Supply Chain Coordination and Influenza Vaccination

Stephen E. Chick* Hamed Mamani† David Simchi-Levi‡

May 2006; Revised October 2006

Abstract

Annual influenza outbreaks incur great expenses in both human and monetary terms, and billions of dollars are being allocated for influenza pandemic preparedness in an attempt to avert even greater potential losses. Vaccination is a primary weapon for fighting influenza outbreaks. The influenza vaccine supply chain has characteristics that resemble the Newsvendor problem, but possesses several characteristics that distinguish it from typical supply chains. Differences include a nonlinear value of sales (caused by the nonlinear health benefits of vaccination that are due to infection dynamics) and vaccine production yield issues. We show that production risks, taken currently by the vaccine manufacturer, lead to an insufficient supply of vaccine. Unfortunately, several supply contracts that coordinate buyer (governmental public health service) and supplier (vaccine manufacturer) incentives in many other industrial supply chains can not fully coordinate the influenza vaccine supply chain. We design a variant of the cost sharing contract and show that it provides incentives to both parties so that the supply chain achieves global optimization and hence improve the supply of vaccines.

*INSEAD; Technology and Operations Management Area; Boulevard de Constance; 77300 Fontainebleau France. stephen.chick@insead.edu

†MIT Operations Research Center; 77 Massachusetts Ave. Bldg E40-130; Cambridge, MA 02139, USA. hamed@mit.edu

‡MIT Department of Civil and Environmental Engineering, and The Engineering System Division; 77 Mass. Ave. Rm. 1-171; Cambridge, MA 02139, USA. dslevi@mit.edu

1 Influenza: Overview, Control and Operational Challenges

Influenza is an acute respiratory illness that spreads rapidly in seasonal epidemics. Globally, annual influenza outbreaks result in 250,000 to 500,000 deaths. The World Health Organization reports that costs in terms of health care, lost days of work and education, and social disruption have been estimated to vary between $1 million and $6 million per 100,000 inhabitants yearly in industrialized countries. A moderate, new influenza pandemic could increase those losses by an order of magnitude (WHO, 2005).

This paper provides background about influenza and vaccination, a key tool for controlling influenza outbreaks, then highlights some operational challenges for delivering those vaccines. One challenge is the design of contracts to coordinate the incentives of actors in a supply chain that crosses the boundary between the public sector (health care service systems) and private sector (vaccine manufacturers).

> Some experts suggest the U.S. government should promise to purchase a fixed amount of flu vaccine—despite the cost and the likelihood that some of the money would end up being wasted. Canada, for instance, has contracts with vaccine makers to cover most of its population. …That takes much of the risk out of the company's business, but still lets it manufacture additional doses for the private market…(Wysocki and Lueck, 2006)

> I recently met with leaders of the vaccine industry. They assured me that they will work with the federal government to expand the vaccine industry, so that our country is better prepared for any pandemic. … I'm requesting a total of $7.1 billion in emergency funding from the United States Congress…(George W. Bush, 2005)

We then present a model of a government's decision of purchase quantities of vaccines, which balances the public health benefits of vaccination and the cost of procuring and administering those vaccines, and a manufacturer's choice of production volume. We characterize the optimal decisions of each in both selfish and system-oriented play, then assess whether several contracts can align their incentives. Due to special features of the influenza value chain, wholesale price and pay back contracts are shown to be unable to fully coordinate decisions. We conclude by demonstrating a variation of a cost sharing contract that can coordinate incentives, improve public health cost-benefit outcomes, increase manufacturer revenues, and increase vaccine production volumes. The Appendices in an Online Companion provide mathematical proofs for the analytical results that are given below.

1.1 Influenza and Influenza Transmission

Influenza is characterized by fever, chills, cough, sore throat, headache, muscle aches and loss of appetite. It is most often a mild viral infection transmitted by respiratory secretions through sneezing or coughing. Complications of influenza include pneumonia due to secondary bacterial infection, which is more common in children and the elderly (e.g., see `http://www.cdc.gov/flu`, or Janeway et al. 2001).

The various strains of influenza experience slight mutations in their genome through time (antigenic drift). This allows for annual outbreaks, as previously acquired adaptive immunity may not cover emerging strains. Every few decades, a highly virulent strain may emerge that causes a global pandemic with high mortality rates. This may be caused by a larger genomic mutation (antigenic shift).

The three pandemics that occurred in the twentieth century came from strains of avian flu. The "Spanish flu" (H1N1) of 1918 killed 20–40 million people worldwide (WHO, 2005), far more than died in World War I. Milder pandemics occurred in 1957 (H2N2) and 1968 (H3N2). The H5N1 virus is the most likely potential culprit for a future pandemic (`http://www.who.int/csr/disease/influenza/`).

1.2 Vaccination as a Control Tool

Vaccines can reduce the risk of infection to exposed individuals that are susceptible to infection, and can reduce the probability of transmission from a vaccinated individual that is infected with influenza (Longini et al., 1978; Smith et al., 1984; Longini et al., 2000). Vaccines therefore act on the basic reproduction number, R_0, the mean number of new infections from a single infected in an otherwise susceptible population (Dietz, 1993). Colloquially, if R_0 can be reduced below 1, then the dynamics of a large outbreak can be averted. Let f^0 be the so-called critical vaccination fraction, the minimum fraction of the population to vaccinate to reduce the reproduction number to 1 when a single infected is introduced to an otherwise susceptible population. Appendix A of the Online Companion provides precise definitions for these terms.

Vaccination is seen as a principal means of preventing influenza. Although vaccination policies may vary from country to country, particular attention is typically paid to those aged 65 or more, health care workers,

and those that may have certain risk factors (WHO, 2005). Vaccination can be complemented with antiviral therapy.

Vaccination is cost effective. Nichol et al. (1994) found that immunization in the elderly saved $117 per person in medical costs. Weycker et al. (2005) argue for the systematic vaccination of children, not only the elderly, as a means to obtain a significant population-wide benefit for vaccination.

1.3 Operational Challenges in the Influenza Vaccine Supply Chain

Gerdil (2003) overviews the highly challenging and time-constrained vaccine production and delivery process. We focus on the predominant method, inactivated virus vaccine production. For the northern hemisphere, the WHO analyzes global surveillance data and in February announces the selection of three virus strains for the fall vaccination program. Samples of the strains are provided to manufacturers. High-volume production of vaccine for each of the three strains then proceeds separately. Production takes place in eleven day old embryonated eggs, so the number of eggs needed must be anticipated well in advance of the production cycle. Blending and clinical trials begin in May-June. Filling and packaging occur in July and August. Governmental certification may be required at various steps for different countries. Shipping occurs in September for vaccination in October-November. Immunity is conferred two weeks after vaccination. The southern hemisphere uses a separate 6-month cycle. Within two 6-month production cycles, almost 250 million doses are delivered to over 100 countries per year.

There are several key operational challenges that are presented by the influenza vaccine value chain.

A challenge at the start of the value chain is antigenic drift, which requires that influenza vaccines be reformulated each year. Influenza vaccines are one-time Newsvendor products, as opposed to all other vaccines, which closely resemble (perishable) EOQ-type products. Not only are production volumes hard to predict, but the selection of the target strains is a challenge. Wu et al. (2005) develop an optimization model of antigenic changes. Their results suggest that the current selection policy is reasonably effective. They also identify heuristic policies that may improve the selection process.

Another challenge occurs toward the end of the value chain, after vaccines are produced. That involves the allocation of vaccines to various subpopulations, and the logistics of transhipment to insure appropriate delivery. Hill and Longini (2003) describe a mathematical model to optimally allocate vaccines to several subpopulations with potentially heterogeneously mixing individuals. Weycker et al. (2005) use a different, stochastic simulation model to illustrate the benefits of vaccinating certain subpopulations (children). Those articles do not discuss the logistics of delivery. Yadav and Williams (2005) propose an information clearinghouse for vaccine supply and demand to provide a market overview and to help to eliminate the gaming of orders and price gouging. They also propose the use of demand forecasting tools, and regional vaccine redistribution pools to shift supplies from areas with surpluses to areas experiencing shortages.

This paper is concerned with a challenge in the middle of the value chain: the design of contracts that align manufacturer choices for production volume and the need for profitability, and governmental choices that balance the costs and public health benefits of vaccination programs. Special characteristics of the influenza vaccine supply chain that differentiate it from many other supply chains include a nonlinear value of a sale (the value of averting an infection by vaccination depends upon nonlinear infection dynamics), and a dependence of production yields on the virus strains taht are selected for the vaccine.

Current production technology for inactivated virus vaccines, market forces, and business practices also combine to limit the ability to stockpile vaccines, limit production capacity, and slow the ability to respond to outbreaks. Governmental and industry partnerships may help to improve responsiveness (U.S. GAO, 2001; Pien, 2004; Bush, 2005; Wysocki and Lueck, 2006). The ideal way to structure those partnerships is an open question. This paper addresses one dimension of that multi-faceted question.

1.4 Relation to Operations Management Literature and Overview of Paper

This work relates to the operations management literature in three ways. First, this paper considers the random production yield of influenza vaccine production, a Newsvendor setting. Silver (1976) considers random production yields in an EOQ setting, and shows that the optimal lot size is a slight modification of

usual EOQ. Yano and Lee (1995) review approaches to lot sizing in the presence of five different types of yield randomness. We assume perfect correlation, which is what they call stochastically proportional yield, and which has been studied by Shih (1980) and Henig and Gerchak (1990).

Second, this paper relates to the supply contract literature (Lariviere, 1999; Cachon, 2003). One stream in that literature focuses on optimizing the terms of a contract so as to improve supply chain coordination. Examples include buy-back contracts (Pasternack, 1985), revenue sharing contracts (Cachon and Lariviere, 2005) and option contracts (Barnes-Schuster et al., 2002). The objective is to characterize contracts that allow each party to optimize its own profit but lead to a globally optimized supply chain.

The model that we propose below is similar to the Newsvendor situation with an exchange of demand uncertainty by production uncertainty. Since the buyback contract coordinates the supply chain for the Newsvendor (Pasternack, 1985), one could expect that the corresponding contract with uncertain yield (i.e., a payback contract) should be able to achieve the same. Unfortunately, we will show that this is not the case for our model. On the other hand, we show that a cost sharing contract, where the manufacturer effort is taken into account, can coordinate a supply chain in the presence of yield uncertainty. That is, while contracts like payback, which only depend on the production *output*, do not align the manufacturer's incentive, contracts that take into account the production *effort* are able to do so by shifting enough production risk from the supplier to the buyer.

The third way in which our work relates to the operations management literature is the increasing interest in modeling the intersection of operations management and epidemic or disease modeling. In addition to articles that are cited below, Kaplan et al. (2002) assesses operational decisions for vaccination policy, with capacity constraints, to respond to smallpox bioterrorism attacks. Su and Zenios (2004) examine the role of queueing and patient choice in kidney allocation. Zaric et al. (2006) merge an inventory model with an anthrax outbreak model to assess inventory management decisions for bio-terror preparedness. See also Brandeau et al. (2004) and references therein. As far as we are aware of, the current paper appears to be the

first to link supply contract design with epidemic models.

Section 2 presents a model to assess contractual mechanisms that align manufacturer risks and incentives with governmental health care policy objectives for influenza vaccination. Section 3 and Section 4 analyze the model. A variant of the cost sharing contract, which we show can align incentives for public health benefits and production costs, also increases production volumes. Increased production volumes for annual vaccination are consistent with the recommendations of the Pandemic Influenza Plan of the U.S. Dept. of Health and Human Services (2005). The contract achieves this improvement by carefully balancing payment for output with some payment for effort. Section 5 discusses implications and limitations of the analysis.

2 Joint Epidemic and Supply Chain Model

This section links two distinct streams of literature. The epidemic literature provides epidemic models and cost benefit analysis for interventions such as vaccination (Murray, 1993; Hill and Longini, 2003), but does not address logistical and manufacturing concerns. The supply chain literature addresses logistical and manufacturing concerns in general, but does not address the special characteristics of the influenza vaccine supply chain that are highlighted above.

We use simplified epidemic and supply chain models to focus on contractual issues between a single government and a single manufacturer. The single government is intended to represent centralized aggregate planning decisions for vaccination policy. The government initially announces a fraction f of a population of N individuals to vaccinate. Given the demand by the government, the manufacturer then decides how much to produce. Production volume decisions are indexed by the number of eggs, n_E, a critical factor in influenza vaccine production. Production costs are c per egg. The actual amount produced, $n_E U$, is a random variable that is indexed by a yield, U. The U.S. GAO (2001) reports that the strain can strongly influence the production yield. In this paper, we assume that the yield U has a continuous probability density function $g_U(u)$ with mean μ and variance σ^2, independent of n_E. This assumption means that the yield is affected by the specific strain of the virus, and may vary from year to year, more so than from one statistically

independent batch to the next within a given production campaign.

The manufacturer then sells whatever vaccine is produced, up to the amount initially requested by the government (a maximum of Nfd doses, where N is the population size, and d is the number of doses per individual). Unmet demand is lost, and excess vaccines are discarded (due to antigenic drift or shift).

When acting separately, the government seeks to minimize the variable cost of procuring, p_r, and administering, p_a, each dose, plus the total social cost due to infection, $bT(f)$, where $T(f)$ is the total expected number of infected individuals by the end of the influenza season, and b is the average direct and indirect cost of an influenza infection (Weycker et al., 2005, provide estimates of such costs). Define $\bar{f}$ to be the maximum fraction of the population for which the net benefit of administering more vaccine is positive, and define $\bar{\bar{f}}$ similarly with respect to both vaccine procurement and administration costs,

$$\bar{f} = \sup\{f : bT'(f) + p_a Nd < 0, \text{ for } f \text{ such that } T'(f) \text{ exists}\} \quad (1)$$

$$\bar{\bar{f}} = \sup\{f : bT'(f) + (p_a + p_r)Nd < 0, \text{ for } f \text{ such that } T'(f) \text{ exists}\}. \quad (2)$$

The epidemic model determines the number of individuals, $T(f)$, that are infected by the end of the influenza season. While vaccine effects and health outcomes may vary by subpopulation, and vaccination programs can take advantage of that fact (Weycker et al., 2005), we simplify the model in order to focus on contract issues for production volume, rather than including details about optimal allocation of a given volume. We use a deterministic compartmental model of N homogeneous and randomly mixing individuals that start out Susceptible to infection, but may also be infected and Infectious, or Removed upon recovery from infection, a standard SIR compartmental model that is a reasonable model for the natural history of infection of influenza (Murray, 1993). The fraction of susceptible, infectious and removed individuals ($S(t)$, $I(t)$, and $R(t)$, respectively) in the population varies as a function of time t according to a deterministic differential equation (see Appendix A of the Online Companion).

We assume that a fraction f of the population is vaccinated, and that a fraction ϕ of those vaccinated are are immune to infection (so $R(t) = f\phi$ for $t \leq 0$). At the start of the influenza season, at time $t = 0$,

Table 1: Summary of Notation.

Supply Chain	
n_E	Number of eggs input into vaccine production by the manufacturer
U	Random variable for the yield per egg, with pdf of $g_U(u)$, mean μ, and variance σ^2
d	Doses of vaccine needed per person
c	Unit cost of production for manufacturer, per egg input
p_r	Revenue to the manufacturer from government, per dose of vaccine
p_a	Cost per dose for government to administer vaccine
b	Average total social cost per infected individual
Z	Number of doses sold from manufacturer to government
W	Number of doses administered by government to susceptible population
Infection Transmission	
N	Total number of people in the population
R_0	Basic reproduction number, or expected number of secondary infections caused by one infected in an otherwise susceptible, unvaccinated population
f	fraction of the population to vaccinate announced by government to manufacturer
$T(f)$	Total expected number infected during the infection season, a function of the fraction vaccinated
χ	The fraction of susceptibles that are initially infected due to exogenous exposure
$I(0)$	The initial fraction of infected people introduced to the population
$S(0)$	The initial fraction of susceptible people in the population
ϕ	Vaccine effects on transmission, including susceptibility and infectiousness effects
ψ	Linear approximation to number of direct and indirect infections averted by a vaccination
f^0	The critical vaccination fraction (fraction of population to vaccinate to halt outbreak)
$\bar{f}$	The maximum fraction for which (free) vaccine can be cost-effectively administered
$\bar{\bar{f}}$	The maximum fraction for which vaccine can be cost-effectively procured and administered
k	Relates vaccination fractions and vaccine production inputs, $k = \frac{fNd}{n_E}$

a fraction χ of the remaining susceptible population that becomes infected due to exposure from exogenous sources, so that $S(0) = (1 - f\phi)(1 - \chi)$ and $I(0) = (1 - f\phi)\chi$. The total number that become infected during the influenza season is $T(f) = Np$, where the so-called attack rate p (see the Online Companion or Longini et al. 1978) satisfies

$$p = S(0)(1 + \frac{I(0)}{S(0)} - e^{-R_0 p}). \tag{3}$$

The critical vaccination fraction is $f^0 = (R_0 - 1)/(R_0\phi)$ when $R_0 > 1$ (Hill and Longini, 2003).

Rather than deriving results via such an implicit characterization from the epidemic model, we derive results for a nonincreasing $T(f) \geq 0$ with specific general characteristics. When the values of all of the epidemic and vaccine parameters are known, Appendix C describes why it is reasonable to consider two functional forms: a piecewise linear $T(f)$ when χ is close to 0, or a strictly convex $T(f)$ when χ is sufficiently large. This removes the details of an implicit solution for an epidemic model from the supply chain analysis. Section 3 handles the piecewise linear case. Section 4 handles the convex case.

This generic approach to modeling $T(f)$ is also important because it allows for an analysis when the values of the epidemic and vaccine parameters are not known. In practice, the basic reproduction number R_0, the initial fraction of susceptibles that become infected due to exogenous exposure χ, and the vaccine efficacy ϕ are unknown at the time that the order quantities are decided. It is hard to predict each of these for pandemic influenza, due to the newness and evolutive nature of the strains. Even for annual influenza strains, such as H3N2, a new "cluster" of drift variants tends to appear every 3-5 years (Plotkin et al., 2002; Smith et al., 2004). A prior distribution can be used to describe these types of uncertainty about the epidemic and vaccine parameters parameters, based upon past experience with strains that are similar to those that are selected for the current year's formulation. When parameters are uncertain, let $T(f) = \mathrm{E}[T(f; R_0, \chi, \phi)]$ be the total expected number of infected individuals, averaged over all uncertainty in those parameters, where $T(f; R_0, \chi, \phi)$ makes the parameter values for (3) explicit. For example, when χ is close to 0, $T(f; R_0, \chi, \phi)$ will be shown to approximately a piecewise linear function that is, and an expectation of convex functions makes $T(f) = \mathrm{E}[T(f; R_0, \chi, \phi)]$ a convex function. Results for a piecewise linear and for a strictly convex $T(f)$, then, are also useful when parameters are uncertain.

Before deriving those results, we first complete the statement of the supply chain optimization problems.

2.1 Game setting

The epidemic and supply chain models above define a sequential game. The government announces a fraction f of the population for which it will purchase vaccines. The manufacturer then decides on a production quantity, indexed by n_E, in order to maximize expected profits (minimize expected costs), subject to potential yield losses and market capacity constraints. The **manufacturer problem** is:

$$\begin{array}{lll} \min\limits_{n_E} & MF = \mathrm{E}\left[cn_E - p_r Z\right] & \text{(net manufacturer costs)} \\ \text{s.t.} & Z = \min\{n_E U, fNd\} & \text{(doses sold} \leq \text{yield and demand)} \\ & n_E \geq 0 & \text{(nonnegative production volume).} \end{array} \tag{4}$$

So that the optimal production level is not zero, $n_E^* > 0$, we assume:

Assumption 1 *The expected revenue exceeds the cost per egg, $p_r\mu > c$, so vaccines can be profitable.*

Given that assumption, we characterize the optimal production quantity.

Proposition 1 *For any random egg yield, U, with pdf $g_U(u)$, and given the order quantity $D = fNd$ by government, the optimal production level n_E^* for the manufacturer is determined by*

$$\int_0^{\frac{fNd}{n_E^*}} u g_U(u) du = \frac{c}{p_r}. \tag{5}$$

Proofs of all claims in the main text are provided in Appendix B of the Online Companion.

A useful corollary follows directly.

Corollary 1.1 *If c, p_r, $g_U(u)$, N and d are held constant, then the relationship between the fraction of people to be vaccinated, f, and optimum production level, n_E^*, is linear. That is, there is a fixed constant, k^G, such that $k^G n_E^* = fNd$.*

The **government problem** is to select a fraction f that indexes demand, knowing that the manufacturer will behave optimally, as in (5), and may deliver less, in expectation, than what is ordered due to yield losses. The government may order some excess (even $f > \bar{f}$), in order to account for potential yield losses. In this base model, we assume that the government purchases up to the amount it announced, but will administer only those doses that have a nonnegative cost-health benefit.

$$\begin{array}{lll}
\min\limits_{f} & GF = \mathrm{E}\left[\, bT(\frac{W}{Nd}) + p_a W + p_r Z \,\right] & \text{(net government costs)} \\
\text{s.t.} & Z = \min\{n_E U, fNd\} & \text{(doses bought} \leq \text{yield and demand)} \\
& W = \min\{n_E U, fNd, \bar{f}Nd\} & \text{(doses given} \leq \text{doses bought, cost effective level)} \\
& \displaystyle\int_0^{\frac{fNd}{n_E}} u g_U(u) du = \frac{c}{p_r} & \text{(manufacturer acts optimally)} \\
& 0 \leq f \leq 1 & \text{(fraction of population)} \\
& n_E \geq 0 & \text{(nonnegative production volume)}
\end{array} \tag{6}$$

Such a two-actor game has a Nash equilibrium (Nash, 1951), which we will identify below.

2.2 System setting

The system setting assesses whether the manufacturer and government can collaborate via procurement contracts to reduce the sum of their expected financial and health costs, to a level that is below the sum of

those costs if each player acts individually as in Section 2.1. System costs do not include monetary transfers from government to manufacturer. Formally, the **system problem** is

$$\begin{array}{lll} \min\limits_{f,n_E} & SF = \mathrm{E}\left[bT(\frac{W}{Nd}) + p_a W + cn_E\right] & \text{(total system costs)} \\ \text{s.t.} & W = \min\{n_E U, fNd, \bar{f}Nd\} & \text{(doses given} \leq \text{yield, demand, cost effective level)} \\ & 0 \leq f \leq 1 & \text{(fraction of population)} \\ & n_E \geq 0 & \text{(nonnegative production volume).} \end{array} \tag{7}$$

This formulation does not explicitly link f and n_E together, since we seek system optimal behavior rather than local profit-maximizing behavior.

3 Piecewise Linear Number of Infected

Figure 1 plots the attack rate, p, which is directly proportional to the total number infected, $T(f)$, as a function of the initial fraction of susceptibles that become infected due to exogenous exposure, χ, and reasonable values of R_0 for influenza transmission (Gani et al., 2005). If there are few that are initially infected due to exogenous exposure (small χ, or small $I(0)/S(0)$), then Appendix C in the Online Companion justifies the following piecewise linear approximation for $T(f)$.

$$T(f) = \begin{cases} M - N\psi f, & 0 \leq f \leq f^0 \\ 0, & f^0 \leq f \leq 1, \end{cases} \tag{8}$$

where ψ is interpreted here as the marginal number of infections averted per additional vaccination.

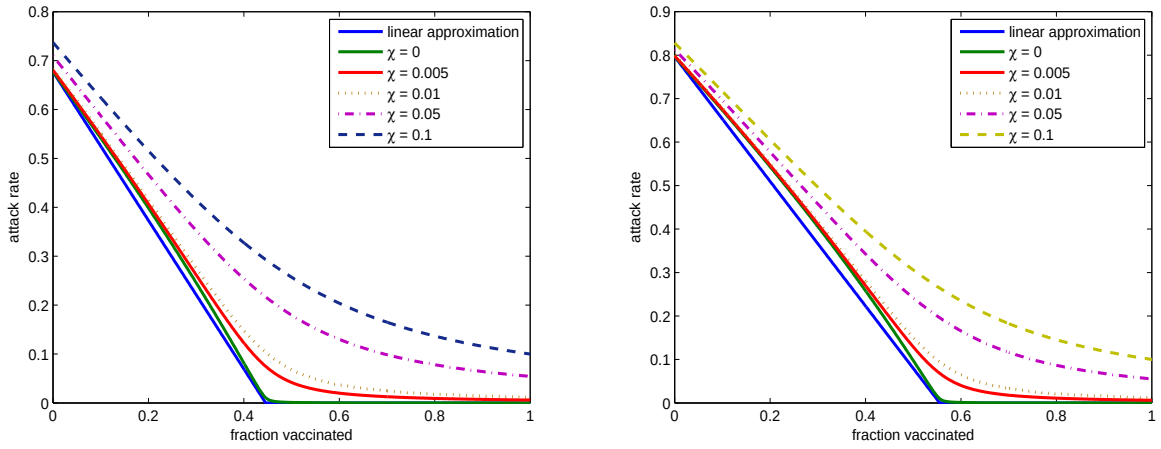
We seek structural results to compare the values of the game equilibrium and system optimum. With this approximation for $T(f)$, the maximum cost-effective number of individuals to vaccinate equals the critical vaccination fraction, $\bar{f} = f^0$. The government's objective function from Problem (6) is

$$GF = \mathrm{E}\left[b\max\{M - \psi\frac{W}{d}, 0\} + p_a W + p_r Z\right]. \tag{9}$$

The manufacturer problem is the same.

The system's objective function from Problem (7) is

$$SF = \mathrm{E}\Big[b\max\{M - \psi\frac{W}{d}, 0\} + p_a W + cn_E\Big]. \tag{10}$$



(a) $R_0 = 1.67$ (b) $R_0 = 2.0$

Figure 1: Attack rate (p) as a function of the fraction vaccinated (f), for different values of the fraction of susceptibles that are initially infected (χ) and the basic reproduction number (R_0).

3.1 Optimal solutions for game and system settings

This section describes the equilibria of the game setting and the optimal system solution for the manufacturer and government. It assumes that the parameters of the model in Section 2 are given. A series of assumptions and results are developed to show that the optional system solution requires a higher vaccine production level than in the game setting. Section 3.2 uses those results to design contracts that create a new game, to get the individual actors to behave in a system optimal way.

If the following assumption were not valid, then even free vaccines would not be cost effective.

Assumption 2 *The expected health benefit of vaccination exceeds the administration cost,* $\psi b - p_a d > 0$.

Proposition 2 *Let* f^S, n_E^S *be optima for the system setting with objective function in (10). If Assumption 2 holds, then (1) all values of* f^S *that are between* f^0 *and* 1 *are optimal; and (2)* n_E^S *satisfies*

$$\int_0^{\frac{f^0 Nd}{n_E^S}} u g_U(u) du = \frac{c}{\frac{\psi b}{d} - p_a}. \tag{11}$$

The next assumption implies that vaccination is cost effective from the government's point of view.

Assumption 3 *The expected health benefit of vaccination exceeds the cost of administering and procuring the doses,* $\psi b - (p_a + p_r)d > 0$.

Observe that if Assumption 3 does not hold, then vaccines at market costs are not cost effective. To see this, set $\tilde{f} = \min\{f, f^0\}$. Then for all $0 \leq f \leq 1$,

$$\begin{aligned} GF(f, n_E) \geq \; & b\int_0^{\frac{\tilde{f}Nd}{n_E}} (M - \psi\frac{n_E u}{d}) g_U(u)du + b(M - N\psi\tilde{f}) \int_{\frac{\tilde{f}Nd}{n_E}}^{\infty} g_U(u)du \\ & + (p_a + p_r)n_E \int_0^{\frac{\tilde{f}Nd}{n_E}} u g_U(u)du + (p_a + p_r)(\tilde{f}Nd) \int_{\frac{\tilde{f}Nd}{n_E}}^{\infty} g_U(u)du \\ = \; & bM + n_E \frac{1}{d}\big((p_a + p_r)d - \psi b\big) \int_0^{\frac{\tilde{f}Nd}{n_E}} u g_U(u)du + \tilde{f}N\big((p_a + p_r)d - \psi b\big) \int_{\frac{\tilde{f}Nd}{n_E}}^{\infty} g_U(u)du. \end{aligned}$$

If $\psi b - (p_a + p_r)d < 0$, then $GF(f, n_E) > bM$ for all $f, n_E > 0$, and $f^G = n_E^G = 0$ would be optimal.

Given Assumption 3 and Proposition 2, we can compare the values of (5) and (11) to obtain Corollary 2.1.

Corollary 2.1 *Let f^S, n_E^S be optimal values of the system problem and define $k^S = \frac{f^0 Nd}{n_E^S}$. Let f^G, n_E^G denote optimal values of the game setting and define $k^G = \frac{f^G Nd}{n_E^G}$. If Assumption 3 holds, then $k^S < k^G$.*

The concept $k = \frac{fNd}{n_E}$ that relates vaccination fractions to vaccine production volumes is useful below.

Proposition 2 characterized the optimal vaccine fraction and production level for the system setting. We now assess optimal behavior in the game setting. (5) indicates that it suffices to characterize the optimal vaccine fraction, which then determines the optimal production level in the game setting.

Proposition 3 *Let f^G, n_E^G be optimal solutions for the game setting, and set $k^G = \frac{f^G Nd}{n_E^G}$. If Assumption 3 holds, then $f^G \geq f^0$. Furthermore, $f^G = f^0$ if and only if*

$$(-\frac{\psi b}{d} + p_a + p_r) \int_0^{k^G} u g_U(u)du + p_r k^G \int_{k^G}^{\infty} g_U(u)du \geq 0. \tag{12}$$

Although it may seem, at first glance, that Condition (12) depends on f^G through k^G, this is not true. Given the problem data, the value of k^G is determined by (5), independently of the values of f^G and n_E^G. The condition in this claim is therefore verifiable by having the initial data of the problem.

Intuitively, the inequality in the second part of Proposition 3, Condition (12), shows that if b is sufficiently higher than the other costs, then the game pushes the government to order a higher amount of vaccine than the amount specified by the critical vaccine fraction, f^0.

Theorem 4 uses our results on the optimal production level in the system setting, Proposition 2, and the game setting, Proposition 3, to prove the main result of this section: optimal production volumes are higher in the system setting than in the game setting.

Theorem 4 *Given Assumption 3 and the setup above,* $n_E^S > n_E^G$.

The intuition behind Theorem 4 is that the manufacturer bears all the risk of uncertain production yields in the game setting and hence is not willing to produce enough.

3.2 Coordinating Contracts

The objective of this section is to design contracts that will align governmental and manufacturer incentives. There are a variety of contracts in use, include wholesale pricing (CDC, 2005). Recent support by the U.S. Dept. of Health and Human Services (2004) to a major manufacturer for the development of a stable egg supply resembles a payment that is proportional to effort, a characteristic that is shared with the cost sharing contract below. We show that wholesale or pay back contracts can not coordinate this supply chain. We then demonstrate a cost sharing contract that is able to do so.

3.2.1 Wholesale price contracts

In wholesale price contract, the supplier and government negotiate a price p_r. Unfortunately, the system optimum can not be fully achieved just by adjusting the value of p_r.

Proposition 5 *There does not exist a wholesale price contract which satisfies the condition in Assumption 3 and coordinates the supply chain.*

3.2.2 Pay back contracts

In a pay back contract, the government agrees to buy any excess production, beyond the desired volume, for a discounted price p_c (with $0 < p_c < p_r$) from the manufacturer. This shifts some risk of excess production from the manufacturer to the government, and would typically increase production.

We show that the pay back contract does not provide sufficient incentive to coordinate the influenza supply chain, unlike typical supply chains, for any reasonable value of p_c. The combination of yield uncertainty and maximal purchase quantities will be shown to prevent full coordination. Assumption 4 defines a reasonable p_c as one that precludes the manufacturer from producing an infinite volume for an infinite profit.

Assumption 4 *The average revenue per egg at the discounted price is less than its cost, $p_c\mu < c$.*

The pay back contract increases the manufacturer's profit by adding the revenue associated with $n_E U - \min\{n_E U, fNd\}$ doses of excess production. This changes the manufacturer problem from Problem (4) to

$$\begin{aligned} \min_{n_E} \quad & MF = \mathrm{E}\Big[cn_E - p_r Z - p_c(n_E U - Z)\Big] \\ \text{s.t.} \quad & Z = \min\{n_E U, fNd\} \\ & n_E \geq 0. \end{aligned}$$

By adapting the argument of Proposition 1, the optimal production level n_E^* can be shown to satisfy

$$\int_0^{\frac{fNd}{n_E^*}} u g_U(u) du = \frac{c - p_c\mu}{p_r - p_c}. \tag{13}$$

The effect of this contract on the government problem in Problem (6) is to change the objective to

$$GF = \mathrm{E}\left[b\max\{M - \psi\frac{W}{d}, 0\} + p_a W + p_r Z + p_c(n_E U - Z)\right],$$

and to change the "manufacturer acts optimally" constraint, which determines the optimal production input quantity n_E as a function of f, from (5) to (13).

Denote the optimal values of this pay back contract problem by f^N, n_E^N. Set $k^N = \frac{f^N Nd}{n_E^N}$.

Proposition 6 *If Assumptions 1, 2 and 4 hold, then there does not exist a pay back contract which could coordinate this supply chain. In fact, under any pay back contract, the resulting production level is less than the optimal system production level, $n_E^N < n_E^S$.*

Proposition 6 suggests that compensating the manufacturer for having excess inventory is not enough to achieve global optimization. Indeed, a pay back contract does not compensate the manufacturer when the production volume, n_E, is high while the yield, $n_E U$ is low. The cost sharing agreement described below is designed to address this issue.

3.2.3 Cost sharing contracts

In a cost sharing contract, the government pays proportional to the production volume n_E at a rate of p_e per each egg. Such an agreement decreases the manufacturer's risk of excess production, and provides an incentive to increase production. Here, we describe a contract that increases production to the system optimum, f^0, n_E^S.

With the cost sharing contract, the manufacturer problem is:

$$\begin{aligned} \min_{n_E} \quad & MF = \mathrm{E}\big[(c-p_e)n_E - p_r Z\big] \\ \text{s.t.} \quad & Z = \min\{n_E U, fNd\} \\ & n_E \geq 0. \end{aligned}$$

The optimality condition for n_E given f follows immediately, as for the original problem,

$$\int_0^{\frac{fNd}{n_E^*}} u g_U(u)du = \frac{c-p_e}{p_r}. \tag{14}$$

Cost sharing increases the governments costs, changing its objective function to:

$$GF = \mathrm{E}\left[b\max\{M-\psi\frac{W}{d},0\} + p_a W + p_r Z + p_e n_E\right], \tag{15}$$

and resulting in the following optimization problem:

$$\begin{aligned} \min_{f} \quad & GF = \mathrm{E}\left[b\max\{M-\psi\tfrac{W}{d},0\} + p_a W + p_r Z + p_e n_E\right] \\ \text{s.t.} \quad & Z = \min\{n_E U, fNd\} \\ & W = \min\{n_E U, fNd, f^0 Nd\} \\ & \int_0^{\frac{fNd}{n_E}} u g_U(u)du = \frac{c-p_e}{p_r} \\ & 0 \leq f \leq 1 \\ & n_E \geq 0. \end{aligned}$$

Denote the optimal solutions of this problem by f^e, n_E^e, and set $k^e = \frac{f^e Nd}{n_E^e}$.

For any given p_r, choose $p_e > 0$ so that $\frac{c-p_e}{p_r} = \frac{c}{\frac{\psi b}{d}-p_a}$. Such a p_e exists since $p_r < \frac{\psi b}{d} - p_a$. If p_e is chosen this way, then $k^e = k^S$. Further, if p_r satisfies Assumption 3, such a p_e not only moves k^e to k^S, but it aligns the vaccination fractions and production volumes, as in Theorem 7.

Theorem 7 *If Assumption 3 holds and p_e is chosen so that $\frac{c-p_e}{p_r} = \frac{c}{\frac{\psi b}{d} - p_a}$, then the optimal values (f^e, n_E^e) for Problem (15) equal (f^0, n_E^S), so this cost sharing contract will coordinate the supply chain.*

The cost sharing contract can coordinate incentives, unlike the pay back contract, because the manufacturer's risk of both excess and insufficient yield can be handled by the contract's balance between paying for outputs (via p_r) and for effort (via p_e).

4 Strictly Convex Number of Infected

This section presumes that $T(f)$ is strictly convex. While $T(f)$ may not be convex for all choices of the parameters of the infection model, it is strictly convex for sufficiently large χ and values of R_0 that are representative of influenza (see Appendix C). This corresponds to a larger initial exposure to members of the population, such as may occur in an initial pandemic wave.

Below we explore the game equilibrium and the optimal system solution. We then show that a variation of the cost sharing contract can coordinate the supply chain.

4.1 Optimal solutions for game and system settings

The solution to the manufacturer problem in Problem (4) with convex $T(f)$ remains the same as above, as the manufacturer's objective function does not depend upon $T(f)$. The analysis of the government problem in Problem (6) and the system problem in Problem (7) is somewhat more complicated when $T(f)$ is strictly convex, but the general ideas are similar to those in the linear model.

For the system setting, the following analog of Proposition 2 holds.

Proposition 8 *If $T(f)$ is strictly convex, $\bar{f}$ is the solution of (1), and the optimum values of the system problem in Problem (7) are denoted by f^S, n_E^S, then (a) f^S can be picked to be any value between $\bar{f}$ and 1; and (b) n_E^S is the solution of the following equation: $\int_0^{\frac{\bar{f}Nd}{n_E^S}} \Big[\frac{b}{Nd} T'(\frac{n_E^S u}{Nd}) + p_a\Big] u g_U(u) du + c = 0$.*

The following analog of Proposition 3 for convex $T(f)$ characterizes the set of the game equilibria.

Proposition 9 *Let f^G, n_E^G denote the game solution, let $k^G = \frac{f^G Nd}{n_E^G}$ and set $\bar{n}_E = \frac{\bar{f} Nd}{k^G}$. If $T(f)$ is strictly convex, then (a) $\int_0^{k^G} u g_U(u)du = \frac{c}{p_r}$; and (b) $f^G \leq \bar{f}$ if and only if*

$$\int_0^{k^G} \Big[\frac{b}{Nd} T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \geq 0. \tag{16}$$

The inequality in Condition (16) shows that if b is not sufficiently higher than the other costs, such that the marginal health benefit obtained by vaccination do not cover the vaccine costs, then the game pushes the government to order less vaccine than is required to vaccinate a fraction $\bar{f}$ of the population.

Theorem 10, the main result of this section, shows that, as in the linear case, the system optimal production level exceeds that of the game equilibrium.

Theorem 10 *Let n_E^S and n_E^G denote the production level under the system optimum and game equilibrium, respectively. For all nonincreasing strictly convex $T(f)$, we have $n_E^S > n_E^G$.*

The theorem says that the production level that is set by the manufacturer, n_E, is below the amount that is optimal for the system. Hence, there is an opportunity for an effective contract to align incentives.

4.2 Coordinating Contracts

This section constructs a contract which can coordinate this supply chain. Unfortunately, the cost sharing contract of Section 3.2.3, defined by the pair p_r, p_e, does not coordinate the supply chain. Observe that in the piecewise linear case, the government orders enough, i.e., $f^G \geq f^0$, even without a coordinating contract. This may not be true for the convex case, where without the contract, f^G maybe smaller than $\bar{f} \leq f^S$, as shown by Proposition 8 and Proposition 9.

Thus, a coordinating contract should provide an incentive for the government to vaccinate a higher fraction of the population, and provide a manufacturer with an incentive to produce enough. Section 4.2.1 shows that this goal can be achieved using a whole-unit discount for the vaccine purchased by the government. In return, the government will pay the manufacturer a portion of the production cost. The relation between

the whole-unit discount and the cost sharing portion is such that the more people the government plans to vaccinate, the greater the discount they get and the higher its participation in the production cost.

4.2.1 Whole-unit discount/cost sharing contract

Consider a contract where the vaccine price depends on the fraction of the population the government plans to vaccinate, that is, the government pays the manufacturer $p_r(f)$ per dose. The cost sharing component of the contract is such that the government pays proportional to the production level, n_E. The per unit price paid by the government, $p_e(f)$ depends on f.

This section first constructs a specific class of pricing policies. It then shows how the original game is modified by the pricing policy, and that the given pricing policies indeed align incentives.

The following two assumptions constrain the set of pricing policies of interest.

Assumption 5 *The price $p_r(f) \geq 0$ has the following characteristics:*

1. *There is a whole-unit discount, i.e., $p_r'(f) \leq 0$.*
2. *The total vaccine cost ($p_r(f)fNd$) is nondecreasing in f,*
   (a) $(p_r(f)fNd)' = p_r'(f)fNd + p_r(f)Nd \geq 0$ *for all* $0 \leq f \leq \bar{f}$.
   (b) $p_r'(\bar{f})\bar{f}Nd + p_r(\bar{f})Nd = 0$.
3. *The total cost to the government, excluding the cost sharing component, is convex in f,*
   (a) $bT''(f) + p_r''(f)fNd + 2p_r'(f)Nd \geq 0$ *for all* $0 \leq f \leq \bar{f}$.
4. *There are no further volume discounts beyond a certain threshold, $p_r(f) = p_r(\bar{f})$ for all $\bar{f} \leq f \leq 1$.*

If the derivative $p_r'(f)$ does not exist at $f = \bar{f}$, then use the left derivative in Assumption 5.

The first two characteristics in Assumption 5 allow for many pricing policies. The third characteristic restricts pricing policies to ones for which the total cost of vaccine procurement and social costs are convex.

Assumption 6 *Given $p_r(f)$, let $p_e(f) \geq 0$ satisfy $\dfrac{c - p_e(f)}{p_r(f)} = \displaystyle\int_0^{k^S} u g_U(u)du$ for all $f \in [0,1]$.*

In Assumption 6, $k^S = \frac{\bar{f}Nd}{n_E^S}$ is the same as before, where $\bar{f}, n_E^S$ are the solutions for the system setting.

Before proceeding, we show first that the set of the conditions in Assumptions 5 and 6 results in a feasible set. We give an example that satisfies the conditions in Assumption 5, then modify it to obtain functions that satisfy all of the conditions in both assumptions. Consider the following pricing strategy,

$$p_r(f) = \begin{cases} \kappa \dfrac{b}{fNd}\big[-T(f) + T'(\bar{f})f + T(0)\big], & 0 \leq f \leq \bar{f} \\ p_r(\bar{f}), & \bar{f} < f \leq 1. \end{cases} \tag{17}$$

Claim 11 *If $0 < \kappa < 1$, then the pricing strategy introduced in (17) gives a nonnegative price for any f and satisfies all the conditions in Assumption 5.*

Now we show that for some κ, (17) satisfies Assumption 6. If we set $p_e(f) = c - p_r(f)\int_0^{k^S} u g_U(u)du$, then it suffices to show that $p_e(f) \geq 0$ for all f. Since $p_r(f)$ is nonincreasing in f, we only need to show that $p_r(0)\int_0^{k^S} u g_U(u)du \leq c$. For any $p_r(f)$ that satisfies (17),

$$\begin{aligned} p_r(0) = \lim_{f\to 0} p_r(f) &= \lim_{f\to 0} \kappa \frac{b}{fNd}\big[-T(f) + T'(\bar{f})f + T(0)\big] \\ &= \kappa \frac{b}{Nd}\big[T'(\bar{f}) - \lim_{f\to 0}\big(\frac{T(f) - T(0)}{f}\big)\big] \\ &= \kappa \frac{b}{Nd}\big[T'(\bar{f}) - T'(0)\big] \end{aligned}$$

Observe that $\int_0^{k^S} u g_U(u)du \leq \mu$. We therefore satisfy Assumption 6 if $\kappa \frac{b}{Nd}\big[T'(\bar{f}) - T'(0)\big]\mu \leq c$. This justifies Claim 12: pricing strategies exist that satisfy both assumptions.

Claim 12 *If $0 < \kappa < \min\{1, \frac{c}{[T'(\bar{f}) - T'(0)]\mu}\}$, then the pricing strategy $p_r(f)$ in (17) satisfies Assumptions 5 and 6.*

All the ingredients are in place to build a coordinating contract. The key idea is to keep the relationship between the optimal production level and order quantity linear. Assumption 6 accomplishes this. To see this,

observe that this contract changes the manufacturer objective, for a given f, to:

$$MF(n_E) = (c - p_e(f))n_E - p_r(f)n_E \int_0^{\frac{fNd}{n_E}} ug_U(u)du - p_r(f)fNd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du.$$

By taking the derivatives, we have:

$$\begin{aligned} \frac{\partial MF(n_E)}{\partial n_E} &= (c - p_e(f)) - p_r(f) \int_0^{\frac{fNd}{n_E}} ug_U(u)du \\ \frac{\partial^2 MF(n_E)}{\partial n_E^2} &= p_r(f)\frac{fNd}{n_E^2}(\frac{fNd}{n_E})g_U(\frac{fNd}{n_E}) \geq 0. \end{aligned}$$

Therefore, this MF is convex in n_E, and the optimal n_E satisfies $\int_0^{\frac{fNd}{n_E^*}} ug_U(u)du = \frac{c-p_e(f)}{p_r(f)}$. Together with Assumption 6, this implies that $\int_0^{\frac{fNd}{n_E^*}} ug_U(u)du = \int_0^{k^S} ug_U(u)du$. So for any given f, the optimal production level for the manufacturer is linear in f, with

$$n_E^* = \frac{fNd}{k^S}. \tag{18}$$

Therefore this contract changes the government objective to

$$\min_f \quad GF = E\big[\, bT(\tfrac{W}{Nd}) + p_a W + p_r(f)Z + p_e(f)n_E \,\big], \tag{19}$$

and changes the manufacturing constraint to $\frac{fNd}{n_E} = k^S$. This restatement of the game setting for the whole-unit discout/cost sharing contract permits the statement of the main result of this section.

Theorem 13 *For any $p_e(f), p_r(f)$ that satisfy Assumptions 5 and 6, the optimal values of Problem (19), denoted by (f^c, n_E^c), are equal to $(\bar{f}, n_E^S)$. That is, this cost sharing contract coordinates the supply chain.*

4.2.2 Coordinating Contract: Numerical Application

This section uses the idea behind Theorem 13, together with estimates of parameters from the influenza literature, in order to develop a contract that can coordinate the supply chain empirically, even though the actual $T(f)$ may slightly deviate from strict convexity.

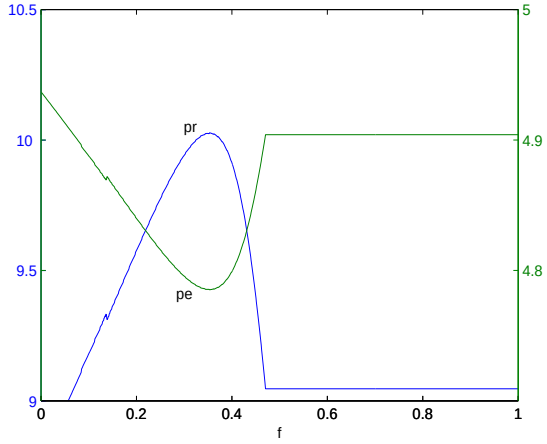
Longini et al. (2004) estimate $R_0 = 1.68$ and Weycker et al. (2005) argue that $\phi = 0.90$ is a reasonable value for vaccine effects, so we chose ϕ to vary between $[0.85, 0.95]$ and R_0 between $[1.5, 2]$. Weycker

et al. (2005) estimate the direct costs (not indirect) of each infected individual with $b = \$95$ on average over the different groups. In our experiments, b takes values between 70 and 120. The vaccine price is set to $p_r = \$12$ (CDC, 2005). For vaccine administration costs, we tested each of $p_a = \$20$, approximately the value in Pisano (2006) for Medicare reimbursement; $p_a = \$40$, the value that Weniger et al. (1998) estimated for pediatric vaccines, based on the cost of a doctor visit; and $p_a = \$60$, which accounts for inflation and provides a sensitivity analysis. We used $d = 1$ dose of vaccine, the usual value, per adult vaccinated. We are not aware of published estimates of the variance of vaccine production yields, although it is clear that variable vaccine yields are significant enough to cause noticeable fluctuations in the quantity of vaccine delivered (U.S. GAO, 2001). We assumed that U has a gamma distribution with mean $\mu = 1$ (Palese, 2006) and tested different values for the variance, $\sigma^2 = 0.025, 0.05, 0.06, 0.1, 0.2$. We assumed a population of $N = 3 \times 10^8$ individuals and a production cost of $c = \$6$ (not necessarily the actual value).

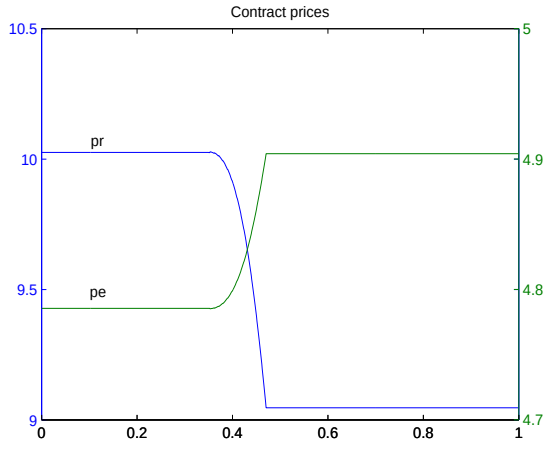
We implemented the whole-unit discount/cost sharing contract in Section 4.2.1 for cases of $T(f)$ that are based upon the above parameters and using $\chi = 0.01$. For example, Figure 2 depicts the contract prices, government costs, and manufacturer profit when $b = \$95$, $p_a = \$60$, $\sigma^2 = 0.2$ and $\kappa = 0.145$. While $T(f)$ in this case is not precisely convex, a strict application of the prices implied by (17) and Assumption 6 leads to a whole-unit discount price, $p_r(f)$ and cost sharing price $p_e(f)$ that coordinates incentives. Figure 2(a) shows that the wholesale price obtained by (17) is not monotone in this case. We can show that a modification of the wholesale price in which the increasing part of the price is replaces by a constant value equal to the maximum wholesale price, as in Figure 2(b), is still coordinating. This makes our proposed contract coordinating even for a larger class of attack rates in which $T(f)$ is first concave and after a point becomes convex. The appendix shows that $T(f)$ has this behavior for almost all of the parameters which are valid for influenza. In this case, our assumption of having a convex $T(f)$ reduces to the assumption $bT'(0) + p_a N d < 0$.

In this example, the manufacturer's effort is increased from 194M eggs to 213M eggs ($\sim 10\%$ increase), and its profit increases from $\$8.74 \times 10^8$ to $\$9.33 \times 10^8$ (by \$59M, or 6.75%). The government's order

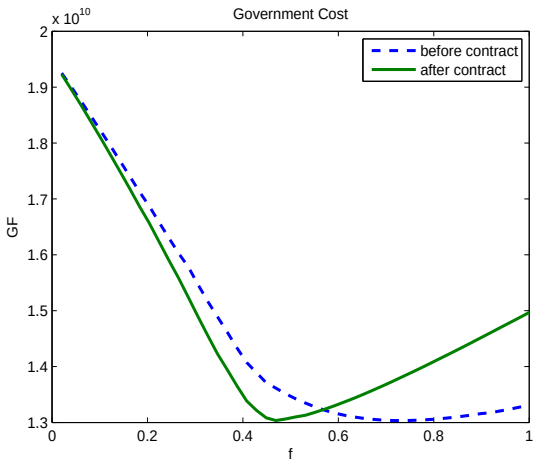
changes from 0.73 to 0.47, and its vaccine and social costs increase from $\$9.847 \times 10^9$ to $\$1.016 \times 10^{10}$ (3.21% increase) and $\$3.20 \times 10^9$ to $\$2.856 \times 10^9$ (10.75% decrease), respectively. The total governmental outlay decreases from $\$1.304 \times 10^{10}$ to $\$1.301 \times 10^{10}$ (by \$30M, or 0.3%).



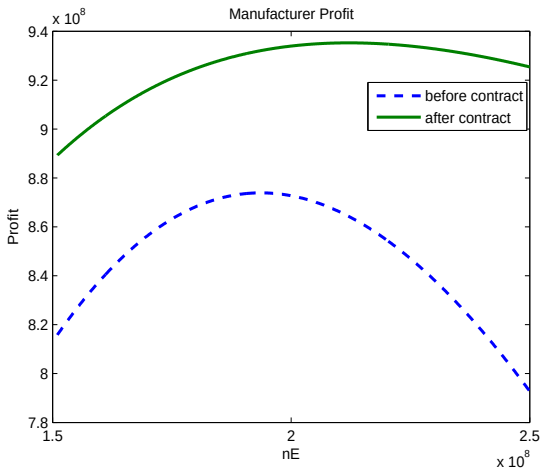
(a) Contract prices before modification (left vertical axis has $p_r(f)$, right vertical axis gives $p_e(f)$).



(b) Cost per dose, $p_r(f)$ (scale on left vertical axis), and per unit production effort, $p_e(f)$ (scale on right vertical axis).



(c) Governmental vaccine procurement, vaccine administration, and health costs, $GF(f)$.



(d) Manufacturer profit, $-MF(n_E)$.

Figure 2: Cost sharing/whole-unit discount contract.

Table 2 provides a sensitivity analysis with respect to the model's parameters. The particular choice of κ in the fourth column of Table 2 insures that both government and manufacturer are better off under the contract. The following three columns show the increase in the manufacturer cost and decrease in the

Table 2: Sensitivity analysis for contract outcomes, with $R_0 = 1.68$ and $\phi = 0.9$

b	p_a	σ^2	κ	manufacturer profit increase	gov. vaccine cost increase	gov. social cost decrease	before contract		after contract	
							f	n_E	f	n_E
95	20	0.2	0.097	6.91%	5.28%	15.42%	0.87	2.32×10^8	0.55	2.58×10^8
95	60	0.2	0.145	6.75%	3.35%	10.33%	0.73	1.94×10^8	0.47	2.13×10^8
120	20	0.2	0.077	6.51%	4.88%	14.57%	0.95	2.53×10^8	0.56	2.79×10^8
120	40	0.2	0.087	5.31%	2.69%	8.82%	0.91	2.42×10^8	0.51	2.46×10^8
70	40	0.2	0.18	7.42%	4.5%	12.64%	0.68	1.82×10^8	0.47	2.01×10^8
95	20	0.1	0.091	4.18%	2.74%	10.03%	0.74	2.09×10^8	0.55	2.21×10^8
120	20	0.1	0.071	3.91%	2.52%	9.29%	0.79	2.23×10^8	0.56	2.35×10^8
120	60	0.06	0.086	4.55%	2.43%	10.16%	0.62	1.78×10^8	0.48	1.93×10^8
95	40	0.06	0.102	1.5%	1.46%	6.46%	0.63	1.83×10^8	0.50	1.91×10^8

government social (cost of the infected population) and vaccine costs (procurement, administration, and cost sharing costs), respectively, when the contract is implemented. Notice that the government vaccine cost denotes all the vaccine procurement, administration and cost sharing costs (i.e. terms related to $p_r,\ p_a,\ p_e$) and the government social costs represents only the social costs of the disease (i.e. the term related to b). Although the government is better off after the contract in each row of the table, this benefit is primarily through a reduction in social costs due to increased vaccination expenditures. In our tests, we observed:

- There are always choices for κ so that with the contract, (a) the manufacturer's profit increases, (b) the government's social cost decreases, (c) the government's vaccination cost increases.
- Higher variability in yield leads to greater manufacturer profit and higher government vaccine costs, but also to lower government social costs
- For a given set of parameters, a larger κ increases the manufacturer's profit and governmental vaccine costs, but the governmental social costs do not fluctuate that much.

5 Discussion and Model Limitations

This work derived the equilibrium state of an interaction between a government and a manufacturer, with the realistic feature that a manufacturer bears the risk of uncertain production yields. The model shows that *a rational manufacturer will always underproduce influenza vaccines* in that setting, relative to the levels that provide an optimal system-wide cost-benefit tradeoff.

When the levels of exogenous introduction of influenza into a population are small, and good estimates for the infection transmission parameters are available, the piecewise linear approximation for $T(f)$ in Section 3 is appropriate. A relatively simple cost sharing contract can coordinate the incentives of the actors to obtain a system optimal solution.

When the levels of exogenous introduction of influenza into a population are somewhat large, as in a large-wave pandemic situation, or when the function $T(f)$ is estimated by averaging over prior distributions for unknown parameter values, the analysis of Section 4 is more appropriate. The simple cost sharing contract must be modified to account for the nonlinear population-level health benefits that are provided by influenza vaccination programs. It is therefore not surprising that the whole-unit discount/cost sharing contracts that can align incentives depend on the expected number of infections averted by a given magnitude of the vaccination program effort.

There are several limitations of this model. Some of the limitations can be handled with existing methods. Other limitations could lead to interesting future work, but do not limit the value of insights above regarding contract design for governmental/industry collaboration for influenza outbreak preparedness.

One, an epidemic model with homogeneous and homogeneously mixing populations ignores the potential to target specific critical subpopulations, such as children or the elderly. In the short run, the contractual designs here that determine production volumes could be accompanied in a second stage analysis with other work (e.g., Hill and Longini, 2003) that can optimally allocate vaccines to different subpopulations. The generality of the analysis for piecewise linear or convex $T(f)$ allows some flexibility in adapting the incentive alignment results above to more complex epidemic models that prioritize certain subgroups.

Two, the coupling of drift variants and residual immunity from previous vaccination or past infection can complicate the multi-year dynamics of influenza vaccination (Plotkin et al., 2002; Smith et al., 2004; Duschoff et al., 2004). In a given year, information about previous strains can in principle be used to update prior information about the parameters of the next outbreak. The current formulation does not examine any

multiyear benefits from vaccination that may accrue from projecting vaccine strains for multiple years. This paper presents a positive first step for approaching the first-order effects of the current year's outbreak.

Three, the analysis above assumes that the per person benefit b and the cost to administer p_a are constant. A conjecture is that the results generalize nicely to the case of variable marginal benefits of vaccination, $b(f)$, as long as $b(f)T(f)$ is convex and decreasing. Terms like $bT'(f)$ in the definition of $\bar{f}$ would be replaced with $(b(f)T(f))' = b'(f)T(f) + b(f)T'(f)$. A convex increasing administration cost, $p_a(f)$, may also be appropriate. The net effect of these two changes is expected to decrease the optimal vaccination fraction.

Four, the model assumes that health consequences can be quantified by direct and indirect monetary costs, but a multi-attribute approach might be desired to more fully examine issues like the number of deaths or hospitalizations. These features can be modeled indirectly with our proposed model by assessing the number infected and applying the relevant morbidity and mortality rates.

Five, the analysis assumes that the government is risk neutral, but in fact a government may wish to specify a higher level of vaccines in order to prepare for a worst case scenarios. One approach to account for that would be to perform the above optimization with the added constraint that the fraction to vaccinate that is announced by the government be at least as large as some threshold. Another would be to inflate the value of the parameter b that is used to model the cost per infection, to reflect a penalty for having too many infections.

Six, the model assumes that the government can precisely specify the number of individuals to vaccinate. This is potential drawback of the other epidemic models mentioned in this paper, too. The inclusion of individual's choice to become vaccinated would also require much additional complexity.

Seven, the model currently examines a single manufacturer and a single government, and assumes that all parameters are known to all parties. The cost per dose and yield distributions are not likely to be public information, and there are several providers and many purchasers. Nevertheless the equilibrium might still be modeled as an outcome of interactions between two rational actors of the model. Multiple buyers and

suppliers would be an interesting extension. Contracts in the presence of multiple manufacturers and/or suppliers could be complicated, to avoid collusion on the part of a subset of the players.

6 Conclusion

This work developed the first integrated supply-chain/health economics model of two key players in the influenza vaccine supply chain: a government that purchases and administers vaccines in order to achieve an efficient cost-benefit tradeoff, and a manufacturer that optimizes production input levels to achieve a cost-effective delivery of vaccines in the presence of yield uncertainty. The model indicates that a lack of coordination leaves the manufacturer with production yield risks. That lack of coordination results in vaccine production shortfalls if the players in the model act rationally.

We show that a global social optimum cannot be fully attained by changing the vaccine price alone, or by reducing the risk of production yields by having the government contractually pay a reduced rate for doses that are produced in excess of the original demand. A variation of the cost sharing contract is one option that can align incentives to achieve a social optimum. That contract shares production yield risks, which are initially carried by the manufacturer, with the public payor, by carefully balancing the price per dose of vaccine delivered with a small charge for production effort by the manufacturer.

Acknowledgments: We are grateful for informative discussions regarding vaccine production with representatives of a leading influenza vaccine manufacturer during the preparation of this manuscript. We appreciate the support of the Booz Allen fund to INSEAD.

REFERENCES

ANDERSON, R. AND R. MAY. 1991. *Infectious Diseases of Humans: Dynamics and Control.* Oxford University Press.

BARNES-SCHUSTER, D., Y. BASSOK., AND R. ANUPINDI. 2002. Coordination and flexibility in supply contracts with options. *Manufacturing & Service Operations Management 4*(3), 171–207.

BRANDEAU, M., F. SAINFORT., AND W. PIERSKALLA. (eds) 2004. *Operations Research and Health Care: A Handbook of Methods and Applications*. Kluwer Academic Publishers.

BUSH, G. W. 2005. President outlines pandemic influenza preparations and response. Accessed 22 January 2006 at http://www.whitehouse.gov/news/releases/2005/11/20051101-1.html.

CACHON, G. AND M. LARIVIERE. 2005. Supply chain coordination with revenue sharing contracts. *Management Science 51*(1), 30.

CACHON, G. P. 2003. Supply chain coordination with contracts. In S. Graves and T. de Kok (Eds.), *Handbook in Operations Research and Management Science: Supply Chain Management*. Elsevier.

CDC 2005. Vaccine price list. Available from http://www.cdc.gov/nip/vfc/cdc_vac_price_list.htm.

DIETZ, K. 1993. The estimation of the basic reproduction number for infectious diseases. *Statistical Methods in Medical Research 2*, 23–41.

DUSCHOFF, J., J. PLOTKIN., S. LEVIN., AND D. EARN. 2004. Dynamical resonance can account for seasonality of influenza epidemics. *Proc Natl Acad Sci USA 101*, 16915–16916.

GANI, R., H. HUGHES., D. FLEMING., T. GRIFFIN., J. MEDLOCK., AND S. LEACH. 2005. Potential impact of antiviral drug use during influenza pandemic. *Emerging Infectious Diseases 11*(9), 1355–1362.

GERDIL, C. 2003. The annual production cycle for influenza vaccine. *Vaccine 21*, 1776–1779.

HENIG, M. AND Y. GERCHAK. 1990. The structure of periodic review policies in the presence of random yield. *Operations Research 38*(4), 634–643.

HILL, A. N. AND I. M. LONGINI. 2003. The critical fraction for heterogeneous epidemic models. *Mathematical Biosciences 181*, 85–106.

JANEWAY, C. A., P. TRAVERS., M. WALPORT., AND M. SHLOMCHIK. 2001. *Immunobiology* (5th ed.). New York: Garland Publishing.

KAPLAN, E. H., D. L. CRAFT., AND L. M. WEIN. 2002. Emergency response to a smallpox attack: The case for mass vaccination. *Proc Nat Acad Sci U S 99*(16), 10935–10940.

LARIVIERE, M. 1999. Supply chain contracting and coordination with stochastic demand. In *Quantitative Models of Supply Chain Management* (S. Tayur, M. Magazine and R. Ganeshan ed.). Kluwer Academic.

LONGINI, I. M., E. ACKERMAN., AND L. R. ELVEBACK. 1978. An optimization model for influenza A epidemics. *Mathematical Biosciences 38*, 141–157.

LONGINI, I. M., M. E. HALLORAN., A. NIZAM., M. WOLFF., P. M. MENDELMAN., P. E. FAST., AND R. B. BELSHE. 2000. Estimation of the efficacy of live, attenuated influenza vaccine from a two-year, multi-center vaccine trial: implications for influenza epidemic control. *Vaccine 18*, 1902–1909.

LONGINI, I. M., M. E. HALLORAN., A. NIZAM., AND Y. YANG. 2004. Containing pandemic influenza with antiviral agents. *Am. J. Epidemiol. 159*(7), 623–633.

MURRAY, J. D. 1993. *Mathematical Biology* (2nd ed.). Springer.

NASH, J. F. 1951. Non-cooperative games. *Annals of Mathematics 54*, 286–295.

NICHOL, K. L., K. L. MARGOLIS., J. WUORENMA., AND T. V. STERNBERG. 1994. The efficacy and cost effectiveness of vaccination against influenza among elderly persons living in the community. *The New England Journal of Medicine 331*(12), 778–784.

PALESE, P. 2006. Making better influenza virus vaccines. *Emerging Infectious Diseases 12*(1), 61–65.

PASTERNACK, B. 1985. Optimal pricing and returns policies for perishable commodities. *Marketing Science 4*(2), 166–176.

PIEN, H. 2004. Statement presented to Committee on Aging, United States Senate, by President and CEO, Chiron Corporation. Accessed 22 January 2006 at http://aging.senate.gov/_files/hr133hp.pdf.

PISANO, W. 2006. Keys to strengthening the supply of routinely recommended vaccines: View from industry. *Clinical Infectious Diseases 42*, S111–S117.

PLOTKIN, J., J. DUSCHOFF., AND S. LEVIN. 2002. Hemagglutinin sequence clusters and the antigenic evolution of influenza A virus. *Proc Natl Acad Sci USA 99*, 6263–6268.

SHIH, W. 1980. Optimal inventory policies when stockouts result from defective products. *International Journal of Production Research 18*(6), 677–685.

SILVER, E. A. 1976. Establishing the reorder quantity when the amount received is uncertain. *INFOR 14*, 32–39.

SMITH, D., A. LAPEDES., J. DE JONG., T. BESTEBROER., G. RIMMELZWAAN., AND OTHERS. 2004. Mapping the antigenic and genetic evolution of influenza virus. *Science 305*, 3713–376.

SMITH, P. G., L. C. RODRIGUES., AND P. E. FINE. 1984. Assessment of the protective efficacy of vaccines against common diseases using case-control and cohort studies. *International Journal of Epidemiology 13*(1), 87–93.

SU, X. AND S. ZENIOS. 2004. Patient choice in kidney allocation: The role of the queueing discipline. *Manufacturing and Service Operations Management 6*(4), 280–301.

U.S. DEPT. OF HEALTH AND HUMAN SERVICES 2004. Secretary Thompson announces contract to secure future egg supply for flu vaccines. At http://www.dhhs.gov/news/press/2004pres/20041109a.html.

U.S. DEPT. OF HEALTH AND HUMAN SERVICES 2005. HHS Pandemic influenza plan: Supplement 6–Vaccine distribution and use. Accessed 26 January 2006 at http://www.dhhs.gov/pandemicflu/plan/.

U.S. GAO 2001. Flu vaccine: Supply problems heighten need to ensure access for high-risk people. U.S. General Accounting Office Report To Congressional Requesters, GAO-01-624.

WENIGER, B. G., R. T. CHEN., S. H. JACOBSON., E. C. SEWELL., R. DEMON., J. R. LIVENGOOD., AND W. A. ORENSTEIN. 1998. Addressing the challenges to immunization practice with an economic algorithm for vaccine selection. *Vaccine 16*(9), 1885–1897.

WEYCKER, D., J. EDELSBERG., M. E. HALLORAN., I. M. LONGINI., A. NIZAM., V. CIURYLA., AND G. OSTER. 2005. Population-wide benefits of routine vaccination of children against influenza. *Vaccine 23*, 1284–1293.

WHO 2005. Influenza vaccines. *Weekly Epidemiological Record 80*(33), 277–287, Accessed 21 Jan 2006 at http://www.who.int/wer/2005/wer8033.pdf.

WU, J. T., L. M. WEIN., AND A. S. PERELSON. 2005. Optimization of influenza vaccine selection. *Operations Research 53*(3), 456–476.

WYSOCKI, B. AND S. LUECK. 2006. Margin of safety: Just-in-time inventories make U.S. vulnerable in a pandemic. Wall Street Journal. 12 January 2006.

YADAV, P. AND D. WILLIAMS. 2005. Value of creating a redistribution network for influenza vaccine in the U.S. Presentation at INFORMS 2006 Annual Conference, San Francisco.

YANO, C. A. AND H. L. LEE. 1995. Lot sizing with random yields: A review. *Operations Research 43*(2), 311–334.

ZARIC, G., M. BRANDEAU., D. BRAVATA., J.-E. HOLTY., E. WILHELM., K. MCDONALD., AND D. OWENS. 2006. Modeling the logistics of response to anthrax bioterrorism. *working paper*.

Online Companion For:

Supply Chain Coordination and Influenza Vaccination

Stephen E. Chick
INSEAD, Technology
and Operations Management Area
Boulevard de Constance
77305 Fontainebleau FRANCE
stephen.chick@insead.edu

Hamed Mamani
MIT Operations Research Center
77 Massachusetts Ave. Bldg
E40-130
Cambridge, MA 02139 USA
hamed@mit.edu

David Simchi-Levi
MIT Department of Civil and Environmental
Engineering & The Engineering System Division
77 Mass. Ave. Rm. 1-171
Cambridge, MA 02139 USA
dslevi@mit.edu

Appendix A provides background on the specific epidemic model that was used to determine results for the paper, "Supply Chain Coordination and Influenza Vaccination". Appendix B provides the mathematical proofs for the propositions, theorems, and lemmas that were presented in the paper. Appendix C justifies why the paper considers piecewise-linear and convex functions for the number of individuals that are ultimately infected in an influenza outbreak, $T(f)$, as a function of the fraction of individuals that are vaccinated.

A Epidemic Model

At a high level, the epidemic model drives the analysis for supply chain behavior through the function $T(f)$, which models the expected number of infected individuals in the population, as a function of the fraction of the population that is vaccinated. The details of the underlying epidemic model are decoupled from the analysis.

This section recalls one specific epidemic model in detail, the closed-population SIR model, and an analysis of that model which is not an advance to the literature *per se*, but that fixes ideas for the paper. It also provides some structural results for the SIR model with an initial vaccination (a nonzero $R(0)$) that are not readily accessible in standard texts. That model gives rise to the formula for the attack rate p in (3).

A standard formulation for the SIR epidemic model in a closed population of N individuals is:

$$\frac{dS}{dt} = -c\beta SI/N \tag{20}$$

$$\frac{dI}{dt} = +c\beta SI/N - I/d \tag{21}$$

$$\frac{dR}{dt} = +I/d, \tag{22}$$

where $c > 0$ is the number of contacts per unit time, $\beta \in [0, 1]$ is the probability of infection per contact, $d > 0$ is the duration of infection, and I/N is the probability that a contact is infectious. Timely vaccination followed by the onset of (instantaneous) infections from exogenous sources results in initial conditions $R(0) = Nf\phi$, $S(0) = N(1 - f\phi)(1 - \chi)$, $I(0) = N(1 - f\phi)\chi$.

If $S(0), I(0), R(0)$ are given initial conditions, then Murray (1993) defines an outbreak by $dI(0)/dt > 0$, which happens if and only if $c\beta dS(0)/N > 1$ (in the notation here). Once the derivative is negative, it stays negative. Murray (1993) calls $c\beta dS(0)/N$ the basic reproductive number. With the stated initial conditions, an outbreak occurs if and only if

$$\frac{c\beta d - \frac{1}{1-\chi}}{c\beta d\phi} > f. \tag{23}$$

In the main paper, an outbreak refers to the transmission of influenza during a single season, following its introduction at time $t = 0$, whether $dI(0)/dt > 0$ or not. This allows for a seasonal influenza outbreak to be stunted by a successful vaccination program. A large outbreak refers to an outbreak with $dI(0)/dt > 0$.

What we have defined as the basic reproduction number, $R_0 = c\beta d$, corresponds to the common epidemiological interpretation of R_0 as the expected number of individuals that are infected by a single infectious individual in an otherwise susceptible population (Anderson and May, 1991). This definition of R_0 is also consistent with the definition of Murray (1993) in the limit as $S(0)/N \to 1$ (a single infected in a large population).

Our definition of the critical vaccine fraction,

$$f^0 = \frac{R_0 - 1}{R_0\phi},$$

corresponds to setting f to the left hand side of (23), and letting $\chi \to 0$. Operationally, this f^0 corresponds to the (limiting) fraction of the population that must be vaccinated in order to halt an outbreak for *any* nonzero level for the fraction of individuals that are infected from exogenous sources.

We now analyze that formulation. From (20) and (22),

$$\begin{aligned} \frac{dS}{dR} &= -\frac{c\beta\frac{SI}{N}}{I/d} = -c\beta d\frac{S}{N} = -\frac{R_0}{N}S \\ &\Rightarrow \quad S = \left[S(0)\exp(\frac{R_0}{N}R(0))\right]\exp(-\frac{R_0}{N}R). \end{aligned} \tag{24}$$

The constant $S(0)\exp(\frac{R_0}{N}R(0))$ comes from solving for initial conditions. Using (22), the conservation of the total population size ($N = S + I + R$ is constant, from adding equations (20) through (22)), and (24), we get an equation for dR/dt that only involves R and constants:

$$\begin{aligned} \frac{dR}{dt} &= \frac{I}{d} = \frac{1}{d}(N - R - S) \\ &= \frac{1}{d}(N - R - S(0)\exp(-\frac{R_0}{N}(R - R(0)))) \end{aligned} \tag{25}$$

At the end of the epidemic, the number that are ultimately infected is $R(\infty)$ and the derivative in (25) converges to 0. Set (25) to 0 and multiply by d to get:

$$R(\infty) = N - S(0)\exp(-\frac{R_0}{N}(R(\infty) - R(0))). \tag{26}$$

Rescaling to $N = 1$, to obtain fractions of the population, the above formula is:

$$R(\infty) = 1 - S(0)\exp(-R_0(R(\infty) - R(0))). \tag{27}$$

We now subtract out the fraction of those that were vaccinated, $R(0) = 1 - S(0) - I(0)$, to obtain the attack rate $p = R(\infty) - R(0)$, the fraction infected during the outbreak.

$$p = R(\infty) - R(0) = S(0) + I(0) - S(0)\exp(-R_0 p) \tag{28}$$

That justifies (3) from the main paper.

B Proofs of Mathematical Results

Proposition 1. *Proof:* The expected cost function for the manufacturer is

$$\begin{aligned} MF(n_E) &= cn_E - p_r \mathrm{E}\big[\min\{n_E U, fNd\}\big] \\ &= cn_E - p_r n_E \mathrm{E}\big[\min\{U, \frac{fNd}{n_E}\}\big] \\ &= cn_E - p_r n_E \Big(\int_0^{\frac{fNd}{n_E}} u g_U(u)du + \int_{\frac{fNd}{n_E}}^{\infty} \frac{fNd}{n_E} g_U(u)du\Big) \\ &= cn_E - p_r n_E \int_0^{\frac{fNd}{n_E}} u g_U(u)du - p_r fNd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du \end{aligned}$$

So to get the minimum of MF we need to see the behavior of its derivative:

$$\begin{aligned} \frac{\partial MF}{\partial n_E} &= c - p_r \int_0^{\frac{fNd}{n_E}} u g_U(u)du - p_r n_E \Big[(\frac{fNd}{n_E}) g_U(\frac{fNd}{n_E})(-\frac{fNd}{n_E^2})\Big] - p_r fNd\Big[-g_U(\frac{fNd}{n_E})(-\frac{fNd}{n_E^2})\Big] \\ &= c - p_r \int_0^{\frac{fNd}{n_E}} u g_U(u)du + p_r \frac{(fNd)^2}{n_E^2} g_U(\frac{fNd}{n_E}) - p_r \frac{(fNd)^2}{n_E^2} g_U(\frac{fNd}{n_E}) \\ &= c - p_r \int_0^{\frac{fNd}{n_E}} u g_U(u)du \end{aligned}$$

Note that $\frac{\partial^2 MF}{\partial n_E^2} = p_r[(\frac{(fNd)^2}{n_E^3}) g_U(\frac{fNd}{n_E})] \geq 0$ so the first order optimality condition is sufficient. Hence the optimum production quantity n_E^* is solution of the following equation: $\int_0^{\frac{fNd}{n_E^*}} u g_U(u)du = \frac{c}{p_r}$. □

Corollary 1.1. *Proof:* Immediate upon inspection of the values of the parameters. □

Proposition 2. *Proof:* To show these results, we analyze SF in two different regions, $f \leq f^0$ and $f \geq f^0$.

Let $SF_1(f, n_E)$ denotes the value of SF when $f \leq f^0$, and likewise $SF_2(f, n_E)$ is the value of SF where $f \geq f^0$. Note that if $f \leq f^0$ then $W = Z = \min\{n_E U, fNd\}$, and the value of SF_1 is

$$\begin{aligned} SF_1(f, n_E) =& b \int_0^{\frac{fNd}{n_E}} (M - \psi \frac{n_E u}{d}) g_U(u)du + b(M - N\psi f) \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du \\ &+ p_a n_E \int_0^{\frac{fNd}{n_E}} u g_U(u)du + p_a(fNd) \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + cn_E \qquad (f \leq f^0). \end{aligned} \tag{29}$$

For $f > f^0$, given that $M - N\psi f = M - N\psi f^0 = 0$, the value of SF is

$$\begin{aligned} SF_2(f, n_E) =& b \int_0^{\frac{f^0 Nd}{n_E}} (M - \psi \frac{n_E u}{d}) g_U(u)du + p_a n_E \int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du \\ &+ p_a(f^0 Nd) \int_{\frac{f^0 Nd}{n_E}}^{\infty} g_U(u)du + cn_E \qquad (f \geq f^0). \end{aligned} \tag{30}$$

The limits of integration in the right hand side of (30) use f^0, not f. In order to get the overall optimal values for f^S, n_E^S, we solve the following two subproblems.

$$\begin{array}{lll} SF1 = & \min & SF_1 \\ & \text{s.t.} & 0 \le f \le f^0 \\ & & n_E \ge 0 \end{array} \qquad \begin{array}{lll} SF2 = & \min & SF_2 \\ & \text{s.t.} & f^0 \le f \le 1 \\ & & n_E \ge 0 \end{array}$$

Optimality conditions for subproblem $SF1$**:** The KKT conditions, if $f \le f^0$, are,

$$-N\psi b \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + p_a Nd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + \xi - \theta_0 = 0$$

$$-\frac{\psi b}{d} \int_0^{\frac{fNd}{n_E}} u g_U(u)du + p_a \int_0^{\frac{fNd}{n_E}} u g_U(u)du + c - \varphi = 0$$

$$\xi(f - f^0) = \theta_0 f = \varphi n_E = 0 \quad ; \quad \xi, \theta_0, \varphi \ge 0,$$

where the first equation is obtained by taking the derivative with respect to f and the second equation is obtained by taking the derivative with respect to the n_E. Moreover ξ, θ_0, φ are KKT multipliers of constraints $f \le f^0,\ f \ge 0,\ n_E \ge 0$, respectively. Note that if Assumption 2 were not valid, then the second equation of KKT conditions would require $\varphi > 0$, and the third equation would imply that $n_E^* = 0$.

We are interested in the case where $n_E > 0, f > 0$ which is a conclusion of Assumption 2. This implies that $\theta_0 = \varphi = 0$, and the KKT conditions simplify:

$$\begin{gathered} \big[-N\psi b + p_a Nd \big] \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + \xi = 0 \\ \big[-\frac{\psi b}{d} + p_a \big] \int_0^{\frac{fNd}{n_E}} u g_U(u)du + c = 0 \\ \xi(f - f^0) = 0 \quad ; \quad \xi \ge 0 \end{gathered}$$

In the first equation above, Assumption 2 suggests that $\xi > 0$. If $\xi > 0$, the last of the KKT conditions would give rise to $f^* = f^0$. So SF_1 will always get its minimum at the extreme f^0. The optimal n_E in this case can be obtained from the second equation of the KKT conditions and using the fact that $f^* = f^0$, and

$$\int_0^{\frac{f^0 Nd}{n_E^*}} u g_U(u)du = \frac{c}{\frac{\psi b}{d} - p_a}. \tag{31}$$

Optimality conditions for the problem $SF2$: If $f \geq f^0$, then SF_2 does not depend on f (the vaccine fraction declared by the government does not change the value of objective function). It follows that all values $f^0 \leq f \leq 1$ are optimum and so the first part of the claim is proved.

Now SF_2 is a function of n_E only and the derivative of GF with respect to n_E is

$$\frac{\partial SF_2}{\partial n_E} = (-\frac{\psi b}{d} + p_a) \int_0^{\frac{f^0 Nd}{n_E}} u g_U(u) du + c.$$

Note that $\frac{\partial^2 SF_2}{\partial n_E^2} = (\frac{\psi b}{d} - p_a)(\frac{f^0 Nd}{n_E^2})(\frac{f^0 Nd}{n_E}) g_U(\frac{f^0 Nd}{n_E})$, which is nonnegative by Assumption 2, hence $SF_2(n_E)$ is a convex function on n_E and the first order optimality condition is sufficient. By getting the root of the derivative of SF_2 above, we can see that the optimum n_E for SF_2 is the same as the solution of (31). So the optimum value for n_E^S satisfies the same equation in both cases. $\square$

Proposition 3. *Proof:* We break this into two subproblems, as with $SF1$ and $SF2$ above. Define GF_1 to be the objective function for subproblem $GF1$, which handles the case where $f \leq f^0$. Then

$$\begin{aligned}
GF_1(f, n_E) = \; & b \int_0^{k^G} (M - \psi \frac{n_E u}{d}) g_U(u) du + b(M - N\psi f) \int_{k^G}^{\infty} g_U(u) du \\
& + (p_a + p_r) n_E \int_0^{k^G} u g_U(u) du + (p_a + p_r)(fNd) \int_{k^G}^{\infty} g_U(u) du \\
= \; & bM - \frac{\psi b}{d} n_E \int_0^{k^G} u g_U(u) du - N\psi b f \int_{k^G}^{\infty} g_U(u) du \\
& + (p_a + p_r) n_E \int_0^{k^G} u g_U(u) du + (p_a + p_r) f Nd \int_{k^G}^{\infty} g_U(u) du \qquad (\textstyle\int_0^{\infty} g_U(u) du = 1) \\
= \; & bM - \frac{\psi b}{d} n_E \int_0^{k^G} u g_U(u) du - \psi b \frac{n_E k^G}{d} \int_{k^G}^{\infty} g_U(u) du \\
& + (p_a + p_r) n_E \int_0^{k^G} u g_U(u) du + (p_a + p_r) n_E k^G \int_{k^G}^{\infty} g_U(u) du \qquad (fNd = n_E k^G) \\
= \; & bM + n_E (-\frac{\psi b}{d} + p_a + p_r) \big[\int_0^{k^G} u g_U(u) du + k^G \int_{k^G}^{\infty} g_U(u) du \big].
\end{aligned}$$

By Assumption 3, the coefficient of n_E in the last equality is negative, so the optimum value for n_E in GF_1 lies on the upper boundary, where $f = f^0$. This proves the first part of the claim.

For the second part, similarly define GF_2 to be the government objective function for the case $f \geq f^0$.

Use the fact that $T(f)=0$ for all $f \geq f^0$, and the optimal manufacturing constraint, $f=\frac{n_E k^G}{Nd}$, to obtain

$$\begin{aligned}
GF_2(f,n_E) &= b\int_0^{\frac{f^0Nd}{n_E}}(M-\psi\frac{n_E u}{d})g_U(u)du + p_a n_E\int_0^{\frac{f^0Nd}{n_E}} ug_U(u)du + p_r n_E\int_0^{k^G} ug_U(u)du \\
&\quad + p_a(f^0Nd)\int_{\frac{f^0Nd}{n_E}}^{\infty} g_U(u)du + p_r(fNd)\int_{k^G}^{\infty} g_U(u)du \\
&= b\int_0^{\frac{f^0Nd}{n_E}}(M-\psi\frac{n_E u}{d})g_U(u)du + p_a n_E\int_0^{\frac{f^0Nd}{n_E}} ug_U(u)du + p_a(f^0Nd)\int_{\frac{f^0Nd}{n_E}}^{\infty} g_U(u)du \\
&\quad + p_r n_E\big[\int_0^{k^G} ug_U(u)du + k^G\int_{k^G}^{\infty} g_U(u)du\big] \\
\frac{\partial GF_2}{\partial n_E} &= (-\frac{\psi b}{d}+p_a)\int_0^{\frac{f^0Nd}{n_E}} ug_U(u)du + p_r\int_0^{k^G} ug_U(u)du + p_r k^G\int_{k^G}^{\infty} g_U(u)du \\
\frac{\partial^2 GF_2}{\partial n_E^2} &= (\frac{\psi b}{d}-p_a)\frac{f^0Nd}{n_E^2}g_U(\frac{f^0Nd}{n_E})
\end{aligned}$$

for $f \geq f^0$. Note that $\frac{f^0Nd}{n_E} \leq k^G$. By Assumption 2, $\frac{\partial^2 GF_2}{\partial n_E^2} \geq 0$, so GF_2 is a convex function of n_E. To find the minimum it suffices to look at the sign of its first derivative. If Condition (12) holds, then Assumption 2 implies that $\frac{\partial GF_2}{\partial n_E} \geq 0$ on $f \geq f^0$, so that the minimum of GF_2 for $f \in [f^0,1]$ is obtained at f^0. The optimum for both GF_1 and GF_2 lead to the claimed optimum, namely $f^G = f^0$.

If Condition (12) does not hold (i.e. $(-\frac{\psi b}{d}+p_a+p_r)\int_0^{k^G} ug_U(u)du + p_r k^G\int_{k^G}^{\infty} g_U(u)du < 0$); then because of the convexity of function GF_2 on n_E (non-decreasing derivative), there are two cases:

Case 1: $\exists \tilde{n}_E$; $\frac{\partial GF_2}{\partial \tilde{n}_E^G}=0$. In this case clearly the optimum values for the f, n_E are the following: $n_E^G = \tilde{n}_E$, $f^G = k^G n_E^G/Nd$.

Case 2: If $n_E(1)$ denotes the maximum n_E corresponding to $f=1$ (i.e. $n_E(1)=\frac{1Nd}{k^G}$) and still $\frac{\partial GF_2}{\partial n_E^G} < 0$ then $f^G = 1$, $n_E^G = n_E(1)$.

Combined, the two cases complete the proof. $\square$

Theorem 4. *Proof:* Proposition 2 shows that $f^G \geq f^0$. We consider the two cases $f^G = f^0$ and $f^G > f^0$ separately, and prove that both cases lead to the relation $n_E^S > n_E^G$.

Case 1: $f^G = f^0$**.** Using the inequality in Corollary 2.1 (i.e. $k^S < k^G$) and using the definitions of k^G, k^S it immediately follows that $n_E^S > n_E^G$, as desired.

Case 2: $f^G > f^0$**.** (Proof by contradiction.) Assume to the contrary that $n_E^S \leq n_E^G$. First of all we obtain the sign of $\left[\frac{\partial GF_2}{\partial n_E}\right]_{n_E^G}$. As in the proof of Proposition 3, there are two cases for n_E^G. If the condition in case 1 of Proposition 3 holds, then $\left[\frac{\partial GF_2}{\partial n_E}\right]_{n_E^G} = 0$. If case 2 holds, then $\left[\frac{\partial GF_2}{\partial n_E}\right]_{n_E^G} \leq 0$. In either case, the following relation is true:

$$\left[\frac{\partial GF_2}{\partial n_E}\right]_{n_E^G} \leq 0 \tag{32}$$

On the other hand,

$$\begin{aligned}
\left[\frac{\partial GF_2}{\partial n_E}\right]_{n_E^G} &= (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E^G}} ug_U(u)du + p_r\int_0^{k^G} ug_U(u)du + p_r k^G\int_{k^G}^{\infty} g_U(u)du \\
&\geq (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E^S}} ug_U(u)du + p_r\int_0^{k^G} ug_U(u)du + p_r k^G\int_{k^G}^{\infty} g_U(u)du \\
&= (-\frac{\psi b}{d} + p_a)(\frac{c}{\frac{\psi b}{d} - p_a}) + c + p_r k^G\int_{k^G}^{\infty} g_U(u)du \\
&= p_r k^G\int_{k^G}^{\infty} g_U(u)du > 0
\end{aligned}$$

The inequality in the second line comes from the assumption $n_E^S \leq n_E^G$, and with Assumption 2. The third line is valid by (5) and Proposition 2. But the last inequality contradicts (32), so $n_E^G \geq n_E^S$ is false. □

Proposition 5. *Proof:* The proof of Theorem 4 shows that there does not exist a wholesale contract which coordinate this supply chain. That proof proceeded in two cases. The first case requires $n_E^S > n_E^G$. For full coordination, we require $n_E^S = n_E^G$ for some p_r. In case 2, $n_E^S = n_E^G$ for some p_r implies that $\frac{\partial GF}{\partial n_E}|_{n_E^G} > 0$, which would not be true for the optimizer of GF. □

Proposition 6. *Proof:* Note that $\int_0^{k^N} ug_U(u)du = \frac{c - p_c\mu}{p_r - p_c}$. By rewriting the GF in terms of values of f, n_E and by replacing $f = \frac{k^N n_E}{Nd}$ we have:

$$\begin{aligned}
GF(n_E) = b\int_0^{\frac{f^0 Nd}{n_E}} (M - \psi\frac{n_E u}{d})g_U(u)du + p_a n_E\int_0^{\frac{f^0 Nd}{n_E}} ug_U(u)du + p_a(f^0 Nd)\int_{\frac{f^0 Nd}{n_E}}^{\infty} g_U(u)du \\
+(p_r - p_c)n_E\int_0^{k^N} ug_U(u)du + (p_r - p_c)(k^N n_E)\int_{k^N}^{\infty} g_U(u)du + p_c\mu n_E
\end{aligned}$$

By Assumption 2, $\frac{\partial^2 GF_2}{\partial n_E^2} = (\frac{\psi b}{d} - p_a)\frac{f^0 Nd}{n_E^2} g_U(\frac{f^0 Nd}{n_E}) \geq 0$, so GF is a convex function on n_E. The optimal value of GF can therefore be found by setting its derivative to zero:

$$\begin{aligned}
\frac{\partial GF}{\partial n_E} &= (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du + p_c\mu \\
&\quad +(p_r - p_c)\Big[\int_0^{k^N} u g_U(u)du + k^N \int_{k^N}^{\infty} g_U(u)du\Big] \\
&= (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du + c + (p_r - p_c)k^N \int_{k^N}^{\infty} g_U(u)du
\end{aligned}$$

The last inequality comes from (13). The last term indicates implicitly that $n_E^N < n_E^S$. To see this, plug n_E^S into the last terms, use Proposition 2 and using the fact that $p_r > p_c$, to obtain $\frac{\partial GF}{\partial n_E}\Big|_{n_E^S} = (p_r - p_c)k^N \int_{k^N}^{\infty} g_U(u)du > 0$. That implies that $n_E^N < n_E^S$. $\square$

Theorem 7. *Proof:* First we show that $f^e \geq f^0$ by showing that optimum value for GF_1 for $f \in [0, f^0]$ is always obtained at f^0. By replacing $f = \frac{k^e n_E}{Nd}$ we get GF_1 to be only a function of n_E:

$$\begin{aligned}
GF_1(n_E) = \; & b\int_0^{k^e} (M - \psi\frac{n_E u}{d}) g_U(u)du + b(M - N\psi\frac{n_E k^e}{Nd})\int_{k^e}^{\infty} g_U(u)du \\
& +(p_a + p_r)n_E \int_0^{k^e} u g_U(u)du + (p_a + p_r)(k^e n_E)\int_{k^e}^{\infty} g_U(u)du + p_e n_E
\end{aligned}$$

Now by taking the derivative of GF_1 with respect to n_E we obtain that:

$$\begin{aligned}
\frac{\partial GF_1}{\partial n_E} &= -\frac{\psi b}{d}\int_0^{k^e} u g_U(u)du - \frac{\psi b}{d}k^e \int_{k^e}^{\infty} g_U(u)du \\
&\quad +(p_a + p_r)\int_0^{k^e} u g_U(u)du + (p_a + p_r)k^e \int_{k^e}^{\infty} g_U(u)du + p_e \\
&= (-\frac{\psi b}{d} + p_a)\int_0^{k^S} u g_U(u)du + p_r \int_0^{k^e} u g_U(u)du \qquad (33) \\
&\quad + (-\frac{\psi b}{d} + p_a + p_r)k^e \int_{k^e}^{\infty} g_U(u)du + p_e \\
&= -c + (c - p_e) + (-\frac{\psi b}{d} + p_a + p_r)k^e \int_{k^e}^{\infty} g_U(u)du + p_e \qquad (34) \\
&= (-\frac{\psi b}{d} + p_a + p_r)k^e \int_{k^e}^{\infty} g_U(u)du, \qquad (35)
\end{aligned}$$

in which (33) is obtained because $k^e = k^S$, and (34) is obtained using Proposition 2 and (14). On the other hand (35) is negative by Assumption 3, so that GF_1 is decreasing for all eligible n_E. Hence f^0 and the

corresponding n_E (i.e. $n_E = \frac{f^0 Nd}{k^e} = \frac{f^0 Nd}{k^S}$) are optimal in this case. So $f^e \geq f^0$. Because $k^e = k^S$, it immediately follows that $n_E^e \geq n_E^S$.

Now we show that the optimum of GF_2, for $f \in [f^0, 1]$, also occurs at f^0, completing the proof. Note that $f \geq f^0$ and $k^e = k^S$ imply that $n_E \geq n_E^S$. Consider GF_2.

$$GF_2(n_E) = b\int_0^{\frac{f^0 Nd}{n_E}} (M - \psi\frac{n_E u}{d}) g_U(u)du + p_a n_E \int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du + p_a f^0 Nd \int_{\frac{f^0 Nd}{n_E}}^{\infty} g_U(u)du$$
$$+ p_r n_E \int_0^{k^e} u g_U(u)du + p_r(k^e n_E)\int_{k^e}^{\infty} g_U(u)du + p_e n_E$$

The derivative is nonnegative,

$$\begin{aligned}
\frac{\partial GF_2}{\partial n_E} &= (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du + p_r \int_0^{k^e} u g_U(u)du + p_r k^e \int_{k^e}^{\infty} g_U(u)du + p_e \\
&= (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E}} u g_U(u)du + c + p_r k^e \int_{k^e}^{\infty} g_U(u)du \qquad (36)\\
&\geq (-\frac{\psi b}{d} + p_a)\int_0^{\frac{f^0 Nd}{n_E^S}} u g_U(u)du + c + p_r k^e \int_{k^e}^{\infty} g_U(u)du \qquad (37)\\
&= p_r k^e \int_{k^e}^{\infty} g_U(u)du \geq 0 \qquad (38)
\end{aligned}$$

(36) comes from (14). As before, (37) comes from Assumption 2 and the fact that $n_E \geq n_E^S$. Finally, (38) is true by Proposition 2. The last inequality shows that the optimum value for GF_2 occurs at f^0 hence $f^e = f^0$ and because of the fact that $k^e = k^S$, we obtain $n_E^e = n_E^S$. $\square$

Proposition 8. *Proof:* The proof resembles the proof of Proposition 2, except for the change in role of f^0 to $\bar{f}$, and the definitions of $SF1, SF_1$ and $SF2, SF_2$. We first show that the optimum value of SF_1 always occurs at the border, i.e. $f^* = \bar{f}$, by examining the KKT condition for SF_1:

$$bT'(f)\int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + p_a Nd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + \xi = 0$$
$$-\frac{b}{Nd}\int_0^{\frac{fNd}{n_E}} T'(\frac{n_E u}{Nd}) u g_U(u)du + p_a \int_0^{\frac{fNd}{n_E}} u g_U(u)du + c = 0$$
$$\xi(f - \bar{f}) = 0 \quad ; \quad \xi \geq 0$$

If $f < \bar{f}$, then by the convexity of $T(f)$ and the definition in (1), we conclude that $bT'(f) + p_a Nd < 0$. So the first equation forces $\xi > 0$, then by the third equation we obtain $f^* = \bar{f}$. So the optimum value for

SF_1 occurs at the border which is $\bar{f}$. Since SF does not change as f varies in $[\bar{f}, 1]$, we have shown the first part of the claim. The optimum value for n_E^* in this case can be obtained using the second equation of the KKT conditions and the fact that $f^* = \bar{f}$. Namely, the optimum n_E solves the following equation: $\int_0^{\frac{\bar{f}Nd}{n_E^*}} \Big[\frac{b}{Nd}T'(\frac{n_E^* u}{Nd}) + p_a\Big] u g_U(u)du + c = 0$, as claimed.

It is now enough to show that in the second case where $f \geq \bar{f}$, the same relation holds for the optimum production level. To show this, note that first of all, SF_2 is a function of n_E only, hence to get the optimum it suffices to find the root of its derivative:

$$\frac{\partial SF_2}{\partial n_E} = \int_0^{\frac{\bar{f}Nd}{n_E}} \Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\Big] u g_U(u)du + c$$

By setting this equation to zero we will end up by the same type of relation for n_E^* which we obtained before from SF_1, hence always $\int_0^{\frac{\bar{f}Nd}{n_E^S}} \Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big] u g_U(u)du + c = 0$. $\square$

Proposition 9. *Proof:* The first part of this claim is just the optimality condition for the manufacturer. As above, this does not depend on the shape of $T(f)$ so this relation remains the same. The fraction k^G is therefore determined by the values of c, p_r and the egg yield variability, and are assumed to be known.

To prove the second part, note that if $\int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du < 0$, then by replacing $f = \frac{n_E k^G}{Nd}$, we can rewrite GF_1 just as a function of n_E as follows:

$$\begin{aligned} GF_1(n_E) = \; & b\int_0^{k^G} T(\frac{n_E u}{Nd}) g_U(u)du + bT(\frac{n_E k^G}{Nd}) \int_{k^G}^{\infty} g_U(u)du \\ & + (p_a + p_r) n_E \int_0^{k^G} u g_U(u)du + (p_a + p_r)(n_E k^G) \int_{k^G}^{\infty} g_U(u)du \end{aligned}$$

$GF_1(n_E)$ is a convex function of n_E so the first derivative shows the behavior of this function completely:

$$\begin{aligned} \frac{\partial GF_1}{\partial n_E} = \; & \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\Big] u g_U(u)du + p_r \int_0^{k^G} u g_U(u)du \\ & + \frac{k^G}{Nd}\Big[bT'(\frac{n_E k^G}{Nd}) + p_a Nd\Big] \int_{k^G}^{\infty} g_U(u)du + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ = \; & \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ & + \frac{k^G}{Nd}\Big[bT'(\frac{n_E k^G}{Nd}) + p_a Nd\Big] \int_{k^G}^{\infty} g_U(u)du \end{aligned} \tag{39}$$

However, note that the function GF_1 is a convex function so clearly for every $f \leq \bar{f}$ or equivalently $n_E \leq \bar{n}_E$ we have: $\frac{\partial GF_1}{\partial n_E} \leq \Big[\frac{\partial GF_1}{\partial n_E}\Big]_{n_E=\bar{n}_E}$. On the other hand if we plug $\bar{n}_E$ into (39) we have:

$$\begin{aligned}\Big[\frac{\partial GF_1}{\partial n_E}\Big]_{n_E=\bar{n}_E} &= \int_0^{k^G}\Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd})+p_a\Big]ug_U(u)du + c + p_r k^G\int_{k^G}^{\infty} g_U(u)du \\ &\quad + \frac{k^G}{Nd}\Big[bT'(\frac{\bar{n}_E k^G}{Nd})+p_a Nd\Big]\int_{k^G}^{\infty} g_U(u)du \\ &= \int_0^{k^G}\Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd})+p_a\Big]ug_U(u)du + c + p_r k^G\int_{k^G}^{\infty} g_U(u)du\end{aligned}$$

in which the last equality comes from the fact that $\bar{n}_E = \frac{\bar{f}Nd}{k^G}$, and recalling (1). Note that the last expression is less than zero by assumption, so the optimum of GF_1 occurs at its border, $f^* = \bar{f}$. Because the inequality is strict, optimum of GF_2 also is greater than $\bar{f}$, so $f^G > \bar{f}$.

To show the reverse direction, we first show that the function

$$H(n_E) = \int_0^{\frac{\bar{f}Nd}{n_E}}\Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd})+p_a\Big]ug_U(u)du$$

is a nondecreasing function on n_E.

$$\begin{aligned}\frac{\partial H}{\partial n_E} &= \int_0^{\frac{\bar{f}Nd}{n_E}}\Big[\frac{b}{(Nd)^2}T''(\frac{n_E u}{Nd})\Big]u^2 g_U(u)du + \big[\frac{b}{Nd}T'(\bar{f})+p_a\big]\frac{\bar{f}Nd}{n_E}g_U(\frac{\bar{f}Nd}{n_E})\times(-\frac{\bar{f}Nd}{n_E^2}) \\ &= \int_0^{\frac{\bar{f}Nd}{n_E}}\Big[\frac{b}{(Nd)^2}T''(\frac{n_E u}{Nd})\Big]u^2 g_U(u)du \geq 0\end{aligned}$$

The second equation follows from the definition of $\bar{f}$, and the last inequality is due to the convexity of $T(f)$ in f. Hence we have $H(n_E) \geq H(\bar{n}_E)$ for all $n_E \geq \bar{n}_E$. By replacing $H(n_E)$ with its definition,

$$\int_0^{\frac{\bar{f}Nd}{n_E}}\Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd})+p_a\Big]ug_U(u)du \geq \int_0^{\frac{\bar{f}Nd}{\bar{n}_E}}\Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd})+p_a\Big]ug_U(u)du \qquad ;\forall n_E \geq \bar{n}_E \quad (40)$$

If we assume $\int_0^{k^G}\Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd})+p_a\Big]ug_U(u)du + c + p_r k^G\int_{k^G}^{\infty} g_U(u)du \geq 0$ we will show that $f^G \leq \bar{f}$, which is the reverse direction of part 2 of the claim.

Because this is the game setting, f can be replaced by $\frac{n_E k^G}{Nd}$, and

$$\begin{aligned} GF_2(n_E) &= b\int_0^{\frac{\bar{f}Nd}{n_E}} T(\frac{n_E u}{Nd}) g_U(u)du + bT(\frac{n_E k^G}{Nd}) \int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du + p_r n_E \int_0^{k^G} u g_U(u)du \quad (41) \\ &\quad + p_r(n_E k^G)\int_{k^G}^{\infty} g_U(u)du + p_a n_E \int_0^{\frac{\bar{f}Nd}{n_E}} u g_U(u)du + p_a(\bar{f}Nd)\int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du \\ \frac{\partial GF_2}{\partial n_E} &= \int_0^{\frac{\bar{f}Nd}{n_E}} \Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\Big] u g_U(u)du + p_r \int_0^{k^G} u g_U(u)du + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ &= \int_0^{\frac{\bar{f}Nd}{n_E}} \Big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \quad (42) \\ &\geq \int_0^{\frac{\bar{f}Nd}{\bar{n}_E}} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \geq 0 \end{aligned}$$

The second equality for $\frac{\partial GF_2}{\partial n_E}$ comes from (5). If $f \geq \bar{f}$ then $n_E \geq \bar{n}_E$, so the inequality in the third line is justified by (40). Finally the last inequality comes by assumption, and implies that for every $f \geq \bar{f}$ the function GF_2 is nondecreasing under the stated assumptions, so the optimum f^* for GF_2 can be obtained at $f^* = \bar{f}$. Hence $f^G \leq \bar{f}$, completing the proof. $\square$

The proof of **Theorem 10** requires the following three lemmas.

Lemma 14 *If $n_E^G \geq n_E^S$, then $f^G \leq \bar{f}$.*

Proof: To proof this lemma we show that the function GF_2 obtains its minimum at its border ($\bar{f}$). We use the function $H(n_E)$ that was defined in the proof of Proposition 9, which was shown to be nondecreasing, and $n_E^G \geq n_E^S$ to conclude that

$$\int_0^{\frac{\bar{f}Nd}{n_E^G}} \Big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\Big] u g_U(u)du \geq \int_0^{\frac{\bar{f}Nd}{n_E^S}} \Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big] u g_U(u)du.$$

By plugging n_E^G into the derivative function of GF_2 in (42), and using the above relation,

$$\begin{aligned} \Big[\frac{\partial GF_2}{\partial n_E}\Big]_{n_E = n_E^G} &= \int_0^{\frac{\bar{f}Nd}{n_E^G}} \Big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ &\geq \int_0^{\frac{\bar{f}Nd}{n_E^S}} \Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ &= p_r k^G \int_{k^G}^{\infty} g_U(u)du > 0 \end{aligned}$$

The equality in the third line comes from (8). The last inequality shows that the derivative of the function GF_2 at the optimum point n_E^G is strictly positive, which is not possible unless n_E^G is at its lower extreme, $n_E^G = \bar{n}_E$, where $\bar{n}_E$ introduced earlier. $\square$

Lemma 15 *Let $\bar{\bar{f}}$ be the solution of $bT'(\bar{\bar{f}}) + (p_a + p_r)Nd = 0$. Then $f^G > \bar{\bar{f}}$.*

Proof: By the definitions of $\bar{f}$ and $\bar{\bar{f}}$, and strict convexity of $T(f)$, we have $\bar{\bar{f}} < \bar{f}$. Let $\bar{\bar{n}}_E = \frac{\bar{\bar{f}}Nd}{k^G}$. Because $\bar{\bar{f}} < \bar{f}$, we examine the government subproblem $GF1$ with objective function GF_1 to analyze the pair $(\bar{\bar{f}}, \bar{\bar{n}}_E)$.

$$\begin{aligned}\Big[\frac{\partial GF_1}{\partial n_E}\Big]_{n_E=\bar{\bar{n}}_E} &= \int_0^{k^G}\Big[\frac{b}{Nd}T'(\frac{\bar{\bar{n}}_E u}{Nd}) + p_a + p_r\Big]ug_U(u)du \\ &\quad + \frac{k^G}{Nd}\Big[bT'(\frac{\bar{\bar{n}}_E k^G}{Nd}) + p_a Nd + p_r Nd\Big]\int_{k^G}^{\infty} g_U(u)dus \\ &= \int_0^{k^G}\Big[\frac{b}{Nd}T'(\frac{\bar{\bar{n}}_E u}{Nd}) + p_a + p_r\Big]ug_U(u)du < 0\end{aligned}$$

The second equality is true because the second term in the derivative is zero, by the definition of $\bar{\bar{f}}, \bar{\bar{n}}_E$. The last inequality comes from the strict convexity of $T(f)$, so $T'(f) < T'(\bar{\bar{f}})$; for all $f < \bar{\bar{f}}$. The derivative of GF_1 is negative at $\bar{\bar{f}}$. By the convexity of $T(f)$, it follows that the optimum of GF_1 is attained for a point bigger than $\bar{\bar{f}}$ (since $\bar{\bar{f}} < \bar{f}$), and so $f^G > \bar{\bar{f}}$. $\square$

Lemma 16 *Let $k^S = \frac{\bar{f}Nd}{n_E^S}$. Then for all $k > 0$,*

$$\int_0^{k^S}\Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big]ug_U(u)du \le \int_0^{k}\Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big]ug_U(u)du.$$

Proof: To prove the lemma, we show that $I(k) = \displaystyle\int_0^{k}\Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big]ug_U(u)du$ attains its minimum at $k^* = k^S$. The derivative of $I(k)$ is $\frac{\partial I}{\partial k} = \big[\frac{b}{Nd}T'(\frac{n_E^S k}{Nd}) + p_a\big]kg_U(k)$. Note that for $k < k^S$, we have $\frac{kn_E^S}{Nd} < \frac{k^S n_E^S}{Nd} = \bar{f}$, and so by the definition of $\bar{f}$, the derivative of $I(k)$ is negative. So $I(k)$ is decreasing for $k < k^S$. If $k > k^S$, then $\frac{kn_E^S}{Nd} > \bar{f}$ so $\frac{\partial I}{\partial k} > 0$, and $I(k)$ is increasing. Therefore $I(k)$ attains its minimum at k^S. $\square$

Theorem 10. *Proof:* Equipped with the three lemmas, we turn to a proof of Theorem 10 by contradiction. Let us assume that $n_E^G \geq n_E^S$. First of all by Lemma 14, we have $f^G \leq \bar{f}$. We consider two cases:

Case 1: $f^G < \bar{f}, n_E^G \geq n_E^S$**.** In this case, the optimum solution (f^G, n_E^G) would occur in the middle of the region for GF_1, so that $\Big[\frac{\partial GF_1}{\partial n_E}\Big]_{n_E=n_E^G} = 0$. By plugging n_E^G into (39), we have

$$\begin{aligned}
0 &= \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\Big] u g_U(u)du + c \\
&\quad + \frac{k^G}{Nd}\Big[bT'(\frac{n_E^G k^G}{Nd}) + p_a Nd + p_r Nd\Big] \int_{k^G}^{\infty} g_U(u)du \\
&= \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\Big] u g_U(u)du + c \\
&\quad + \frac{k^G}{Nd}\Big[bT'(f^G) + p_a Nd + p_r Nd\Big] \int_{k^G}^{\infty} g_U(u)du \quad ; \text{(because of } n_E^G k^G = f^G Nd) \\
&> \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\Big] u g_U(u)du + c \qquad ; \text{(Lemma 15 and convexity of } T(f)) \qquad (43)
\end{aligned}$$

On the other hand note that the function $J(n_E) = \int_0^k \big[\frac{b}{Nd}T'(\frac{n_E u}{Nd}) + p_a\big] u g_U(u)du$ is an increasing function of n_E. This is because $\frac{\partial J}{\partial n_E} = \int_0^k \big[\frac{b}{(Nd)^2}T''(\frac{n_E u}{Nd})\big] u^2 g_U(u)du \geq 0$, as $T(f)$ is a convex function. So $n_E^G \geq n_E^S$, means that $J(n_E^G) \geq J(n_E^S)$. By the definition of $J(n_E)$, and for $k = k^G$,

$$\int_0^{k^G} \big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\big] u g_U(u)du \geq \int_0^{k^G} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du \qquad (44)$$

If $\int_0^{k^G} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du \geq \int_0^{k^S} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du$, then (44) implies

$$\begin{aligned}
\int_0^{k^G} \big[\frac{b}{Nd}T'(\frac{n_E^G u}{Nd}) + p_a\big] u g_U(u)du + c &\geq \int_0^{k^S} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du + c \\
&= 0 \qquad ; \text{(by Proposition 8),}
\end{aligned}$$

which contradicts (43). So we should have:

$$\int_0^{k^G} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du < \int_0^{k^S} \big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\big] u g_U(u)du$$

but this inequality also contradicts Lemma 16. So case 1 results in a contradiction.

Case 2: $f^G = \bar{f}, n_E^G \geq n_E^S$**.** In this case, the production level would be $n_E^G = \bar{n}_E = \frac{\bar{f}Nd}{k^G}$. As $(\bar{f}, \bar{n}_E)$ is the optimum pair for GF_1, we should have: $\left[\frac{\partial GF_1}{\partial n_E}\right]_{\bar{n}_E} \leq 0$ or equivalently:

$$\begin{aligned} 0 &\geq \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ &\quad + \frac{k^G}{Nd}\Big[bT'(\frac{\bar{n}_E k^G}{Nd}) + p_a Nd\Big] \int_{k^G}^{\infty} g_U(u)du \\ &= \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c + p_r k^G \int_{k^G}^{\infty} g_U(u)du \\ &> \int_0^{k^G} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c \end{aligned}$$

On the other hand, the last expression can be written as:

$$\int_0^{\frac{\bar{f}Nd}{\bar{n}_E}} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c < 0 \tag{45}$$

Note however that $\bar{n}_E = n_E^G \geq n_E^S$. By the monotonicity of the function $H(n_E)$ from Proposition 9,

$$\int_0^{\frac{\bar{f}Nd}{\bar{n}_E}} \Big[\frac{b}{Nd}T'(\frac{\bar{n}_E u}{Nd}) + p_a\Big] u g_U(u)du + c \quad \geq \quad \int_0^{\frac{\bar{f}Nd}{n_E^S}} \Big[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a\Big] u g_U(u)du + c = 0,$$

which contradicts (45). Since both cases lead to a contradiction, the claim is proven. □

Claim 11. *Proof:* First we show that $p_r(f) \geq 0$. Note that the function $-T(f) + T'(\bar{f})f + T(0)$ is an increasing function of f on $[0, \bar{f}]$, as its derivative $-T'(f) + T'(\bar{f})$ exceeds 0 for all $f < \bar{f}$ because $T(f)$ is strictly convex. Further, its value is zero at $f = 0$, so $-T(f) + T'(\bar{f})f + T(0)$ is a nonnegative function over $[0, \bar{f}]$. Therefore $p_r(f) \geq 0$ for $f \in [0, \bar{f}]$. For $f \in (\bar{f} < f]$, it is clear that $p_r(f) = p_r(\bar{f}) \geq 0$.

We show that this $p_r(f)$ satisfies all the conditions in assumption in the reverse order. Multiplying $p_r(f)$ by fNd and taking the second derivative implies $\big(p_r(f)fNd\big)'' = -\kappa bT''(f)$. So $bT''(f) + \big(p_r(f)fNd\big)'' = (1-\kappa)bT''(f)$. But $bT''(f) + \big(p_r(f)fNd\big)''$ is the left hand side of the third condition in Assumption 5. By the strict convexity of $T(f)$,

$$bT''(f) + p_r''(f)fNd + 2p_r'(f)Nd = \frac{b}{2}T''(f) \geq 0; \quad \forall\, 0 \leq f \leq \bar{f}$$

For all $\bar{f} < f \leq 1$ we have $bT''(f) + p_r''(f)fNd + 2p_r'(f) = bT''(f) \geq 0$.

To prove validity of the second part of assumption, by taking the derivative of $\big(p_r(f)fNd\big)$ we have: $\big(p_r(f)fNd\big)' = \kappa b\big[-T'(f) + T'(\bar{f})\big]$ which is nonnegative for $0 \leq f \leq \bar{f}$ (by convexity of $T(f)$) and is zero for $f = \bar{f}$. For $\bar{f} < f \leq 1$; $\big(p_r(f)fNd\big)' = p_r(\bar{f})Nd \geq 0$.

Finally to show the first part we take the derivative of $p_r(f)$ for $0 \leq f \leq \bar{f}$:

$$\frac{\partial p_r}{\partial f} = -\kappa b\Big[\frac{T'(f)f - T(f) + T(0)}{f^2}\Big]$$

The numerator in the bracket is positive due to convexity of $T(f)$ indicating the desired result for $0 \leq f \leq \bar{f}$. Finally, for $\bar{f} < f \leq 1$ we have $p_r'(f) = 0$. $\square$

Theorem 13. *Proof:* In order to analyze Problem (19), we again split it into two separate subproblems.

Case 1 (*$0 \leq f \leq \bar{f}$*): In this case the optimization problem would be

$$\begin{aligned}\min_f \quad GF_1 = \Big[\, & b\int_0^{\frac{fNd}{n_E}} T(\frac{n_E u}{Nd}) g_U(u)du + bT(f)\int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + p_a n_E \int_0^{\frac{fNd}{n_E}} u g_U(u)du \\ & + p_a fNd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du + \underbrace{p_e(f)n_E + p_r(f)n_E \int_0^{\frac{fNd}{n_E}} u g_U(u)du}_{= cn_E \text{(by Assumption 6)}} \\ & + p_r(f)(fNd)\int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du \,\Big],\end{aligned}$$

subject to the constraints $fNd = k^S n_E$; $0 \leq f \leq \bar{f}$; and $n_E \geq 0$. Substituting the constraint $n_E = \frac{fNd}{k^S}$ into the objective function gives

$$\begin{aligned}\min_f \quad GF_1 = \Big[\, & b\textstyle\int_0^{k^S} T(\frac{f}{k^S}u) g_U(u)du + bT(f)\int_{k^S}^{\infty} g_U(u)du + p_a \frac{fNd}{k^S}\int_0^{k^S} u g_U(u)du \\ & + p_a fNd \displaystyle\int_{k^S}^{\infty} g_U(u)du + c\frac{fNd}{k^S} + p_r(f)fNd\int_{k^S}^{\infty} g_U(u)du \,\Big] \\ \text{s.t.} \quad & 0 \leq f \leq \bar{f}.\end{aligned}$$

We show that in this case the optimum value is at $\bar{f}$. For this purpose, it is enough to analyze the first

derivative of GF_1:

$$\begin{aligned}\frac{\partial GF_1}{\partial f} &= \Big[\ \frac{b}{k^S}\int_0^{k^S} T'(\frac{f}{k^S}u)ug_U(u)du + bT'(f)\int_{k^S}^{\infty} g_U(u)du + p_a\frac{Nd}{k^S}\int_0^{k^S} ug_U(u)du \\ &\quad + p_aNd\int_{k^S}^{\infty} g_U(u)du + c\frac{Nd}{k^S} + p_r(f)Nd\int_{k^S}^{\infty} g_U(u)du + p_r'(f)fNd\int_{k^S}^{\infty} g_U(u)du\ \Big] \\ &= \frac{Nd}{k^S}\Big(\int_0^{k^S}[\frac{b}{Nd}T'(\frac{f}{k^S}u) + p_a]ug_U(u)du + c\Big) \qquad (46) \\ &\quad + \big[\ bT'(f) + p_aNd + p_r(f)Nd + p_r'(f)fNd\ \big]\int_{k^S}^{\infty} g_U(u)du.\end{aligned}$$

We show that each of the two components in (46) is negative, making the derivative of GF_1 negative for all $0 \le f \le \bar{f}$. To see this, first note that the function $J(f) = \int_0^{k^S}[\frac{b}{Nd}T'(\frac{f}{k^S}u) + p_a]ug_U(u)du$ is an increasing function of f, as $J'(f) = \int_0^{k^S}[\frac{b}{Ndk^S}T''(\frac{f}{k^S}u)]u^2g_U(u) \ge 0$. Hence $J(f) \le J(\bar{f}),\ \forall f \le \bar{f}$. However, using $\bar{f}Nd = n_E^S k^S$, we get $J(\bar{f}) = \int_0^{k^S}[\frac{b}{Nd}T'(\frac{n_E^S u}{Nd}) + p_a]ug_U(u) = -c$ (by Proposition 8). As a result $J(f) + c \le 0$, so

$$\int_0^{k^S}[\frac{b}{Nd}T'(\frac{f}{k^S}u) + p_a]ug_U(u)du + c \le 0, \qquad \forall\, 0 \le f \le \bar{f}.$$

This shows that the first term in parenthesis in (46) is negative. To show that the second term of the derivative of GF_1 is also negative, we consider the term $bT'(f) + p_aNd + p_r(f)Nd + p_r'(f)fNd$. The derivative of this expression is $bT''(f) + p_r''(f)fNd + 2p_r'(f)Nd$, which is positive using the third part of Assumption 5. This means that $bT'(f) + p_aNd + p_r(f)Nd + p_r'(f)fNd \le bT'(\bar{f}) + p_aNd + p_r(\bar{f})Nd + p_r'(\bar{f})\bar{f}Nd$ for all $0 \le f \le \bar{f}$. Note that $bT'(\bar{f}) + p_aNd = 0$ by the definition of $\bar{f}$, and that $p_r(\bar{f})Nd + p_r'(\bar{f})fNd = 0$ by the second part of Assumption 5. This suggests

$$bT'(f) + p_aNd + p_r(f)Nd + p_r'(f)fNd \le 0, \qquad \forall\, 0 \le f \le \bar{f},$$

which shows that the second term of the derivative of GF_1 is also negative. By the strict convexity of $T(f)$, equality occurs only at $\bar{f}$. Hence (46) implies that $GF_1(f) \le 0$ for all $0 \le f \le \bar{f}$, meaning that the minimum of GF_1 is attained at $\bar{f}$. The corresponding production value to $\bar{f}$ is n_E^S (using 18). So in this case, the only candidate for optimality is the system optimal solution.

Case 2 ($\bar{f} \le f \le 1$): In this case, using the definition of $p_r(f)$, $p_r(f) = p_r(\bar{f})$, and hence $p_e(f) = p_e(\bar{f})$ for all $f \ge \bar{f}$. As a result, the government objective becomes:

$$\begin{aligned} GF_2 \;=\; & \Big[\, b\int_0^{\frac{\bar{f}Nd}{n_E}} T(\frac{n_E u}{Nd}) g_U(u)du + bT(\bar{f})\int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du + p_a n_E \int_0^{\frac{\bar{f}Nd}{n_E}} u g_U(u)du \\ & + p_a \bar{f} Nd \int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du + \underbrace{p_e(\bar{f})n_E + p_r(\bar{f})n_E \int_0^{\frac{fNd}{n_E}} u g_U(u)du}_{= cn_E \text{(by Assumption 5)}} \\ & + p_r(\bar{f}) fNd \int_{\frac{fNd}{n_E}}^{\infty} g_U(u)du \,\Big], \end{aligned}$$

subject to the constraints $fNd = k^S n_E$; $\bar{f} \le f \le 1$; and $n_E \ge 0$. Substituting the constraint $fNd = k^S n_E$ to remove f from the objective gives:

$$\begin{aligned} GF_2 = & \Big[\, b\int_0^{\frac{\bar{f}Nd}{n_E}} T(\frac{n_E u}{Nd}) g_U(u)du + bT(\bar{f})\int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du + p_a n_E \int_0^{\frac{\bar{f}Nd}{n_E}} u g_U(u)du \\ & + p_a \bar{f} Nd \int_{\frac{\bar{f}Nd}{n_E}}^{\infty} g_U(u)du + cn_E + p_r(\bar{f})n_E k^S \int_{k^S}^{\infty} g_U(u)du \,\Big] \end{aligned}$$

with the constraint $\bar{f} \le f$ replaced by the constraint $n_E \ge n_E^S$.

We now show that the derivative of the objective function in this case is positive, and hence GF_2 is minimized when that constraint is tight, $n_E = n_E^S$. Consider:

$$\frac{\partial GF_2}{\partial n_E} \;= \int_0^{\frac{\bar{f}Nd}{n_E}} [\frac{b}{Nd} T'(\frac{n_E u}{Nd}) + p_a] u g_U(u)du + c + p_r(\bar{f}) k^S \int_{k^S}^{\infty} g_U(u)du. \tag{47}$$

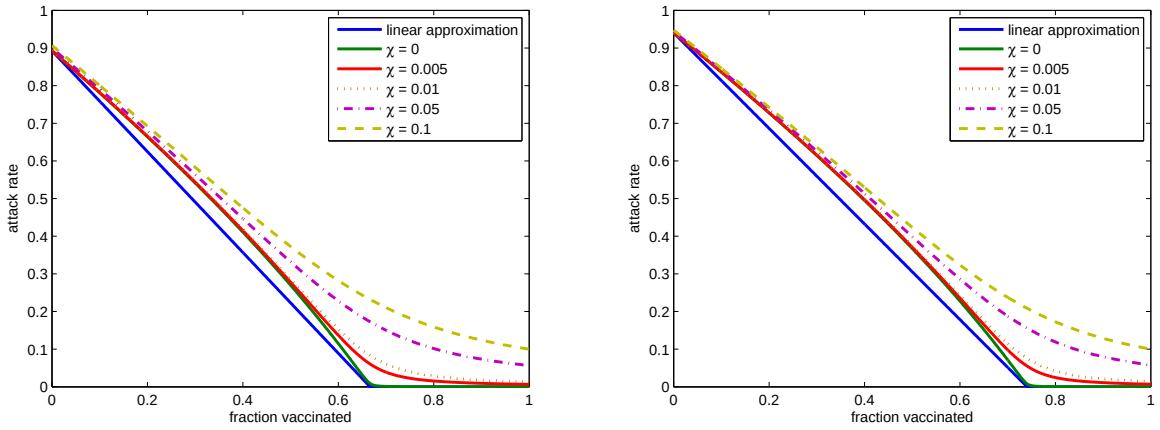
The first term above is exactly the function $H(n_E)$ introduced in the proof of Proposition 9. By using its nondecreasing property, we get $H(n_E) \ge H(n_E^S)$ for all $n_E \ge n_E^S$. Recall that Proposition 8 suggests $H(n_E^S) = \int_0^{\frac{\bar{f}Nd}{n_E^S}} [\frac{b}{Nd} T'(\frac{n_E^S u}{Nd}) + p_a] u g_U(u)du = -c$. This implies that

$$\int_0^{\frac{\bar{f}Nd}{n_E}} [\frac{b}{Nd} T'(\frac{n_E u}{Nd}) + p_a] u g_U(u)du + c \ge 0; \qquad \forall\, n_E \ge n_E^S.$$

By using this result with (47), we obtain the desired result,

$$\begin{aligned} \frac{\partial GF_2}{\partial n_E} \;&= \int_0^{\frac{\bar{f}Nd}{n_E}} [\frac{b}{Nd} T'(\frac{n_E u}{Nd}) + p_a] u g_U(u)du + c + p_r(\bar{f}) k^S \int_{k^S}^{\infty} g_U(u)du \\ &\ge p_r(\bar{f}) k^S \int_{k^S}^{\infty} g_U(u)du \ge 0. \end{aligned}$$

In both case 1 and case 2, the optimum values for the game setting are $\bar{f}, n_E^S$. $\square$



(a) $R_0 = 2.5$ (b) $R_0 = 3.0$

Figure 3: Attack rate (p) as a function of the fraction vaccinated (f), for different values of the fraction of susceptibles that are initially infected (χ) and the basic reproduction number (R_0).

C Justification Why Linear and Convex $T(f)$ are of Interest

Figures 1 and 3 show the shape of $T(f)$ with respect to different values of the initial fraction of susceptibles that become infected due to exogenous exposure, χ, and the expected number of secondary transmissions caused by one infected in an otherwise susceptible population, R_0. The four graphs correspond to $R_0 = 1.67, 2.0, 2.5, 3.0$, which are the range for R_0 for the different flu pandemics (Gani et al., 2005). In each graph, $T(f)$ is drawn for $\chi = 0, 0.005, 0.01, 0.05, 0.1$. The graphs look like a piecewise linear function as χ moves towards smaller values (thick blue curve). If χ is sufficiently large, then $T(f)$ looks strictly convex. The function $T(f)$ may also appear convex when averaging over unknown parameter values. Finally, it may be empirically sufficient for the contract in Section 4.2.2 to coordinate if $T(f)$ is convex for all sufficiently large f, even if it is concave for small f.

This section formalizes those statements.

Piecewise linear. If the initial fraction of the population that is infected is due to a very small exogenous exposure (small χ, so $I(0)$ is close to 0), then we can replace $I(0)/S(0)$ by zero in (3) and conclude:

$$\frac{p}{1 - e^{-R_0 p}} = S(0) = 1 - \phi f \tag{48}$$

Note that the function $\frac{p}{1-e^{-R_0 p}}$ looks like a linear function if R_0 is not very large, which is the case for influenza. So the relationship between f and p is almost linear.

By replacing $\frac{p}{1-e^{-R_0 p}}$ with its Taylor series expansion around zero we have

$$\frac{p}{1-e^{-R_0 p}} \approx \lim_{p_0 \to 0} \Big[\frac{p_0}{1-e^{-R_0 p_0}}\Big] + \lim_{p_0 \to 0} \Big[\frac{1-(1+R_0 p_0)e^{-R_0 p_0}}{(1-e^{-R_0 p_0})^2}\Big](p-0).$$

In order to find the limits we use the Taylor approximation $1 - R_0 p$ for $e^{-R_0 p}$ around zero. Substitute this approximation into the Taylor series expansion above to obtain

$$\begin{aligned} \frac{p}{1-e^{-R_0 p}} &\approx \lim_{p_0 \to 0} \Big[\frac{p_0}{1-1-R_0 p_0}\Big] + \lim_{p_0 \to 0} \Big[\frac{1-(1+R_0 p_0)(1-R_0 p_0)}{(1-1-R_0 p_0)^2}\Big](p-0) \\ &= \frac{1}{R_0} + p. \end{aligned}$$

Hence by plugging this last equation instead of $\frac{p}{1-e^{-R_0 p}}$ into (48) we have the following linear relationship between attack rate and vaccination fraction:

$$p = (1 - \frac{1}{R_0}) - \phi f$$

Note that the above line has a zero intercept at $f = \frac{R_0 - 1}{R_0 \phi}$, which is exactly the critical vaccine fraction in the case of homogeneous population (Hill and Longini, 2003). So clearly p remains zero for the case where f is greater than the critical vaccination fraction as the attack rate is a nonnegative parameter, and $T(f)$ is approximated by

$$T(f) = \begin{cases} N(1-1/R_0) - N\phi f, & 0 \le f \le f^0 \\ 0, & f^0 \le f \le 1 \end{cases}$$

While this equation has an epidemiologically attractive interpretation, it estimates the actual $T(0)$ poorly due to the Taylor series approximations. However, the f- and p-axis intercepts of the roughly linear plot when $I(0) \approx 0$ can be more accurately modeled by replacing $N(1-1/R_0)$ with $M = Np_0$, where p_0 solves (48) when $f = 0$; and by replacing the usual individual-level vaccine effect parameter, ϕ, with a parameter ψ

Table 3: Comparison between the linear approximation and actual $T(f)$

	f^G	n_E^G	Game Problem	f^S	n_E^S	System Problem
Actual $T(f)$	0.98	2.61×10^8	1.00×10^{10}	0.527	2.90×10^8	8.84×10^9
Linear approximation	0.91	2.40×10^8	9.68×10^9	0.517	2.82×10^8	8.52×10^9
Gap	7.1%	8%	3.2%	1.9%	2.7%	3.6%

that represents the number of infections averted in the population, ψ, by one additional vaccination. (The parameters ϕ and ψ are not necessarily the same, due to nonlinear infection dynamics.)

Hence we define the adjusted piecewise linear approximation for total number infected to be

$$T(f) = \begin{cases} Np_0 - N\psi f, & 0 \le f \le f^0 \\ 0, & f^0 \le f \le 1, \end{cases}$$

where $\psi = p_0 \frac{R_0\phi}{R_0 - 1}$ is chosen such that $T(f)$ hits the f axis at the critical vaccine fraction. Figure 1 and Figure 3 show the linear approximation versus the actual values of I_0. We have tested this approximation on a variety of parameters which are reasonable for the case of influenza. The gap between the actual $T(f)$ and the piecewise linear approximation for the game and the system problems is less than $2 \sim 3\%$ and the gap between the optimal decision variables is typically less than $4 \sim 5\%$. Table 3 illustrates these outcomes for the following parameter values: $\phi = 0.9,\ N = 3 \times 10^8,\ d = 1,\ c = 6,\ R_0 = 1.87,\ \chi \to 0,\ p_r = 12,\ b = 95,\ p_a = 40,\ U \sim gamma[5, 1/5]$).

Convex case. We now derive some of the properties of $T(f) = Np$ to argue that it is convex when χ, and therefore $I(0) = (1 - \phi f)\chi$, is sufficiently large. Recall (3) and that $S(0) = (1 - \phi f)(1 - \chi)$ to obtain

$$p = (1 - \phi f) - (1 - \phi f)(1 - \chi)e^{-R_0 p} \tag{49}$$

Our goal is to show that p is a convex function of f. Notice that in (49) p is an implicit function of f and to find its second derivative we will use the following lemma from calculus.

Lemma 17 *If $y = f(x)$ then*

$$\frac{\partial^2 f^{-1}(y)}{\partial y^2} = -\frac{1}{\left(\frac{\partial f(x)}{\partial x}\right)^3} \frac{\partial^2 f(x)}{\partial x^2}$$

By rearranging terms in (49), we can solve for f in terms of p:

$$f = \frac{1}{\phi}\Big[1 - \frac{p}{1-(1-\chi)e^{-R_0 p}}\Big].$$

Hence by taking the derivative:

$$f'(p) = -\frac{1}{\phi}\Big[\frac{1-(1+R_0 p)(1-\chi)e^{-R_0 p}}{(1-(1-\chi)e^{-R_0 p})^2}\Big]$$

First of all we show that $f'(p) \leq 0$. It is enough to show that the numerator in $f'(p)$ is positive. But we know that:

$$\begin{aligned} 1-(1+R_0 p)(1-\chi)e^{-R_0 p} &\geq 1-(1+R_0 p)e^{-R_0 p} \\ &\geq 0 \end{aligned}$$

The last inequality is based on the fact that the function $(1+x)e^x$ obtains its maximum at zero in the interval $[0,1]$. Hence by basic calculus since $\frac{\partial f}{\partial p}$ is negative so is $\frac{\partial p}{\partial f}$. So far we have shown that p is a decreasing function of f, when $\chi > 0$.

The second piece of the puzzle is to find the relationship for $f''(p)$ or the sign of it. By taking the second derivative of f we have:

$$f''(p) = -\frac{1}{\phi}\left[\frac{R_0(1-\chi)e^{-R_0 p}\big[R_0 p - 2 + (R_0 p+2)(1-\chi)e^{-R_0 p}\big]}{\big[1-(1-\chi)e^{-R_0 p}\big]^3}\right]$$

Note that if the second derivative of $f(p)$ were nonnegative, then by the nonpositivity of $f'(p)$ and using the above lemma, we would have $\frac{\partial^2 p}{\partial f^2} \geq 0$, which is the desired result in this part.

We will show that $f''(p)$ is not always positive, but that $f''(p) \geq 0$ for values of χ far enough from zero and small enough values of p. To show this, we note that the denominator is positive, and we evaluate the sign of the $f''(p)$'s numerator. Since $R_0(1-\chi)e^{-R_0 p}$ is always positive, we find the sign of $(R_0 p - 2) + (R_0 p + 2)(1-\chi)e^{-R_0 p}$.

Note that if $R_0 p \geq 2$, then $f(p)$ would be concave (since the numerator would be positive), and by the lemma, $T(f)$ would be concave, independent of the value of χ. The numerator may also be positive if p is

big enough (so that f is small enough). On the other hand, many estimates of R_0 for influenza are less than 2, and for those estimate that are larger than 2, $R_0 p < 2$ for even small to moderate values of f. We observe two things, numerically. (i) If $R_0 \leq 2$ and χ big enough, then attack rate is convex independent of f. (ii) Otherwise attack rate is convex for big enough values of f (which lead to small enough p). The reason is that for sufficiently small p, terms in the numerator that contain p become negligible, so the numerator is negative, making f convex. The related statement for the attack rate is that for big enough values of f (small enough p), the attack rate is a convex function of f.

This is sufficient to show that all of our results still hold true even when attack rate is not completely convex, i.e., the contract is still coordinating for the case where the function $T(f)$ is first a concave but after some point convex function of f, when the optimal f is in the region where $T(f)$ is convex. In fact our examples in man paper are based on this property, since $T(f)$ is not precisely convex for those sets of values of the parameters.

Uncertain outbreak parameters. Throughout the main paper, it is assumed that exact values of the reproduction number (R_0), the vaccine efficiency (ϕ) and the fraction of susceptible individuals that are initially infected (χ) are known. This might not be the case for a real influenza season. Although in general R_0, ϕ, χ are random variables, our analysis only depends on these parameters through the function $T(f)$. In this section we show that the shape of function $T(f)$ can still be convex, even with uncertainty about the values of epidemic parameters.

In order to incorporate this randomness, the definition of $T(f)$ should be the expected number of infected population, where expectation is taken over the uncertain R_0, ϕ, χ. This way the definition of $\bar{f}$ requires that the marginal (expected) benefit balance with the (expected) marginal cost. For this purpose we take a mixture of $T_{R_0,\phi,\chi}(f)$ (the number infected, given the specified R_0, ϕ, χ), in order to obtain $T(f) = E_{R_0,\phi,\chi}[T_{R_0,\phi,\chi}(f)]$. So it is not surprising that if each individual $T_{R_0,\phi,\chi}(f)$ is convex then $T(f)$ is convex as well, since integration preserves convexity. To illustrate this, Figure 4 shows the graph $T(f)$ when

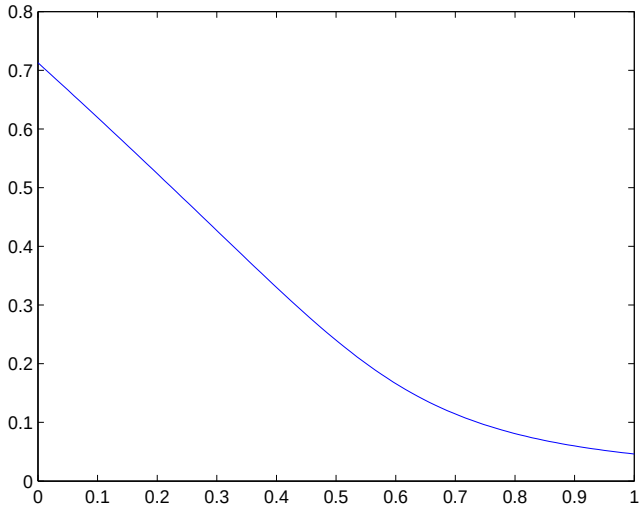


Figure 4: Graph of $T(f)$ by averaging over random $R_0,\ \phi,\ \chi$.

$R_0 \sim$ uniform$[1.5, 2],\ \phi \sim$ beta$(\alpha = 15, \beta = 5)$ (i.e. $\mu = 0.75,\ \sigma^2 = 0.0945$), and $\chi \sim$ beta$(\alpha = 0.96, \beta = 47.04)$ (i.e. $\mu = 0.02,\ \sigma^2 = 0.02^2$). For this purpose we sampled 99 observation of each random variable, corresponding to CDF values at $0.005, 0.015, 0.025, \cdots, 0.995$ then by taking the inverse integral obtained the corresponding values, and numerically averaged to find the resulting $T(f)$.