{1+\pi_{t+1}}{1+\bar{\pi}}\right)^{\phi};\phi\geq 0;\phi\neq 1 \tag{27}$$

which contains two policy parameters: ϕ (the Taylor coefficient) and $(1+\bar{\imath})/(1+\bar{\pi})^{\phi}$ (the target level). With this form of intervention, whether the Taylor coefficient is above or below 1, we found again (in our numerical example below) two steady states – one stable and one unstable – just like in the basic, real economy of Section 1.

No-monetary intervention mode: we define "no intervention" as a policy by which the consolidated government cum central bank keeps constant the *nominal* amount of money in circulation per capita, or, equivalently, grows the *nominal* amount of money in circulation at the rate of population growth, irrespective of the actual rate of growth of the economy or the rate of inflation:

$$\begin{aligned} m_{2,t+1}P_{t+1} &= m_{2,t}P_t \\ M_{2,t+1}P_{t+1} &= (1+n)\,M_{2,t}P_t \end{aligned}$$

Under this specification, we expect the steady-state rate of inflation to be equal to zero. Two steady states satisfy that restriction, both of them unstable. Of the two, one leads to a negative steady-state value for the sum of debt and money. We select the other steady state.

Numerical example and comparison of policies:

Example 7. *(Example 1 continued with the addition of money) In table 3, we describe the saddle paths obtained with money, under no intervention and with various values of intervention parameters, by exhibiting the initial point (for the higher initial capital stock* $k_1=0.12$*) and the ultimate point (the unstable steady state) of a saddle path, for different values of the policy parameters* ϕ *and* $\bar{\imath}$*, while keeping the inflation parameter of the Taylor rule at* 0.02*/year (i.e.,* $\bar{\pi}=(1+0.02)^{25}-1$*). The transaction technology coefficient* ν *(*$\nu=0.003$*) is selected so that the equilibrium amount of money balances is of the same order of magnitude as annual output, which is empirically true in developed countries.*

Focusing on debt capacity in this monetary economy, four sets of observations can be made on the basis of Table 3 *for the cases of intervention.* First, at the initial points of the saddle paths, the capital stock is given ($k_1=0.12$), so that the real rate of interest r_1 (column (7)), output and the wage rate are invariant with respect to the policy parameters ϕ and $\bar{\imath}$. As a mechanical effect of the Taylor rule:

$$1+r_t=\frac{1+i_t}{1+\pi_t}=\frac{1+\bar{\imath}}{(1+\bar{\pi})^{\phi}}(1+\pi_t)^{\phi-1} \tag{28}$$

the rate of inflation π_t (column (9)) and seigniorage revenues (column (5)) increase when $\bar{\imath}$ increases, if the Taylor coefficient $\phi<1$, whereas they decrease if $\phi>1$. The first two columns show the amount of debt outstanding (which is the initial debt capacity, this being the saddle path) and the amount of money balances in circulation. Notice that these two vary in opposite directions as the policy parameter $\bar{\imath}$ is increased. If $\phi<1$, debt capacity increases and money balances decrease. If $\phi>1$, the opposite is true. Money and bonds are to some extent substitute liabilities of the government. In the presence of money, one can venture to define financing capacity as the sum of the two. If one summed columns (1) and (2), one would find that *total financing capacity* increases with $\bar{\imath}$ if $\phi<1$ and decreases if $\phi>1$, entirely because of the prospect of seigniorage revenues being affected in that same way, while the social security deficit, which is based on wages,[45] remains almost unchanged. Financing capacity is clearly higher for $\phi=2$ than for $\phi=1/2$. Debt capacity viewed separately relies additionally on the prospect of money issuance (column (4)). Debt issuance is not shown in the table but one would find that the sum of money and debt issuances is almost constant, confirming the substitution between debt and money.

Second, at the (unstable) steady state, the real rate of interest r_t (column (7)), output and the wage rate differ a lot depending on the policy parameter ϕ but are almost invariant to the policy parameters $\bar{\imath}$. Hence the Taylor rule mechanics (28), which we described à propos the initial point, produce here the same effects

[45] It is based on wages at two successive points in time.

	Initial point ($k_1 = 0.12$)									Steady state								
col	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
No intervention	0.856	1.310	0.058	0.005	0.017	0.036	0.003	0.020	0.016	1.241	0.927	0.000	0.024	0	-0.023	0.030	0.030	0
Intervention																		
$\phi = 0.5$																		
$\bar{\imath}$																		
0.024	0.563	1.162	0.049	0.052	0.019	-0.022	0.003	0.024	0.021	-0.831	2.411	0.000458	0.062	-0.068	0.007	0.025	0.003	-0.021
0.026	0.756	1.114	0.051	0.040	0.020	-0.009	0.003	0.028	0.025	-0.093	1.810	0.000458	0.046	-0.047	0.001	0.026	0.006	-0.020
0.028	0.909	1.074	0.053	0.032	0.022	-0.001	0.003	0.032	0.029	0.317	1.504	0.000458	0.039	-0.034	-0.004	0.027	0.009	-0.018
0.030	1.034	1.042	0.055	0.027	0.023	0.005	0.003	0.036	0.033	0.587	1.318	0.000458	0.034	-0.025	-0.008	0.028	0.012	-0.015
0.035	1.267	0.984	0.058	0.020	0.026	0.013	0.003	0.047	0.043	0.991	1.065	0.000458	0.027	-0.010	-0.016	0.029	0.020	-0.009
$\phi = 2$																		
$\bar{\imath}$																		
0.024	1.157	1.251	0.063	-0.005	0.019	0.049	0.003	0.023	0.020	1.663	0.728	0.000458	0.019	0.020	-0.038	0.032	0.083	0.049
0.026	1.095	1.293	0.063	-0.006	0.018	0.051	0.003	0.021	0.018	1.656	0.731	0.000458	0.019	0.020	-0.038	0.032	0.081	0.047
0.028	1.024	1.342	0.063	-0.008	0.017	0.053	0.003	0.019	0.016	1.649	0.734	0.000458	0.019	0.020	-0.038	0.032	0.079	0.045
0.030	0.943	1.399	0.062	-0.010	0.016	0.056	0.003	0.017	0.014	1.641	0.737	0.000458	0.019	0.019	-0.038	0.032	0.076	0.043
0.035	0.668	1.605	0.062	-0.017	0.013	0.066	0.003	0.012	0.009	1.619	0.747	0.000458	0.019	0.018	-0.037	0.032	0.071	0.038

Table 3: **Saddle paths with money.** Parameter values are: $A = 1$, $n = (1+0.02)^{25} - 1$, $\alpha = 0.2$, $\beta = 0.99^{25}$, $\theta = 0.165$, $\tau = 0.1$, $\bar{\pi} = (1+0.02)^{25} - 1$ and $\nu = 0.003$. Initial capital per head of labor is $k_1 = 0.12$, which is a high amount of capital in the context of our model illustrations. Debt starts at debt-capacity, on the saddle path, which ends at the unstable steady state. col. (1) = debt/annual output; col (2) = money balances/annual ouput; col (3) = social security deficit/ouput; col (4) = money issuance/output; col (5) = seigniorage revenue/output; col. (6) = primary deficit/output = (3) - (4) - (5); col. (7) = real rate of interest/year; col. (8) = nominal rate of interest/year; col. (9) = rate of inflation/year = (1+(8))/(1+(7)) - 1.

across values of $\bar{\imath}$, albeit at very different levels. It is true again that, when $\bar{\imath}$ increases, the rate of inflation π_t and seigniorage revenues increase if the Taylor coefficient ϕ is less than 1 whereas they decrease if $\phi > 1$.[46] And, steady-state total financing capacity increases again with $\bar{\imath}$ if $\phi = 1/2$ and decreases (slightly) if $\phi = 2$.

For the range of values of $\bar{\imath}$ considered in the table, the same Taylor rule mechanics produce a negative rate of inflation and negative seigniorage revenues when $\phi = 1/2$ and positive ones when $\phi = 2$. Examples of positive primary *surpluses* occur when $\phi = 1/2$. With that value of the policy parameter ϕ, steady-state inflation is negative (column (9) in the top right part of the table) and seigniorage revenues are, therefore, also negative. The primary deficit (column (6)) (and debt capacity (column (1))) are nonetheless small and can even turn to a surplus thanks to the large money creation (column (4)). The latter is large because, with negative inflation, money demand and, therefore, money balances (column (2)) are large and money creation in steady state is proportional to money balances (with a factor $1 - 1/(1+n)$). Here again, debt issuance largely offsets money issuance.

In short, $\phi = 1/2$ leads to a deflationary steady state and $\phi = 2$ to a moderately inflationary one.

Third, the first row of Table 3 reveals that, in all dimensions, the case of no intervention falls neatly in between the intervention cases $\phi = 1/2$ and $\phi = 2$. The strong implication to be drawn from this observation is that monetary intervention *per se* does not lead to an increase in total financing capacity (the sum of columns (1) and (2)) but that the capacity is larger for an intervention with Taylor coefficient above 1 than it is for the case of no intervention and for the case of intervention with $\phi < 1$.

Finally, the value of the steady-state capital stock as reflected in the real rate of interest (column (7)), is quite different from what it was in the absence of money, and different as well across values of the policy parameter ϕ. In all monetary cases, however, including the case of no intervention, the steady-state real rate of interest is greater than the real rate of growth set at 2%/year ($r > n$). This means that, relative to the initial point with $k = 0.12$, the stock of capital has dropped more than it did in the case without money and has *overshot* the Golden Rule. Money, in addition to debt, crowds out capital. When $r > n$, the rules of valuation that we have developed in previous sections are overturned: a government that runs a positive primary deficit in steady state must now hold assets (negative debt) and one that runs a positive primary surplus can have debt. That rule is reflected in columns (1) and (6) of the steady-state section of the table. The presence of money seems to stifle over time the $r < n$ roll-over miracle.

5.2 Innovation

In the model developed so far, the only source of perpetual growth is population growth (which can also be viewed as exogenous technical progress). However, in today's economy, productivity, thanks to innovation, keeps rising. As a second extension, we would like to know whether the government can increase its debt capacity by subsidizing innovation and fostering growth. To determine to what extent innovation can modify a government's debt capacity, we borrow a model from the literature on endogenous growth but let the government finance R&D, in addition to paying for social security. Growth by innovation is truly endogenous when R&D is decided by the private sector. Our focus, however, is on government expenditure. We choose a model of innovation that allows policy to have a direct effect on growth.[47]

The modifications to the specification of the economy are as follows.

The households/investors: intermediate goods are produced in varieties i, the cardinal number B_t of which grows like population

$$B_t = L_t \tag{29}$$

This assumption captures the fact that more people generate more varieties; see Jones (1999). For continuity with the previous specifications, in which the population growth rate n stood for all exogenous forms of growth, we now reduce that number to make room for endogenous growth.[48]

[46] As far as the ratio of the social-security deficit to output is concerned, its steady-state value is an invariant quantity. See Equation (22). This ratio is equal to what it was in the previous sections in the absence of money.

[47] See Aghion and Howitt (1998), Dinopoulos and Thompson (1998), Peretto (1998), and Young (1998). We use a reduced form formulated by Jones (1999) that is also used in the empirical paper of Laincz and Peretto (2006). Ferraro and Peretto (2020) develop a model of quality-improving innovation and firm entry and exit that allows them to study the evolution of government debt, like we do. However, they consider the case of no physical capital and of infinitely-lived households that satisfy a transversality condition. They find one steady state.

[48] That is, we solve for a new number n such that, when research activity is at a zero level, the overall growth rate, including population growth and varieties growth, remains what it was before. See Footnote 49.

Before they can be consumed or reinvested, varieties of intermediate goods are turned into a final good, the amount produced being Y_t:

$$Y_t = \left(\int_0^{B_t} X_{i,t}^{\frac{1}{m_y}} \, di \right)^{m_y}$$

where $m_y > 1$, X_{it} is the input of each variety of intermediate good and $\int_0^{B_t} di = B_t$. We look for an equilibrium that is symmetric across varieties: $\int_0^{B_t} X_{i,t} di = B_t X_{i,t} = X_t$. In other words, total production of intermediate goods is $X_t = B_t X_{i,t}$.

The production function for each variety i is

$$X_{i,t} = A_t F\left(K_{i,t}, L_{i,Y,t}\right)$$

where $K_{i,t}$ and $L_{i,Y,t}$ are the inputs of physical capital and labor into the production process of variety i, $A_t > 0$ is knowledge capital applicable in a non rival way to the production of all varieties. With symmetric use of capital and labor for the production of each variety,

$$X_{i,t} = A_t \frac{F\left(K_t, L_{Y,t}\right)}{B_t}$$

so that

$$\begin{aligned} X_t &= A_t F\left(K_t, L_{Y,t}\right) \\ Y_t &= \left(B_t X_{i,t}^{\frac{1}{m_y}} \right)^{m_y} = B_t^{m_y} X_{i,t} = B_t^{m_y - 1} A_t F\left(K_t, L_{Y,t}\right) \end{aligned}$$

The production and accumulation of knowledge capital is controlled by government expenditure. It evolves as

$$A_{t+1} - A_t = \frac{m_A L_{A,t} A_t}{B_t} \tag{30}$$

where $m_A > 0$ is the productivity of labor in knowledge production and $L_{A,t}$ is the amount of labor devoted by the government to knowledge production. As the number of varieties rises, more research labor is required to increase knowledge.

Taxation and spending: the government budget constraint becomes

$$-G_{t+1} + \theta w_{t-1} L_{t-1} + w_t L_{A,t} = \tau w_t L_t - \left(1 + r_t\right) G_t$$

Market clearing: the labor market clears

$$L_{Y,t} + L_{A,t} = L_t$$

and the financial market clears as in Equation (6).

Difference equations and steady states: Suppose that the government pays for a constant proportion h of labor to be involved in R&D: $L_{A,t} = h \times L_t$. With that, the only modification to the difference-equations system governing the evolution of the economy, stated on a per capita basis, is the replacement of the first-order conditions for capital and labor and Equation (4) with the following:

$$B_t^{m_y - 1} A_t f'\left(\frac{k_t}{1-h}\right) - 1 = r_t \tag{31}$$

$$B_t^{m_y - 1} A_t \left[f\left(\frac{k_t}{1-h}\right) - \frac{k_t}{1-h} f'\left(\frac{k_t}{1-h}\right) \right] = w_t \tag{32}$$

$$-\left(1+n\right) g_{t+1} + \theta w_{t-1} \frac{1}{1+n} + w_t h = \tau w_t - \left(1 + r_t\right) g_t \tag{33}$$

Equations (31) and (32) allow us to define r_t and w_t as functions $\mathfrak{r}(k_t, A_t, B_t)$ and $\mathfrak{w}(k_t, A_t, B_t)$. The savings function $s_t = \mathfrak{s}(w_t, r_{t+1})$ is unchanged. Proceeding to equate demand and supply, as we did in Section 1, we redefine the function $\mathfrak{d}$ and $\mathfrak{D}$:

$$\begin{aligned}
\mathfrak{d}(k_{t-1}, k_t, A_t, B_t, A_{t-1}, B_{t-1}) &\triangleq \frac{\theta}{1+n}\mathfrak{w}(k_{t-1}, A_{t-1}, B_{t-1}) \\
&\quad - (\tau - h) \times \mathfrak{w}(k_t, A_t, B_t) \\
\mathfrak{D}(k; k_t, k_{t-1}, A_t, B_t, A_{t-1}, B_{t-1}) &\triangleq (1+n) k_t - \mathfrak{s}(\mathfrak{w}(k_t, A_t, B_t), \mathfrak{r}(k, A_{t+1}, B_{t+1})) \\
+\frac{1+\mathfrak{r}(k_t, A_t, B_t)}{1+n} &\left[\mathfrak{s}(\mathfrak{w}(k_{t-1}, A_{t-1}, B_{t-1}), \mathfrak{r}(k_t, A_t, B_t)) - (1+n) k_t\right] \\
&\quad + \mathfrak{d}(k_{t-1}, k_t, A_t, B_t, A_{t-1}, B_{t-1})
\end{aligned}$$

Instead of two steady states, there are two "expansion paths" with the same growth rate $\varpi \neq n$, which we calculate below, but differing interest rates. One of them is stable as all paths that start in the debt-capacity region (to be determined) approach it; the other is unstable as all paths that do not start within the debt-capacity region diverge from it.

We turn to the calculation of growth rates on a steady-state path. The stock of knowledge capital A_t evolves autonomously as does the population. From Equation (30), for a constant policy h, its growth rate, denoted ϖ_A, is equal to $m_A \times h$ (independently of a steady-state assumption).

For example, for a Cobb-Douglas production function, Equation (31) says that $B_t^{m_y-1} A_t \left[k_t/(1-h)\right]^{\alpha-1}$ is constant:

$$(1+\varpi_A)(1+n)^{m_y-1}(1+\varpi)^{\alpha-1} = 1$$

which gives the steady-state rate of growth ϖ of capital per capita k_t (a rate which, in previous sections, was equal to 0):[49]

$$1+\varpi = \left[(1+\varpi_A)(1+n)^{m_y-1}\right]^{\frac{1}{1-\alpha}} = \left[(1+m_A h)(1+n)^{m_y-1}\right]^{\frac{1}{1-\alpha}}$$

Output per capita y, debt per capita g, deficit per capita d and the wage rate w all grow at that same rate at any steady state.

Given that

$$\begin{aligned}
\mathfrak{r}(k_t, A_t, B_t) &= B_t^{m_y-1} A_t \alpha \left(\frac{k_t}{1-h}\right)^{\alpha-1} - 1 \\
\mathfrak{w}(k_t, A_t, B_t) &= B_t^{m_y-1} A_t \times \left(\frac{k_t}{1-h}\right)^{\alpha} - \frac{k_t}{1-h}\left[\mathfrak{r}(k_t, A_t, B_t) + 1\right]
\end{aligned}$$

we can write the equation ($\mathfrak{D}(k_{t+1}; k_t, k_{t-1}, A_t, B_t, A_{t-1}, B_{t-1}) = 0$) for a steady-state interest rate r

$$\begin{aligned}
\frac{1}{(1+n)(1+\varpi)}\mathfrak{s}(1, r) &= \frac{1-h}{1+r}\frac{\alpha}{1-\alpha} \\
&+\frac{1}{(1+n)(1+\varpi)-(1+r)}\left(\frac{\theta}{(1+n)(1+\varpi)} - (\tau - h)\right)
\end{aligned} \tag{34}$$

and calculate the steady-state debt-capacity ratio, which is debt per output at the unstable steady-state:

$$\frac{g}{y} = \underbrace{\frac{1}{(1+n)(1+\varpi)-(1+r)}}_{\text{total growth rate minus interest rate}} \underbrace{\left(\frac{\theta}{(1+n)(1+\varpi)} - (\tau - h)\right)}_{d/w} \underbrace{\frac{1-\alpha}{1-h}}_{w/y} \tag{35}$$

[49] In these equations, as explained above, n is reduced to a number n_{new} such that

$$(1+\varpi|_{h=0})(1+n_{new}) = 1+n_{old}$$

$$(1+n_{new})^{\frac{\theta_Y-1}{1-\alpha}+1} = 1+n_{old}$$

For $1+n_{old} = (1.02)^{25}$, $n_{new} = 0.4183$ (over 25 years), which is 0.01408/year.

Equation (35) relates debt over output to deficit over output, where the deficit ratio d/w is adjusted by $1-\alpha$, as before but also by $1/(1-h)$ for the fact that some labor is diverted from the production of goods to the production of knowledge. That deficit is discounted in a manner that is analogous to Formula (18) of the basic model. The denominator is positive when the rate of interest is below the total growth rate. One would expect two opposing effects: R&D enhances growth but deepens the deficit of the government. Specifically, one would hope for a hump shaped relation where initially increasing h the growth effect of an increase in R&D dominates and thus the debt capacity increases. For some large enough value of h the deficit would then increase faster than growth implying that the debt capacity declines.

Numerical example and policy comparisons:

The plots in Figure 5 confirm this intuition. We display the way debt capacity changes as one varies the policy parameter h. The graphs are drawn for $m_A = 9.4$ (the degree to which the growth of knowledge responds to R&D labor input) and for $m_y = 1.33$, which corresponds to an elasticity of substitution between varieties equal to 4, a number accepted often by macroeconomists (see, for instance, Galí (2015)). The behavior of debt capacity against h is the result of several competing effects. First, in the denominator, an increase in R&D spending opens a race between the total growth rate $(1+n)(1+\varpi)-1$ and the rate of interest r, both of which increase with h, as the right-hand panels of the figure indicate. Second, in the deficit, which is the numerator, the race is between the rate of growth, which lightens the burden of benefits paid to the old and the rate of spending on R&D. We see that the steady-state debt capacity rises from 89.17% (see Example 6) to a highest value of about 94% for about the value of h that minimizes the deficit (second panel on the left-hand side) and also, approximately, for the value that minimizes the discount factor in the denominator (third panel on the left-hand side).[50]

Overall, we do see a hump-shaped relation between g/annual y and h but the hump occurs for small values of h of about 1%. Most high-income countries already spend more than 2% on R&D. Therefore, for most parameter configurations, steady-state debt capacity is not increased, or is even reduced, by an increase in public R&D spending beyond what it is already. Overall, this exercise does not show that public R&D spending miraculously lifts debt capacity. It remains conceivable that spending on education would have a larger impact.[51]

6 Conclusion

We have presented a model of an equilibrium economy in which the possibility of a debt crisis plays a key role. In an overlapping-generations economy with endogenous capital accumulation and interest rates, and a realistic social-security scheme where debt covers deficits from the scheme, debt is welfare improving and can have positive market value even though the government budget is forever in deficit. We showed that this economy, for sufficiently small deficits and for reasonable parameter values, admits two steady states. One is stable and the other unstable.

We stress that the extant literature that discusses the question of the sustainability of government debt has not proposed a definition of debt capacity. It only reports what fiscal measures are needed for the debt not to explode or it eliminates divergent paths. We, instead, focus on divergent paths in order to locate the frontier between convergent and divergent paths.

The model has allowed us to *identify the unstable steady state as the central pillar for the study of debt sustainability.* Others have focused on the comparison between the rate of interest and the rate of growth. It is actually not true that, when the rate of interest is below the rate of growth, the intertemporal budget constraint facing the government, by virtue of perpetual refinancing, no longer binds. Even then, there exists an upper bound on the amount of debt. *We have defined debt capacity as the level of debt that leads to the unstable steady state.* Whenever the market value of debt is below debt capacity, the debt converges to the stable steady state. If it is above, government debt would eventually grow beyond the ability of the young generation to purchase it from the old, so that by anticipation such paths actually unravel, which means that debt is unsustainable.

[50] For $9.4 < m_A \leq 1$, there exists a range of values of h over which the deficit turns into a surplus and the denominator $(1+n)(1+\varpi)-(1+r)$ switches sign. Over that range, that is, the unstable steady-state rate of interest is larger than the total growth rate. We have chosen $m_A = 9.4$ for the reason, invoked earlier, that we expect the governments of high-income countries to remain in deficit forever.

[51] See Akcigit, Pearce, and Prato (2020).

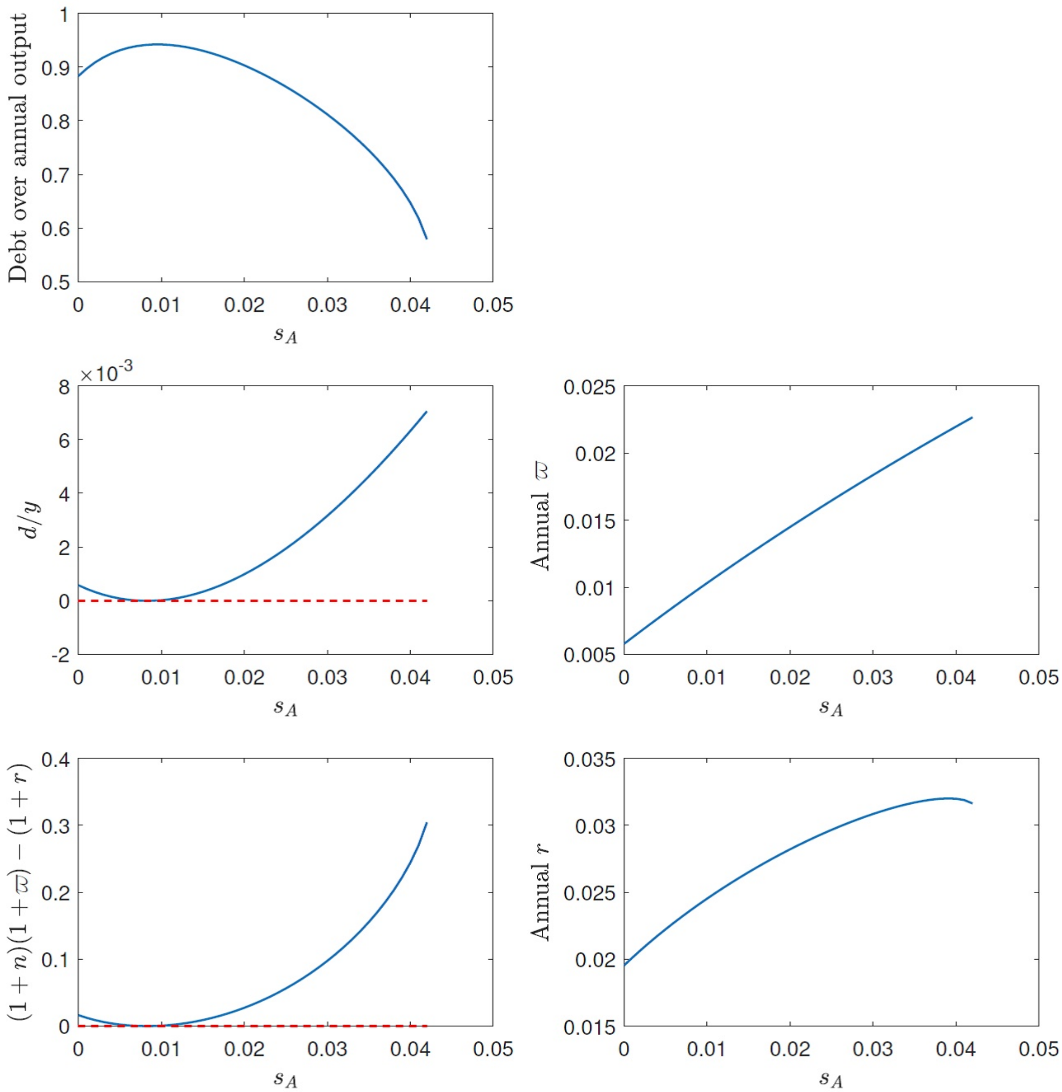

Figure 5: **Debt capacity over annual output, deficit over output, growth condition, rate of growth per year and rate of interest per year as a function of h at the unstable steady state.** Illustration with log utility and Cobb-Douglas production function. The lines stop at the value of h for which the unstable steady-state does not exist. In these examples we set the population growth rate such that the compounded growth rate is at 0.02/year for $h = 0$; see Footnote 49. We use $\alpha = 0.2$ and $m_Y = 4/3$ implying $n_{new} = 0.01408$/year (0.4183 over 25 years), and $m_A = 9.4$. The other parameter values are identical to what they are in the other figures.

Hence, the main policy implication of our model is that, even in a deterministic economy such as ours, the comparison of the rate of interest to the growth rate is not sufficient to determine whether debt is sustainable. The amount of outstanding debt (captured by the Debt-to-GDP ratio) also matters. It is for this reason that the notion of debt capacity is essential. When there exists a cliff somewhere, it is a good idea to find out where the edge of the cliff is. Our model is an attempt at locating the edge in a realistic policy setting.

We have used this basic idea to run policy experiments. If future policies that are stabilizing are anticipated, wrongly or rightly, debt may start above capacity on a seemingly explosive path. The stabilizing responses that must be implemented sooner or later represent the true fiscal cost of exceeding debt capacity. By way of illustration, we have examined demographic scenarios, which can lead to debt becoming unsustainable.[52]

We have extended the model and its concept of debt capacity to two policy-relevant settings. First, while it is widely believed that asset purchases by the central bank facilitate the expansion of government debt and fuel the risk of inflation,[53] that question taken literally is, in fact, moot because one should consolidate the government and the central bank. We study in a monetary version of the economy the sustainability of the consolidated debt and show that intervention with a Taylor coefficient greater than 1 increases total financing capacity in comparison with the case of no intervention and the case of a coefficient lower than 1.

Second, adding growth by innovation to our model, we have shown that, in all cases considered, a government R&D subsidy raises debt capacity for small subsidy amounts but, for larger subsidy rates, worsens it.

Are the high-income country government debt levels close to debt capacity right now? Careful econometric estimation of our model will have to be carried out before precise, quantitative answers can be given. We see steady increases in deficits and government debt levels relative to GDP. We do not, so far, see that social-security benefits, or other governmental services, or governmental spending more generally, are being reduced. In addition, population growth rates are declining and even turning negative or are, at least, predicted to turn negative. We have provided some illustrative numerical examples; these suggest that the current debt levels of high-income countries are close to debt capacity right now.

The lesson from our paper regarding the work of advisory government agencies is that the set of fiscal scenarios to be considered can and should be greatly amended to allow for a definition of debt sustainability and debt capacity that is grounded on the unstable steady state.

[52] There is evidence suggesting that productivity has declined over the last few decades. The effect of such a decline is similar.

[53] In some countries (such as Japan) government debt is mostly held by the central bank .

Appendixes

A Social security

In this appendix, we provide the rationale for having chosen to incorporate social security in our model as the form of government spending. We verify that, when the rate of interest is below the rate of growth, a social-security scheme, balanced or unbalanced, can be welfare improving, which is the reason for which we chose that form of government spending as an illustration. We focus on the steady-state lifetime welfare, which we define as in Diamond (1965). In either the stable or unstable steady state, the lifetime utility of one person is constant, generation after generation.

Five configurations are considered here, the first two being viewed as benchmarks: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with balanced security and zero government debt as in Blanchard and Fischer (1989), the equilibrium with balanced social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

The Diamond equilibrium is inefficient for the well-known reason that each generation, in order to finance their retirement, saves in excess of what they would if the welfare of all generations were optimized. As there is too much capital,[54] the steady-state utility is strictly smaller than in the Golden-rule equilibrium, which can be reached in the Tirole bubbly, zero-deficit equilibrium. These two facts are reflected in our Figure 6 (Example 1 continued) by the solid green line, which is below the solid blue line.

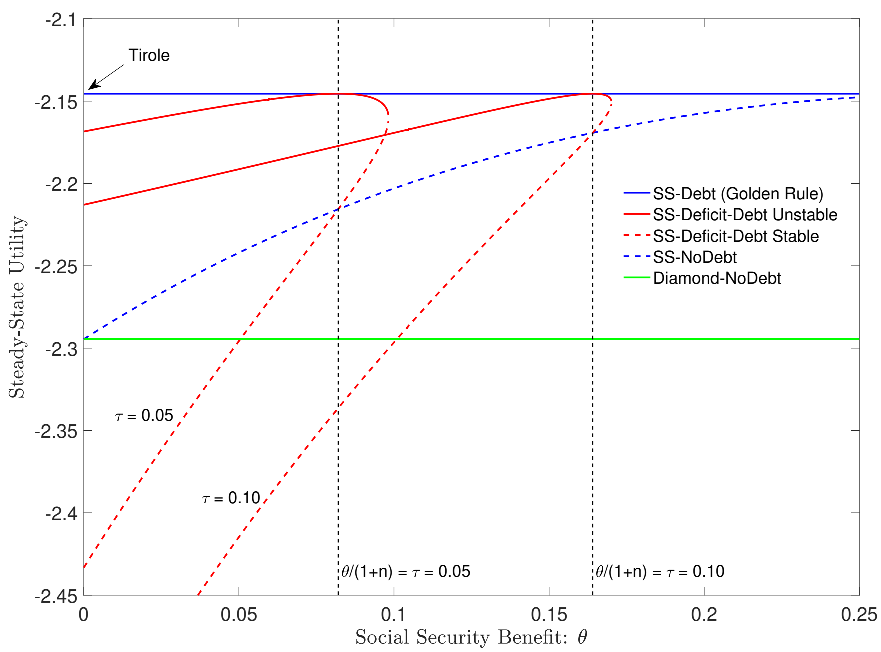

Figure 6: **Steady-state utilities.** Illustration with log utility and Cobb-Douglas production function. The parameter values are identical to what they are in the other figures. In the plot, we vary the social-security benefit, θ, and show the resulting steady-state utilities in the Diamond and Tirole models and in our stable and unstable steady-states, and for a special case without debt. Five configurations are considered: the Diamond (1965) equilibrium with no social security and no debt, the bubbly equilibrium of Tirole (1985) with zero deficit and no debt, the equilibrium with fully funded social security and zero government debt, the equilibrium with fully funded social security with pure roll-over government debt, and finally equilibrium with social security in deficit, financed by government debt.

[54] This comes with the caveat that in case of endogenous productivity growth, there are two kinds of capital: physical capital and knowledge capital. Knowledge capital may be too low, while physical capital is too high. See Section 5.2.

Because the government is infinitely lived, it alone can issue debt that can be perpetually refinanced. When the stock of capital is too high,[55] our Figure 6, – plotted against the level of benefits and for two levels of contributions (taxes) of 5% and 10% –, illustrates the fact that a budget deficit generated by social security and financed by debt can be a welfare-improving form of spending, relative to the competitive Diamond equilibrium.

The figure shows the special case of the equilibrium with balanced social security and no debt. That configuration can only approach the welfare optimum and never be equal to it.

More importantly, the figure shows that, with deficit social security, the unstable steady state, when it exists and assuming one can stay there, produces a larger utility per labor than the competitive equilibrium of Diamond (1965). For the special cases with zero deficit, such as $\theta/(1+n) = \tau = 0.05$ and $\theta/(1+n) = \tau = 0.10$, the unstable steady state with debt can actually reach the Golden Rule equilibrium, as does Tirole's bubble.

The stable steady states of equilibria with deficit social security also produce a welfare improvement, but only for sufficiently high values of the benefits. For lower values of the benefits, it is also possible for the stable steady state to exhibit smaller utility per capita than the competitive equilibrium of Diamond (1965).

B Three lemmata on savings

Lemma 6. *Under the assumptions made for the utility function and the production function, the following statements are both true:*

$$\lim_{k\to+\infty} \frac{\mathfrak{s}\left(w, \mathfrak{r}\left(k\right)\right)}{k} = 0 \tag{36}$$

$$\lim_{k\to+\infty} \frac{\mathfrak{s}\left(\mathfrak{w}\left(k\right), \mathfrak{r}\left(k\right)\right)}{k} = 0 \tag{37}$$

Proof. We have the following inequalities bearing on the savings function (consumption when young and when old are both strictly positive):

$$-\frac{\theta \times w}{f'\left(k\right)} < \mathfrak{s}\left(w, \mathfrak{r}\left(k\right)\right) < \left(1-\tau\right) w$$

This implies:

$$-\frac{\theta}{f'\left(k\right)} \frac{w}{k} < \frac{\mathfrak{s}\left(w, \mathfrak{r}\left(k\right)\right)}{k} < \left(1-\tau\right) \frac{w}{k}$$

For a fixed $w > 0$, the limit of w/k when $k \to \infty$ is 0 and, given the assumptions on the production function, $\lim_{k\to\infty} f'\left(k\right)$ is finite (De la Croix and Michel (2002), Appendix A.1.2, page 307).
For endogenous $\mathfrak{w}\left(k\right) > 0$,

$$-\frac{\theta}{f'\left(k\right)} \frac{\mathfrak{w}\left(k\right)}{k} < \frac{\mathfrak{s}\left(\mathfrak{w}\left(k\right), \mathfrak{r}\left(k\right)\right)}{k} < \left(1-\tau\right) \frac{\mathfrak{w}\left(k\right)}{k}$$

Given the assumptions on the production function, the limit of $\mathfrak{w}\left(k\right)/k$ when $k \to \infty$ is 0, as shown in De la Croix and Michel (2002), Appendix A.1.3, page 309.

Lemma 7.

$$\begin{aligned}\mathfrak{s}'_r\left(w, r\right) &= -\frac{\beta \times u'\left(s \times \left(1+r\right) + \theta \times w\right)}{u''\left(\left(1-\tau\right) w - s\right) + \beta\left(1+r\right)^2 \times u''\left(s \times \left(1+r\right) + \theta \times w\right)} \\ &\quad \times \left(1 - \frac{1}{\mathfrak{e}\left(s, r\right)}\right)\end{aligned}$$

■

[55] If, to the opposite, the stock of capital were below the welfare optimum (the rate of interest is above the one in the welfare optimum), the proceeds of government debt issue could, of course, be used for investment. But that is not the case considered here.

Proof. To obtain the derivative $\mathfrak{s}_r'$, define the function

$$\Upsilon(s,r,w) \triangleq -u'((1-\tau)w - s) + \beta(1+r) \times u'(s \times (1+r) + \theta \times w) \tag{38}$$

Because of the first-order condition attached to (7),

$$\Upsilon(\mathfrak{s}(w,r), r, w) = 0; \forall r$$

which implies

$$\Upsilon_s'(\mathfrak{s}(w,r,w), r)\,\mathfrak{s}_r' + \Upsilon_r'(\mathfrak{s}(w,r), r, w) = 0$$

Differentiate Υ:

$$\Upsilon_s'(s,r,w) = u''((1-\tau)w - s) + \beta(1+r)^2 \times u''(s \times (1+r) + \theta \times w)$$

$$\begin{aligned}
\Upsilon_r'(s,r,w) &= \beta \times u'(s \times (1+r) + \theta \times w) \\
&\quad + \beta(1+r) \times u''(s \times (1+r) + \theta \times w) \times s \\
&= \beta \times u'(s \times (1+r) + \theta \times w)\left(1 - \frac{1}{\mathfrak{e}(s,r)}\right)
\end{aligned}$$

where $1/\mathfrak{e}(s,r)$ is defined in (12). Then $\mathfrak{s}_r'(w,r) = -\Upsilon_r'(s,r,w)/\Upsilon_s'(s,r,w)$ gives the result. ■

Lemma 8. *If*

$$1 - \tau - \beta(1+r) \times \frac{u''(s \times (1+r) + \theta \times w)}{u''((1-\tau)w - s)} \times \theta > 0$$

then

$$\mathfrak{s}_w'(\mathfrak{w}(k_{t-1}), \mathfrak{r}(k_t)) > 0$$

Proof. Recall the function Υ defined in (38). Since

$$\Upsilon(\mathfrak{s}(w,r), r, w) = 0; \forall w$$

$$\mathfrak{s}_w'(w,r) = \frac{1 - \tau - \beta(1+r) \times \frac{u''(s \times (1+r) + \theta \times w)}{u''((1-\tau)w - s)} \times \theta}{1 + \beta(1+r)^2 \times \frac{u''(s \times (1+r) + \theta \times w)}{u''((1-\tau)w - s)}} > 0$$

■

C Proof of existence Theorem 1

This proof is an extension of the proof in De la Croix and Michel (2002), Section 1.5.1.

We first study the sign of $\Delta(k)$ when k tends to $+\infty$.

As a consequence of Lemma (36),

$$\lim_{k \to +\infty} \frac{\Delta(k)}{k} = (1+n)\left(1 + \frac{g}{k}\right) > 0$$

This implies that $\Delta(k)$ is positive for large values of k.

We now study the sign of $\Delta(k)$ when k goes to 0. There are two cases: the first case is one in which $\lim_{k \to 0, k>0} f'(k) = f'(0)$ is finite. Then (because $s \times (1+r) + \theta \times w > 0$)

$$\lim_{k \to 0} \Delta(k) = (1+n)g - \mathfrak{s}(w, \mathfrak{r}(0))$$

which is negative if $(1+n)g < \lim_{k \to 0} \mathfrak{s}(w, \mathfrak{r}(0))$.

In the second case, i.e., $\lim_{k \to 0, k>0} f'(k) = +\infty$. The return on savings becomes infinite as k approaches zero. In this case, savings cannot be negative in the limit because the present value of future benefits approaches 0. But savings can remain positive or tend to zero.

If savings remains positive, as assumed: $\lim_{k\to 0,k>0} \mathfrak{s}(w, \mathfrak{r}(k)) > 0$. This implies that

$$\lim_{k\to 0,k>0} \Delta(k) = (1+n)\,g - \lim_{k\to 0} \mathfrak{s}(w, \mathfrak{r}(k))$$

which is negative if $(1+n)\,g < \lim_{k\to 0} \mathfrak{s}(w, \mathfrak{r}(k))$.

D Proof of Equation (13)

The government budget or debt evolution is

$$-G_{t+1} + \theta_t w_{t-1} L_{t-1} = \tau_t w_t L_t - (1+r_t)\,G_t$$

which we first rewrite as

$$-(1+n)\,g_{t+1} + \theta_t w_{t-1} \frac{1}{1+n} = \tau_t w_t - (1+r_t)\,g_t$$

then as

$$(1+n)\,g_{t+1} = (1+r_t)\,g_t + d_t \qquad \text{where} \qquad d_t = \theta_t w_{t-1} \frac{1}{1+n} - \tau_t w_t$$

Rearranging leads to

$$g_t = \frac{1+n}{1+r_t}\left(g_{t+1} - \frac{d_t}{1+n}\right)$$

Incrementing from $t = 1$ to $\mathcal{T} > 1$ leads to

$$\begin{aligned}
g_1 &= \frac{1+n}{1+r_1}\left(g_2 - \frac{d_1}{1+n}\right) \\
&= \frac{1+n}{1+r_1}\frac{1+n}{1+r_2} g_3 - \frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{d_2}{1+n} - \frac{1+n}{1+r_1}\frac{d_1}{1+n} \\
&= \frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{1+n}{1+r_3} g_4 - \frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{1+n}{1+r_3}\frac{d_3}{1+n} - \frac{1+n}{1+r_1}\frac{1+n}{1+r_2}\frac{d_2}{1+n} - \frac{1+n}{1+r_1}\frac{d_1}{1+n} \\
&\vdots
\end{aligned}$$

$$g_1 = \frac{1}{1+n}\sum_{t=1}^{T-1} \frac{-d_t}{\prod_{u=1}^{t} \frac{1+r_u}{1+n}} + \frac{g_T}{\prod_{u=1}^{T-1} \frac{1+r_u}{1+n}} \qquad \forall T > 1$$

$$g_1 \prod_{u=1}^{T-1} \frac{1+r_u}{1+n} - \frac{1}{1+n}\sum_{t=1}^{T-1}\prod_{u=1}^{T-1}\frac{1+r_u}{1+n}\frac{-d_t}{\prod_{u=1}^{t}\frac{1+r_u}{1+n}} = g_T$$

$$g_T = g_1 \prod_{u=1}^{T-1} \frac{1+r_u}{1+n} + \frac{1}{1+n}\sum_{t=1}^{T-1}\left(\prod_{u=t+1}^{T-1}\frac{1+r_u}{1+n}\right) d_t$$

E On CES production cum isoelastic utility

$$\lim_{k_t\to 0}\frac{\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)}{k_t} = \lim_{k_t\to 0}\left[\frac{1}{f'\left(k_t\right)}\frac{\left(1-\tau\right)\beta^{\frac{1}{\varsigma}}f'\left(k_t\right)^{\frac{1}{\varsigma}}-\theta}{1+\beta^{\frac{1}{\varsigma}}f'\left(k_t\right)^{\frac{1}{\varsigma}}}\right]\frac{f\left(k_t\right)}{k_t}$$

$$\lim_{k\to 0,k>0}f'\left(k\right) = \begin{cases} A\alpha^{\frac{1}{\rho}} & \text{for} \quad \rho<0 \\ \sim A\alpha k^{\alpha-1} & \text{for} \quad \rho=0 \\ \sim A\alpha\left(1-\alpha\right)^{\frac{1-\rho}{\rho}}k^{\rho-1} & \text{for} \quad 0<\rho<1 \end{cases}$$

$$\lim_{k\to 0,k>0}\frac{f\left(k\right)}{k} = \begin{cases} \sim A\times\alpha^{\frac{1}{\rho}} & \text{for} \quad \rho<0 \\ \sim Ak^{\alpha-1} & \text{for} \quad \rho=0 \\ \sim \frac{A\left(1-\alpha\right)^{\frac{1}{\rho}}}{k} & \text{for} \quad 0<\rho<1 \end{cases}$$

F On steady states

F.1 Proof of Lemma 3

k_{GR} is either smaller or greater than k_D (barring the coincidental case of equality).
In the first case, the open interval $\mathcal{E}$ runs from k_{GR} to k_D. At $k_t=k_{GR}$, $n=\mathfrak{r}\left(k_t\right)$, and $\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)/k_t > 1+n$ (debt is positive or $\left(1+n\right)k_t-\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)<0$) because $\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)/k_t$ is assumed to be a strictly decreasing function that takes the value $1+n$ at k_D, which is above k_{GR}. If we increase k_t from the k_{GR} level, thus entering $\mathcal{E}$, $n-\mathfrak{r}\left(k_t\right)$ becomes positive so that $\Delta_0\left(k_t;k_t,k_t\right)$ becomes negative and $\Psi_0\left(k_t\right)>k_t$.
In the second case, the open interval $\mathcal{E}$ runs from k_D to k_{GR}. At k_D, $\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)/k_t = 1+n$, and $n-\mathfrak{r}\left(k_t\right)<0$ because $n-\mathfrak{r}\left(k_t\right)$ is a strictly increasing function that takes the value 0 at k_{GR}, which is higher. If we increase k_t from the k_D level, thus entering $\mathcal{E}$, $\left(1+n\right)k_t-\mathfrak{s}\left(\mathfrak{w}\left(k_t\right),\mathfrak{r}\left(k_t\right)\right)$ becomes positive so that $\Delta_0\left(k_t;k_t,k_t\right)$ again becomes negative and $\Psi_0\left(k_t\right)>k_t$.

F.2 Proof of Theorem 3

We prove by contradiction that the solutions to $\mathfrak{D}\left(k_t;k_t,k_t\right)=0$ are located in between the solutions to $\mathfrak{D}_0\left(k_t;k_t,k_t\right)=0$. Consider a candidate solution k_t of $\left(k_{t-1}=\Psi\left(k_t\right),k_{t-1}=k_t\right)$ outside $\mathcal{E}$.
In the case $k_{GR}<k_D$, the candidate solution would be a value $\check{k}$ of k_t greater than k_D or smaller than k_{GR}.
If $\check{k}>k_D$ (and $>k_{GR}$), $\mathfrak{s}\left(\mathfrak{w}\left(\check{k}\right),\mathfrak{r}\left(\check{k}\right)\right)/\check{k}<1+n$ and $\mathfrak{r}\left(\check{k}\right)<n$ so that $\mathfrak{D}_0\left(\check{k};\check{k},\check{k}\right)>0$ and $\check{k}>\Psi_0\left(\check{k}\right)$.
However, since the line $k_{t-1}=k_t$ is entirely in the region of positive deficit (Lemma 2), it must be, by Lemma 4, that $\Psi\left(\check{k}\right)<\Psi_0\left(\check{k}\right)$. But, then $\check{k}$ cannot be equal to $\Psi\left(\check{k}\right)$.
If $\check{k}<k_{GR}$ (and $<k_D$), $\mathfrak{s}\left(\mathfrak{w}\left(\check{k}\right),\mathfrak{r}\left(\check{k}\right)\right)/\check{k}>1+n$ and $\mathfrak{r}\left(\check{k}\right)>n$ so that $\mathfrak{D}_0\left(\check{k};\check{k},\check{k}\right)>0$ and $\check{k}>\Psi_0\left(\check{k}\right)$.
The same contradiction occurs.
In the case $k_D<k_{GR}$, the candidate solution would be a value $\check{k}$ of k_t smaller than k_D or greater than k_{GR}.
If $\check{k}<k_D$ (and $<k_{GR}$), $\mathfrak{s}\left(\mathfrak{w}\left(\check{k}\right),\mathfrak{r}\left(\check{k}\right)\right)/\check{k}>1+n$ and $\mathfrak{r}\left(\check{k}\right)>n$ so that $\mathfrak{D}_0\left(\check{k};\check{k},\check{k}\right)>0$ and $\check{k}>\Psi_0\left(\check{k}\right)$ and the same contradiction occurs.
If $\check{k}>k_{GR}$ (and $>k_D$), $\mathfrak{s}\left(\mathfrak{w}\left(\check{k}\right),\mathfrak{r}\left(\check{k}\right)\right)/\check{k}<1+n$ and $\mathfrak{r}\left(\check{k}\right)<n$ so that $\mathfrak{D}_0\left(\check{k};\check{k},\check{k}\right)>0$ and $\check{k}>\Psi_0\left(\check{k}\right)$ and the same contradiction occurs.

References

Abel, Andrew B., and S. Panageas, 2022, Optimal rollover of government bonds in a dynamically efficient economy, *Working Paper.*

Aghion, Philippe, and Peter Howitt, 1998, *Endogenous growth theory* (MIT Press).

Akcigit, Ufuk, Jeremy Pearce, and Marta Prato, 2020, Tapping into talent: Coupling education and innovation policies for economic growth, *BFI Working Paper 2020-137.*

Auerbach, Alan J., and Laurence J. Kotlikoff, 1987, *Dynamic Fiscal Policy* (Cambridge University Press).

Bassetto, Marco, and Wei Cui, 2018, The fiscal theory of the price level in a world of low interest rates, *Journal of Economic Dynamics and Control* 89, 5–22.

Blanchard, Olivier Jean, 2019, Public debt and low interest rates, *American Economic Review* 109, 1197–1229.

——— , 2022, *Fiscal Policy under Low Interest Rates* (MIT Press).

——— , and Stanley Fischer, 1989, *Lectures on Macroeconomics* (MIT Press).

Brunnermeier, Markus, Sebastian Merkel, and Yuliy Sannikov, 2022, The fiscal theory of the price level with a bubble, *Working Paper, Princeton University.*

Campbell, John Y., and Robert J. Shiller, 1988, The dividend-price ratio and expectations of future dividends and discount factors, *Review of Financial Studies* 1, 195–228.

Chalk, Nigel A., 2000, The sustainability of bond-financed deficits: An overlapping generations approach, *Journal of Monetary Economics* 45, 293–328.

Cochrane, John H., 2021, $r < g$, *Working Paper.*

Corhay, Alexandre, Thilo Kind, Howard Kung, and Gonzalo Morales, 2021, Discount rates, debt maturity, and the fiscal theory, *Working Paper.*

Croce, Max, Howard Kung, Thien Nguyen, and Lukas Schmid, 2012a, Fiscal policies and asset prices, *Review of Financial Studies* 25, 2635–2672.

——— , 2012b, The market price of fiscal uncertainty, *Journal of Monetary Economics* 59, 401–416.

Croce, Max, Thien Nguyen, Steven Raymond, and Lukas Schmid, 2019, Government debt and the returns to innovation, *Journal of Financial Economics* 132, 205–225.

De la Croix, David, and Philippe Michel, 2002, *A Theory of Economic Growth: Dynamics and Policy in Overlapping Generations* (Cambridge University Press).

Diamond, John W., 2005, Dynamic effects of extending the 2001 and 2003 income tax cuts, *International Tax and Public Finance* 12, 165–192.

——— , and A.D. Viard, 2008, Welfare and macroeconomic effects of deficite financed tax cuts: Lessons from cge models, in A.D. Viard, ed.: *Tax Policy Lessons from the 2000s* . pp. 145–193 (AEI Press, Washington, DC).

Diamond, John W., and George R. Zodrow, 2006, Reflections on the use of life-cycle computable general equilibrium models in analyzing the effects of tax reform, *NTA Network* pp. 3–4.

Diamond, Peter A., 1965, National debt in a neoclassical growth model, *American Economic Review* 55, 1126–1150.

Dinopoulos, Elias, and Peter Thompson, 1998, Schumpeterian growth without scale effects, *Journal of Economic Growth* 3, 313–335.

Eichengreen, Barry, Asmaa El-Ganainy, Rui Pedro Esteves, and Kris James Mitchener, 2019, Public debt through the ages, *NBER Working Paper 25494*.

Elenev, Vadim, Tim Landvoigt, Patrick Shulz, and Stijn Van Nieuwerburgh, 2021, Can monetary policy create fiscal capacity?, *Working Paper*.

Farmer, Roger E. A., and Pawel Zabczyk, 2020, The fiscal theory of the price level in overlapping generations models, *Working Paper*.

Ferraro, Domenico, and Pietro F. Peretto, 2020, Innovation-led growth in a time of debt, *European Economic Review* 121, 103350.

Furman, Jason, and Lawrence Summers, 2020, A reconsideration of fiscal policy in the era of low interest rates, *Working Paper*.

Galí, Jordi, 2015, Hysteresis and the european unemployment problem revisited, in *Inflation and Unemployment in Europe* Proceedings of the ECB Forum on Central Banking pp. 53–79. European Central Bank.

Gersbach, Hans, Jean-Charles Rochet, and Ernst-Ludwig von Thadden, 2022, Public debt, interest rates, and growth with incomplete financial markets, *Working Paper*.

Hardy, Godfrey H., 1973, *Divergent Series* (Oxford at the Clarendon Press).

Hellwig, M. F., 2021, Safe assets, risky assets, and dynamic inefficiency in overlapping-generations economies, *MPI Collective Goods Discussion Paper 2021/10*.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z. Xiaolan, 2021, What drives variation in the u.s. debt/output ratio? the dogs that didn't bark, *Working Paper, Northwestern University*.

Jiang, Zhengyang, Hanno Lustig, Stijn Van Nieuwerburgh, and Mindy Z. Xiaolan, 2021, The U.S. public debt valuation puzzle, *NBER Working Paper 26583*.

Jones, Charles I., 1999, Growth: With or without scale effects?, *American Economic Review, Papers and Proceedings of the American Economic Association* 89, 139–144.

———, 2020, The end of economic growth? Unintended consequences of a declining population, *Working Paper*.

Kaas, Leo, 2016, Public debt and total factor productivity, *Economic Theory* 61, 309–333.

Kelly, Bryan, Ľuboš Pástor, and Pietro Veronesi, 2016, The price of political uncertainty: Theory and evidence from the option markets, *Journal of Finance* 71, 2417–2480.

Kiyotaki, Nobuhiro, and Randall Wright, 1989, On money as a medium of exchange, *Journal of Political Economy* 97, 927–954.

Kocherlakota, Narayana, 1998, Money is memory, *Journal of Economic Theory* 81, 232–251.

Krueger, D., and H. Uhlig, 2022, Neoclassical growth with long-term one-sided commitment contracts, *NBER Working Paper 30518*.

Laincz, Christopher, and Pietro Peretto, 2006, Technological change and population growth, *Journal of Economic Growth* 11, 263–288.

Martin, A., and J. Ventura, 2018, The macroeconomics of rational bubbles: a user's guide, *Annual Review of Economics* 10, 505–539.

Miao, Jianjun, 2014, Introduction to economic theory of bubbles, *Journal of Mathematical Economics* 53, 130–136.

Moore, Rachel, and Brandon Pecoraro, 2018, Macroeconomic implications of modeling the internal revenue code in a heterogeneous-agent framework, *Working Paper*.

Pástor, Ľuboš, and Pietro Veronesi, 2012, Uncertainty about government policy and stock prices, *Journal of Finance* 67, 1219–1264.

Peretto, Pietro, 1998, Technological change and population growth, *Journal of Economic Growth* 3, 283–311.

Reis, Ricardo, 2021, The constraint on public debt when $r < g$ but $g < m$, *Working Paper*.

Samuelson, Paul A., 1958, An exact consumption-loan model of interest with or without the social contrivance of money, *Journal of Political Economy* 66, 467–482.

Santos, Manuel S., and Michael Woodford, 1997, Rational asset pricing bubbles, *Econometrica* 65, 19–57.

Shell, Karl, 1971, Notes on the economics of infinity, *Journal of Political Economy* 79, 1002–1011.

Sims, Christopher, 2020, How to worry about government debt, https://bcf.princeton.edu/events/christopher-sims-how-to-worry-about-government-debt/.

Summers, Lawrence H., 1981, Capital taxation and accumulation in a life cycle growth model, *American Economic Review* 71, 533–544.

Tirole, Jean, 1985, Asset bubbles and overlapping generations, *Econometrica* 53, 1499–1528.

Wallace, Neil, 1980, The overlapping generations model of fiat money, in *Models of Monetary Economies* pp. 2–20. Minneapolis: Federal Reserve Bank of Minneapolis.

Young, Alwyn, 1998, Growth without scale effects, *Journal of Political Economy* 106, 41–63.

Zodrow, George R., and John W. Diamond, 2013, Chapter 11 - dynamic overlapping generations computable general equilibrium models and the analysis of tax policy: The diamond-zodrow model, in Peter B. Dixon, and Dale W. Jorgenson, ed.: *Handbook of Computable General Equilibrium Modeling SET, Vols. 1A and 1B*, vol. 1 . pp. 743–813 (Elsevier).