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for Competitive Optimization:  
Pitfalls to avoid and Conditions to Check**

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## Using Attraction Models for Competitive Optimization: Pitfalls to avoid and Conditions to Check

An important contribution to understanding competition has been the development of market share models based on the relative attractiveness of brands within a competitive set. These models have been used extensively as statistical tools to analyze and represent demand data. They have also been used for analytical studies of competitive optimization but this entails a number of challenges for the researcher. Preliminary guidance to use attraction models for optimization is available when the competing firms make decisions about price alone. However, to be applied to a general marketing context, market share models with multiple marketing instruments need to be specified. Existing research has not considered either the robustness or suitability of market share models for optimization when each firm makes a costly marketing mix decision in addition to setting price. We highlight several problems that arise when specific forms of the attraction model are used for equilibrium analysis. The most important problem relates to whether solutions identified through numerical simulation are unique. Our objective is to explain the origin of these problems and then propose a methodology that avoids them. We propose an approach based on the multinomial logit model that allows an attraction model to be used for competitive optimization. By placing a number of restrictions on the exogenous parameters, a unique solution is guaranteed.

**Key Words:** MCI Models, logit models, attraction models, optimization.

# 1 Introduction

One of the most important contributions to the competitive literature has been the development of market share models based on the relative attractiveness of brands within a competitive set. This has its origins in the work of Luce (1959) and McFadden (1980). These models have been used extensively as statistical tools to analyze demand data (Guadagni and Little 1983, Chintagunta, Jain and Vilcassim 1991 and Allenby and Rossi 1991). The insights provided by these models include improved understanding of the drivers of customer choice, the process of customer choice, and the expected impact of changes to the marketing mix by individual brand managers.

Complementing the popularity of market attraction models to analyze demand data, these models have also been applied to studies of competitive optimization. Attraction models have been developed to gain insights into competitive behavior of firms and markets with analytical focus on equilibrium solutions. For example, the structure of industries in terms of the number of firms to enter a market has been analyzed by Karnani (1983). Another example is the competitive responses by incumbents to new entries into the market (Gruca, Kumar and Sudharshan 1992). In addition, the more recent availability of rich data and of statistical software facilitates the estimation of such attraction models to assess the impact of marketing mix variables on firm's market share. Such information can be useful to managers in terms of the implications for helping them make better marketing mix decisions (Carpenter, Cooper, Hanssens and Midgley 1988 and Choi, Desarbo and Harker 1990). However, when firms make costly decisions about marketing instruments (like advertising, promotion or salesforce effort) in addition to price, the challenge of using market share attraction models for game theoretic analysis is considerable. Attraction models present appealing properties to represent competitive market reactions to marketing mix variables. However, we highlight several problems that arise when specific forms of the attraction model are used for equilibrium analysis. The most important problem relates to whether solutions identified through numerical simulation are unique.

First, we explain the origin of these problems. While some partial discussion of the issues is raised in the literature, no systematic treatment of the problems associated to different model specifications is provided. Second, we propose a methodology that guarantees a unique solution. The methodology is based on the multinomial logit model and it allows an attraction

model structure to be used for competitive optimization. Among the typical forms of the attraction model in the literature, we show that it is the only form for which the existence and uniqueness of solutions can be guaranteed.

The solution we provide is appropriate for a general model specification where competitors make decisions about both price and marketing expenditures (such as advertising, salesforce or distribution). Demonstrating the uniqueness of equilibrium solutions in such contexts is critical for analyzing competitive marketing strategies. Marketing strategy invariably entails decisions about a number of costly marketing mix variables as well as pricing.

## 2 Literature Review

Is the market attraction model solely a good tool to represent and understand demand data or can it also be useful to analyze how managers as well as consumers make decisions?

When comparing specification of market share models, Cooper and Nakanishi (1988) argue that attraction models do not simply provide predictive accuracy but are also meaningful representations of reality i.e., they have construct validity. Gatignon and Hanssens (1987) discuss a similar issue in a different context and argue for the usefulness of market share models based on both behavioral realism and robustness.

Interestingly, in spite of the popularity of market attraction models as representations of demand, their use does not receive overwhelming support in terms of fit or predictive accuracy. The performance of attraction models is not poor; however, simpler models appear to perform as well or better on this criterion (Naert and Weverbergh 1981, Brodie and Kluyver 1984, Ghosh, Neslin and Shoemaker 1984, Foekens, Leeflang and Wittink 1997). The arguments in their favor clearly center on the issues of behavioral realism and optimal robustness. The behavioral realism has been discussed in the context of logical consistency (Naert and Bultez 1973 and Bell, Keeney and Little 1975). In fact, only attraction models ensure that market shares are constrained between zero and one and that the sum of the market shares equals one. Another appealing property of the attraction models is the implicit S-shape of the response to marketing mix variables.<sup>1</sup> Cooper and Nakanishi (1988) develop arguments in support of the value of attraction models based on the behavior of the marketing mix elasticities. With attraction models, it is possible to derive cross-elasticities that by definition are subject to

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<sup>1</sup>The response to many marketing elements is observed to follow an S-shape empirically (Rao and Miller 1975 and Eastlack and Rao 1986).

structural restrictions that guarantee logical consistency. In particular, the cross elasticities are constrained such that the net increase in market share by a brand for a given reduction in price is equivalent to the loss in market share across all other brands.

Optimal marketing mix implications can be derived from these elasticities similar to the approach proposed by Dorfman and Steiner (1954). The implications of these models for equilibrium analysis have also been considered in the literature. Karnani (1985) uses such a model specification to explore strategic issues such as the existence of a firm-specific minimum market share (or firm size) as a function of the firm's cost structure, the relationship between profitability and market share, and the relationship between the marketing/sales ratio and market share.

Besanko, Gupta and Jain (1998) use a logit attraction model and an assumption of Nash competition amongst manufacturers and retailers to estimate the value provided by products in a category. Optimal behavior by each firm provides a basis to estimate the production cost and hence the value provided by each brand (the difference between the observed retail price and the estimated production cost). This study suggests that attraction models may be equally useful as a basis to predict market outcomes (prices) when market elasticities and production costs are exogenous.

Monahan (1987) uses an attraction model to derive optimal effort allocation across independent markets for two firms. This model highlights the potential of attraction models as a basis for analyzing firms that optimize profits in a competitive framework. However, the markets are assumed to have both fixed prices and fixed potential. In addition, all firms are assumed to have a fixed budget to allocate across the markets. In an extension, the author considers a situation where the size of the budget is exogenous. This is closer to the general context that interests us; however, in Monahan's model, prices are fixed and it is unclear whether the solution proposed by the author is unique.

An appealing feature of attraction models is the principle of distinctiveness that affects attraction models (Cooper and Nakanishi 1988). The market share does not depend on the level of a particular marketing mix variable but on the difference across options (brands). Unfortunately, this creates a challenge for the equilibrium analysis of a static market. In particular, if market demand is fixed, all brands in the market can increase their prices without changes in market share. This is not a problem when prices are fixed as in Monahan (1987). However, fixed prices are not typical in most marketing situations. Our objective is

to propose a parsimonious structure that allows for the setting of both prices and marketing effort with logically consistent predictions.

The literature on using attraction models for optimization varies along two dimensions as depicted in Table 1. The first dimension relates to the number of decisions that each firm makes. Most optimization studies using attraction models are based on firms that make decisions about price alone. The importance of being able to analyze markets where firms make decisions about both marketing and pricing is obvious. The second dimension relates to whether the stability and uniqueness of simulated solutions have been investigated. Guaranteeing that simulated solutions are unique is important but has received limited attention from researchers. From an analytical point of view however, when there are multiple solutions, the utility of a model as a normative or managerial tool is limited.

The *only* paper that addresses the existence and uniqueness of equilibria when the competitors make two decisions (prices and marketing effort) is Gruca, Kumar and Sudharshan (1992). However, this paper does not provide comprehensive guidance regarding the existence and uniqueness of equilibria. Gruca et al. argue in favour of the MCI model (for equilibrium analysis) when decisions about marketing effort and prices are made. However, the model of Gruca et al. does not include a no-purchase option. In general, this leads to a problem of multiple equilibria. The authors argue that a unique equilibrium exists if there are upper and lower bounds on all decision variables. Of course, any fixed point can be unique in a sufficiently constrained strategy space. Even with a no-purchase option added, a contribution of our analysis is to show that the MCI model requires unappealing parameter restrictions.

Basuroy and Nguyen (1998) raise a number of concerns regarding the analysis of Gruca et al. (1992) and propose a model where the sales of each product are based on the product of two terms. The first term is a coefficient that determines the overall market size as a function of the average attraction of all brands in the market. The second term is a Logit model where the utility of each product is a linear function of the product's price and a concave function of the product's marketing support. This model has a number of interesting properties but due to the complexity of the mathematical expressions it generates, it is not possible to confirm the uniqueness of equilibria that are identified.<sup>2</sup>

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<sup>2</sup>In order to demonstrate that a putative fixed point is stable for all players, the researcher must demonstrate that the matrix of second order partials is negative semi-definite in the region of the putative fixed point. The matrix must be negative semi-definite throughout the parameter space for the fixed point to be unique. These conditions (the Routh-Hurwitz conditions) cannot be checked in the model of Basuroy and Nguyen (1998) due to their complexity.

Table 1 is a convenient summary of the literature relevant to optimization work based on market attraction models.

**Table 1**  
**Literature Summary**

Decisions Optimized by Each Firm	Existence only is addressed	Existence and Uniqueness are addressed
Prices only or Marketing Spending only*	Monahan (1987) Choi et al (1990)	Rhim and Cooper (2004)
Prices and Marketing Spending	Karnani (1985) Carpenter et. al. (1988) Basuroy and Nguyen (1998)	Gruca, Kumar and Sudharshan (1992)

\*When firms optimize over marketing spending alone, prices are assumed to be fixed.

As follows from our earlier discussion, the key square of interest to us is the lower right and the only paper in this square is Gruca et al. (1992). Because Gruca et al. propose a model with unappealing parameter restrictions, there is a clear need for an approach that a) allows the modeller to consider markets where the firms make pricing and marketing decisions, b) leads to unique equilibrium solutions and c) works with parameter values that are close to those observed empirically.

In this vein and contrary to current wisdom regarding the optimal approach for conducting "attraction model" optimization, we show that *only* an MNL attraction model specification *guarantees* a unique equilibrium. We identify user-friendly conditions that guarantee both the existence and uniqueness of numerically identified fixed points. In fact, the conditions amount to straightforward limits for the equilibrium prices that are functions of the exogenous variables. To summarize, we believe analytical researchers need a methodology that incorporates both pricing and marketing decisions and delivers unique and stable solutions. The objective of the paper is to clearly outline that methodology. In the following section, we discuss problems that are common when using attraction models for optimization.

### 3 Pitfalls of Using Attraction Models in Equilibrium Analysis

Due to the complexity of attraction models, there are significant challenges regarding their use for game theoretic analysis. As noted earlier, a key advantage of a market share formulation is its face validity as a structure for representing the way that consumers make decisions. The disadvantage of such models in equilibrium analysis is that their tractability



and interpretability for optimization is more difficult than typical spatial models or models based on pre-specified parametric demand (such as linear demand). We first present the standard attraction model structure and then demonstrate the source of the problems when using the structure for optimization.

### 3.1 The Standard Structure for Optimizations using an Attraction Model

The basis for any model that involves a market share structure and optimization is the firm level objective function. Here we assume each firm makes a decision about pricing and a level of marketing effort  $x$ . For the  $i$ th firm<sup>3</sup>, the objective function can be written as:

$$\pi_i^o = Q \cdot \text{MS}_i \times (p_i - c_i) - F(x_i) \quad (1)$$

The term  $Q$  is the market size in units,  $\text{MS}_i$  is the unit market share of firm  $i$ ,  $p_i$  and  $c_i$  are the selling price of firm  $i$  and the unit cost for each unit sold respectively and  $F(x_i)$  is a function representing the cost of marketing effort at a level  $x_i$  for firm  $i$ . When an attraction model is used to represent market share, firm  $i$ 's market share is given by:

$$\text{MS}_i = \frac{A_i}{A_{total}} \quad (2)$$

where  $A_i$  is the attraction of firm  $i$  and  $A_{total} = \sum A_i$  is the attraction of all options open to the customer (when a no-purchase option is one of the choices,  $Q$  in equation 1 is the market potential and equation 2 represents product  $i$ 's share of the *potential market*). In the MCI model, the attraction for each firm is given by  $A_i = \alpha_i p_i^{\beta_i} x_i^{\gamma_i}$  where  $\beta_i$  and  $\gamma_i$  are parameters that represent the responsiveness of firm  $i$ 's attraction to its price and marketing levels. In the Logit model, the attraction of each firm is given by  $A_i = \exp(\alpha_i + \beta_i p_i + \gamma_i x_i)$  where the exponent represents the utility derived from the purchase of the product of firm  $i$ . Similar to the MCI model,  $\beta_i$  and  $\gamma_i$  represent the responsiveness of firm  $i$ 's attraction to the price and advertising levels of firm  $i$ .

In order to avoid unnecessary complexity but without loss of generality, we use a normalized form of equation (1) with  $\pi_i = \frac{\pi_i^o}{Q}$  and  $f(x_i) = \frac{F(x_i)}{Q}$ :

$$\pi_i = \text{MS}_i \times (p_i - c_i) - f(x_i) \quad (3)$$

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<sup>3</sup>We refer to the index  $i$  as representing a firm, but it can reflect instead brands that are managed competitively. Most marketing applications of market attraction models are estimated at the brand level (e.g., Carpenter et al. 1988 or Cooper and Nakanishi 1988).

We now consider the current wisdom regarding the use of market share attractiveness models for optimization. As noted earlier, the recommended approach of Gruca et al. (1992) is to use an MCI model structure. This follows from the conclusion of Gruca and Sudharshan (1991) that the Multinomial Logit structure is unsuitable for optimization. We question this conclusion. Gruca and Sudharshan (1991) conclude that the MNL model is inappropriate precisely because marketing costs are linear in the marketing effort chosen. We show that an assumption of convex costs makes the MNL model completely suitable for optimization analysis.<sup>4</sup>

### 3.2 Problems with MCI

The approach of Gruca et al. (1992), which follows tradition, is to specify the MCI model without a no-purchase option. Without such an option, it is clear that increases (or decreases) in the same decision variables of each competitor can be found such that the market shares are not affected. In the case of pricing, this is troubling since the market does not impose a cost on firms that jointly increase prices. Moreover, with an MCI formulation, this can lead to a problem of multiple equilibria.

**Proposition 1** *Without a no-purchase option, the MCI market share formulation may result in an infinite number of simultaneous solutions satisfying first-order conditions.*<sup>5</sup>

The approach to incorporate a no-purchase option into the MCI model is not obvious because "not purchasing" implies that the decision-maker does not pay ( $p = 0$ ). The MCI model is not derived from an underlying utility framework and due to its multiplicative nature, any product with a price of zero has an attraction of zero (so none of the variables affects market shares). As suggested by Bell, Keeney and Little (1975), an outside option with fixed attractiveness can be used to represent a no-purchase option. This is often done by fixing the attraction of the no-purchase option to one.

Even with the inclusion of a no-purchase option, the first-order conditions for a MCI model involving symmetric competitors require a key restriction (for an internal equilibrium) that is inherently unappealing. This restriction is summarized in Proposition 2.

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<sup>4</sup>There is also a strong empirical argument against the assumption of linear marketing costs. For example, the cost to obtain an increase in distribution coverage is not proportional to the size of the increase.

<sup>5</sup>The proofs of this and all following propositions are provided in the appendix.

**Proposition 2** *When an MCI model with  $N$  competing brands, a no-purchase option and zero marginal costs is used;  $\beta$  must be restricted to the interval  $\left(-\frac{N}{-1+N}, -1\right)$  for the existence of a solution with positive real prices.*

As the number of brands becomes large, this interval becomes arbitrarily small. Even with 4 brands, Proposition 2 implies that  $\beta$  is restricted to  $\left(-\frac{4}{3}, -1\right)$ . To illustrate, the Besanko, Gupta and Jain (1998) study is based on the analysis of a category with 4 brands and the measured price elasticities imply values of  $\beta$  that lie outside of this range.<sup>6</sup> Quite simply, the MCI model structure imposes unrealistic constraints upon the price elasticity parameters. For optimization purposes, note that using a price sensitivity parameter outside the interval described in Proposition 2, implies the non-existence of internal fixed points.

In summary, the MCI model with a no-purchase option can be used as the basis for optimization analysis. However, the researcher needs to tolerate price elasticities that are inappropriately inelastic. In addition, the researcher cannot confirm that the fixed points found using search procedures like simplex, Newton-Raphson or quasi-Newton are unique. In the next section, we summarize the necessary requirements for market share attractiveness models to be used efficiently for numerical analysis.

## 4 Requirements for a Robust Model

Market share models have a number of attractive properties so the desire to use them in the context of optimization analysis is self-evident. However, in order to use these models for numerical analysis, at least two properties are necessary. The first is that overall sales in the market be responsive to the average levels of observed decision variables. In the case of market share models with two decision variables (price and advertising), it is necessary for overall sales to be affected by both average levels of pricing and advertising in the market. The second requirement is that the system must have one equilibrium solution (in a game theoretic sense) for any set of exogenous parameters that lie within a reasonable range. Said differently, it is important for the system to generate a unique solution when the implied price and advertising elasticities employed in a simulation are close to empirically observed price and advertising elasticities. We now explore each of these requirements in detail.

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<sup>6</sup>The Besanko, Gupta and Jain (1998) study is based on a logit model but the estimated price elasticities for the brands (Table 2, p.1541) provide a basis for demonstrating a problem were the data to be the basis for an analysis of competitive optimization using an MCI model.

## 4.1 Overall Market is a Function of Decision Parameters

Models without a no-purchase option force the customer to choose a brand. When the sales of a firm are completely determined by equation 2 (perhaps multiplied by a number related to market potential), the overall size of the market is fixed and independent of the price level. In a simple symmetric case (the same cost structure of competitors, same elasticities and otherwise similar products), it follows that any pair of equal prices might satisfy the optimization condition leaving competitors with equal market shares. As noted in Choi, Desarbo and Harker (1990), the price elasticity of total demand equals zero when customers are forced to choose one of the brands. Thus, unless total market demand decreases as a function of average pricing, the profit functions will not satisfy the convexity conditions necessary for optimization.<sup>7</sup>

Following Bell, Keeney and Little (1975), Choi, Desarbo and Harker (1990) solve this problem by specifying a Logit model with a no-purchase option. Recently, even empirical work to analyze demand using a logit formulation, incorporates a no-purchase option (Ofek and Srinivasan 2002). The use of a no-purchase option with a pre-defined fixed level of attractiveness means that the market is responsive to the average levels of decision variables observed in the market. In general, this reduces the problem of multiple solutions. Another solution to this problem is to add a multiplicative term that deflates the market when average prices increase (Basuroy and Nguyen 1998). However, the analytical expressions generated with such an approach are too complex to allow confirmation of the stability or uniqueness of a putative fixed point.

As discussed in the introduction, an inherent strength of market share models is the implied S-shape of the response curve to decisions taken by each competitor. When the response curves of competitors are S-shaped, it is possible for the reaction functions of competitors to cross more than once and a problem of multiple equilibrium may still be present. A key challenge for the researcher is to ensure the uniqueness of a putative equilibrium even when a no-purchase option is included.

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<sup>7</sup>This problem is also evident in representative consumer models with differentiated products. For example with a linear demand system for differentiated product, it is important to impose restrictions on the parameters to ensure convexity (Vives 1999, Godes, Ofek and Sarvary 2004).

## 4.2 The Problem of Uniqueness

The first-order conditions generated by an optimization utilizing an attraction model based on either the MCI model or the Logit model generally do not allow the derivation of explicit expressions for equilibrium prices or marketing effort. As a result, the equilibrium decisions for a set of exogenous parameters such as firm costs (production and marketing) and demand characteristics (customer sensitivities to price and marketing and/or the relative attraction of competing brands) are identified numerically. Regardless of the method that is used to find fixed points for the system of non-linear equations (simplex search methods, Newton-Raphson approximations or quasi-Newton methods), it is critical that the fixed point generated for a set of exogenous parameters be unique. In particular, the equilibrium decisions must not depend on the starting values that are employed in the search routine.<sup>8</sup>

Choi, Desarbo and Harker (1990) demonstrate the existence of Nash equilibria in an optimization problem based on a market share attraction model. However, they do not derive sufficient conditions to claim that an identified equilibrium is unique. Without a guarantee of uniqueness for the fixed points found through simulation, it is dangerous to use a system for analytical work. The primary objective of analytical work is to predict what *will* happen given a set of reasonable assumptions. If there are multiple outcomes and the researcher only reports one of them, it is far from being reliable. Rhim and Cooper (2004) demonstrate an approach to guarantee uniqueness using an optimization model containing an attraction formula; however, the decision makers in the model are restricted to pricing decisions (the model does not incorporate marketing decisions). Here, our objective is to identify a model where both prices and marketing effort are decision variables set by each competitor. This case is essential in order to allow researchers to investigate a richer set of marketing issues using market share attraction models.

Friedman (1986) (pp. 63-107) outlines the necessary conditions for the existence and uniqueness of fixed points in mappings where the decision makers are constrained to making best responses. Not surprisingly, the conditions necessary to show the *existence* of a fixed point are different and less restrictive than those used to demonstrate *uniqueness*. The necessary conditions for proving the uniqueness of a putative equilibrium found through search amount to first demonstrating that second order conditions are satisfied. Second, the researcher must demonstrate that the matrix of second order partials is negative semi-

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<sup>8</sup>See Gragg and Stewart (1976) for a typical approach to solving non-linear equations.

definite for all feasible values in the strategy space (known as the Routh Hurwitz conditions).<sup>9</sup> Alternatively, an indirect method proposed by Friedman (1986) utilizes an approach based on the Banach fixed point theorem and functions that are contraction mappings (Giles 1987). This is the approach we take in order to demonstrate uniqueness.

## 5 A Robust Structure: the MNL Model with an No-Purchase Option and Quadratic Marketing Costs

For optimization work using an attraction model structure, we propose a logit model with a no-purchase option and quadratic marketing costs. As explained earlier, the advantage of using a no-purchase option is that the market size is not fixed (all marketing decisions influence the size of the market). With the Logit model, the attractiveness of not purchasing is represented by  $A_0$  and  $A_0$  is normalized to 1. This has intuitive appeal because having a no-purchase option with an attractiveness of 1, follows from a random utility framework in which any purchased product must provide positive surplus (McFadden 1980). In economic terms, it is analogous to imposing an individual rationality constraint on consumers, i.e., each consumer increases her surplus from purchasing or else she prefers not to buy. For almost all categories this is reasonable. In contrast to the MCI model or models that allow market size to change through a multiplicative inflator, the Logit model is not an *ad hoc* representation of market behavior. The Logit model with a no-purchase option has a straightforward interpretation based on microeconomic utility theory.

With the Logit model, the no-purchase option is preferred to product  $i$  when  $A_0 > \exp(\alpha + \beta p_i)$ . In other words, not purchasing is preferred when  $p_i > -\frac{\alpha}{\beta}$  ( $-\frac{\alpha}{\beta}$  is thus a *de facto* reservation price for product  $i$ ). This reservation price places an entry condition for participating firms: a firm contemplating entry needs a variable cost that is less than the reservation price.

We now demonstrate that the proposed system has a unique Nash equilibrium. In its most general form, the model entails  $N$  firms simultaneously optimizing the following objective function:

$$\pi_i = MS_i(p_i - c_i) - x_i^2 \quad (4)$$

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<sup>9</sup>See Silberberg (1990), pp 656-662. The Routh Hurwitz conditions also ensure that a putative equilibrium is asymptotically stable.

The share of the potential market for the  $i$ th firm is given by the following expression:

$$\text{MS}_i = \frac{e^{(\alpha + \beta p_i + \gamma x_i)}}{1 + \sum_{j=1}^N e^{(\alpha + \beta p_j + \gamma x_j)}} \quad (5)$$

The revenue for each firm is given by  $R_i = \text{MS}_i p_i$  and the cost function is assumed to be  $C_i = \text{MS}_i c_i + x_i^2$ , where  $x_i^2$  is the dollar investment in marketing activities. For example  $x$  in the market share equation (5) could represent advertising gross rating points (GRP's) that would cost  $x^2$  to obtain. Another example would be for  $x$  to represent distribution coverage that can be achieved with an investment of  $x^2$  in salesforce spending. This implies that the cost to obtain a particular level of a marketing variable that affects a brand's attractiveness increases quadratically with the level desired.

Computationally, a linear marketing cost function would be simpler and it has been used in previous research (Carpenter, Cooper, Hanssens and Midgley 1988). However, when the utility of a brand is a linear function of the marketing decision, linear costs can lead to corner solutions of  $x_i = 0$ . In this case, the model reduces to a game of price competition so the researcher gains little by including marketing effort in the model.<sup>10</sup> To guarantee an internal equilibrium, the derivative of the payoff function with respect to marketing expenditure should be positive near zero and negative as marketing expenditures become large. As long as the derivative of the payoff function becomes negative for a  $x_i^* \geq 0$ , there will be at least one internal maximum. As a result, non-linear power functions (power  $>1$ ) can also be used (see the Appendix **A.3**).

The choice of the quadratic form at this stage may appear rather arbitrary; intuition suggests that any increasing function of marketing effort might be satisfactory. As noted in Soberman (2003), any concave function can be approximated with a second-order (i.e. quadratic) Taylor's series expansion. Because the objective of this paper is to outline an analytical tool that can be used easily, the quadratic cost function is the most parsimonious specification for which we show that a unique equilibrium can be guaranteed.

Following the reasoning of Choi, Desarbo and Harker (1990), we demonstrate both nec-

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<sup>10</sup>Unless the impact of a decision,  $x$ , on a brand's attraction is concave, linear marketing costs can generate profit functions that are not convex in marketing effort. For example, linear marketing costs coupled with an impact of marketing on each brand's attraction equal to  $\sqrt{x}$  is the mirror of the structure we propose. The danger of linear marketing costs coupled with a linear impact of marketing on the attraction of each brand is underlined by Gruca and Sudharshan (1991). They find that the profit maximising advertising level is to advertise as much as possible when a brand's market share exceeds 50%. A key contribution of this paper is to propose a cost function for which the convexity of each firm's profit function is guaranteed.

essary (Proposition 3) and sufficient (Proposition 4) conditions for the existence of a Nash equilibrium in an attraction-model optimization with both price and marketing effort decisions. These conditions imply restrictions on the model parameters to ensure that the simulated system of equations has a fixed point. Proposition 3 summarizes these conditions.

**Proposition 3** *If  $\pi_i$  is quasiconcave with respect to  $p_i$  for  $i = 1, 2$ , then with a logit model, a Nash equilibrium exists in the interval  $c_i + \frac{1}{-\beta_i} < p_i < \infty$ .*

On the one hand, Proposition 3 provides a condition that is necessary for a researcher to be certain that a fixed point to a system of equations exists. On the other hand, the proposition is not easy to use because confirming that the profit functions generated by attraction-model based optimizations are quasi-concave is difficult if not impossible. Accordingly, we identify *sufficient* conditions to ensure the existence of a fixed point. These conditions are easy to confirm numerically. Specifically, we impose a restriction of strict concavity on the system (which naturally satisfies the conditions of quasi-concavity). This follows the general approach of Friedman (1986) to confirm the existence of fixed points within a system of equations.

**Proposition 4** *For each firm, sufficient conditions for the existence of a Nash equilibrium are  $p_i - c_i < \frac{1}{-\beta} \left( 2 - \frac{1}{-8\beta} \gamma^2 \right)$ ,  $i = 1, 2$ .*

In simple terms, the implication of Proposition 4 is that for a given market with exogenously measured (or estimated) price and advertising elasticities, there exists a range of prices for which the profit functions of all firms are concave and hence there exists at least one fixed point. Simple rearrangement allows us to also identify a maximum price  $p_i^{Max}$  such that for any  $p_i \leq p_i^{Max}$  a Nash equilibrium exists. The maximum price for each firm is function of its marginal costs as shown in equation 6.

$$p_i^{Max} = c_i + \frac{1}{-\beta} \left( 2 - \frac{1}{-8\beta} \gamma^2 \right) \quad (6)$$

Proposition (4) also implies that

$$\gamma < 2\sqrt{-2\beta} \quad (7)$$

(otherwise the interval for allowable margins will be empty). In other words, for a given  $\beta$ , the higher is the advertising sensitivity  $\gamma$ , the closer  $p_i^{Max}$  gets to the marginal cost and the more difficult it becomes for the researcher to ensure that a fixed point exists for the system.



We now address the issue of uniqueness for optimizations based on the logit formulation. In order to derive necessary conditions for uniqueness, we employ the contraction mapping formulation (Friedman 1986). Uniqueness is guaranteed by the contraction mapping theorem when the strategy space of each player is compact and convex and the payoff vector of each player is a) defined, continuous and bounded for all strategy combinations and b) concave with respect to that player's strategy for all strategy combinations (given the strategies of all other players). These conditions lead to specific restrictions in the case of a competitive logit-based optimization with two decision variables (price and marketing effort). We summarize these restrictions in Proposition 5.

**Proposition 5** *If*

$$(-\beta + \gamma) \frac{-2\beta (p_i - c_i) - 2 + 0.25\gamma}{2\beta^2 (p_i - c_i) - 0.25\gamma^2} < K_i$$

where

$$K_i = 1 + \frac{1}{\exp\left(\alpha + \beta c_j + 0.25 \frac{\gamma^2}{-\beta}\right)}$$

( $i, j = 1, 2 \quad j \neq i$ ), there exists a unique pure-strategy Nash equilibrium for Firms 1 and 2 in terms of price and marketing effort.

The formal proof of Proposition 5 is provided in Appendix D. Interestingly, when  $\gamma = 0$  (implying that the advertising has no effect on the attraction of brands), the condition of Proposition 5 is less restrictive than the condition on the firm's margin ( $p_i - c_i$ ) implied by the existence condition of Proposition 4. However, when  $\gamma$  is larger, the constraint of Proposition 5 is more restrictive than the existence condition. In other words, when advertising has an effect on the attractiveness of brands, checking the condition of Proposition 5 is essential to ensure that a fixed point generated through numerical search is unique.

To evaluate the validity of the model (i.e., the degree to which the model predicts outcomes consistent with the observed behavior of firms), one possibility is to examine demand data and observed prices from published studies and check if the prices satisfy the conditions we have derived. However, the ability to do this is limited by the information that is typically found in empirical studies of demand. Such studies rarely report the per-unit costs for products, advertising costs or advertising elasticity. This makes it difficult to check the condition outlined in Proposition 5.

In order to demonstrate the approach one might take however, we are going to present an example based on the demand data used by Allenby and Rossi (1991). The objective is to

see if the observed prices lie within the range identified by the model for which solutions exist and are unique (based on Propositions 3, 4 and 5). One should be cautious when applying this procedure to a published study for the following reasons.

First, the model requires precise marginal cost information for each product in the data set. Most empirical studies of demand do not provide such estimates. Typically, when marginal cost is needed for analysis (as in empirical IO studies), costs are estimated based on industry norms. But this is only an approximation of the required information.

Second, most estimates of market share elasticities are based on models where the customer chooses one of the products (a no-purchase option is not included). As a result, the reported market share elasticities are likely to be biased downwards compared to the elasticities required for the optimization model. This can lead to prices that appear to fall outside the interval necessary to guarantee uniqueness.

Third, the optimization model is based on independent competitors who each set price optimally for their products. In contrast, most empirical analyses of demand using attraction models is based on scanner data where prices are set by a retailer who acts as a common agent for competing manufacturers. This implies that the observed prices are subject to double marginalization and are set cooperatively by the retailer. Any of these reasons may lead to observed prices that appear inconsistent with the conditions we propose that guarantee both the existence and uniqueness of an equilibrium outcome. Nevertheless, it is useful to illustrate the approach we propose.

A first assumption we make is that the standard margin for grocery items is 40% (the difference between the observed retail price and marginal cost of the product). This is an estimate based on standard margins for grocery products reported in a recent food industry publication (Ontario Government 2005). For our example, we ignore the decentralized nature of the grocery channel and the fact that the margin ( $p - c$ ) is split between the manufacturer and the retailer.<sup>11</sup>

Using the standard formulae for the logit model, we compute the implied  $\beta$  for each brand assuming that the logit structure is reasonable.<sup>12</sup> For example in Allenby and Rossi (1991), the reported market share price elasticity of the largest brand is  $-2.094$ , which implies

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<sup>11</sup>A paper by Lal and Narasimhan (1996) suggests that this assumption is reasonable. In general, the overall margin on grocery products is stable and the division of margin between the manufacturer and the retailer is function of their relative power.

<sup>12</sup>Our model is based on the assumption that competing brands set prices to maximise profit given the pricing decisions of the competitors.

$\beta = -6.66$ . The table below summarizes the bounds for the major brands in the Allenby and Rossi study.

**Table 2: Implied Price Ranges for the leading brands in Allenby and Rossi study**

BRAND	Market share	Estimated price coefficient (average $\beta$ )	Average price of the brand	Lowest price based on S,G&S*	Highest price based on S,G&S*
Parkay Stick	39.5%	-6.66	0.52	0.37	0.74
Blue Bonnet Stick	15.6%	-6.40	0.54	0.38	0.77
House Brand Stick	13.3%	-7.86	0.44	0.31	0.62
Shed Spread Tub	7.1%	-4.16	0.83	0.59	1.18
Generic Stick	7.0%	-9.95	0.34	0.25	0.49

\*S,G & S refers the restrictions we develop in Propositions 3 and 4.

The table shows that the prices reported in the study fit within the bounds needed to guarantee both existence and uniqueness of equilibrium prices. However, this does not mean that the observed prices represent equilibrium behavior on the part of the brands. It simply implies that the restrictions required for a competitive system (based on an attraction model) to have a unique equilibrium imply a range of prices that is reasonable given what is observed empirically.<sup>13</sup>

Our analysis of the Allenby and Rossi study suggests that the model *can* predict pricing behavior close to what might be observed empirically.

A second condition is automatically implied by the logit-based optimization we propose. In order for a product to be more attractive than not purchasing (an arguably reasonable participation constraint for a product),  $A_i > 1$ . Straightforward substitution allows us to derive a maximum price (as a function of the product's advertising) such that  $A_i > 1$ .

$$p_i^{Logit} = -\frac{\alpha}{\beta} - \frac{\gamma}{\beta}x_i \quad (8)$$

Not surprisingly, marketing expense increases the reservation price. It should be noted that the reservation price expressed in equation 8 is an upper limit for the price that a firm can set for its product. Above this limit the utility of not purchasing exceeds the utility offered by the brand. This condition can also be used to derive a maximum marginal cost for a brand

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<sup>13</sup>In fact, the prices reported in Allenby and Rossy fit within the lower and higher bounds of the model for margins between 30% and 50%.

given a firm's advertising:  $c_i \leq \frac{2-\alpha}{\beta} + \frac{\gamma^2}{8\beta^2} - \frac{1}{\beta}\gamma x_i$ . This condition implies that a firm needs to have competitive production costs in order to operate. Similar to the condition for the reservation price (equation 8), this condition increases in the product's advertising  $x_i$ .

## 6 Conclusion

There is compelling evidence that attraction models are effective for explaining the choices that consumers make as a function of marketing activities, prices and product characteristics. We also know that analytical models based on simplified characterizations of demand (such as linear demand and spatial markets) are useful for explaining economic phenomena and making normative predictions about firm behavior. There is however a need to meet half way: a need to allow for a complex representation of demand in the context of analytical modelling. Attraction models have a number of properties that make them attractive for the analysis of competitive marketing strategies. However, their characteristics also make them susceptible to problems in a context of simultaneous optimization. Our analysis highlights the precise nature of these problems.

The problem of multiple equilibria is especially serious. Conclusions drawn from work that only considers one out of numerous possible solutions are dangerously misleading. Few if any published papers address the issue of uniqueness in numerical simulation. In fact, many existing papers avoid consideration of this issue altogether by restricting the nature of the problem studied (by fixing price for example).

Our objective is to address this issue head on. It is certainly not to refute previous work. Instead, it is to identify a path for the development of attraction-models in optimization problems. We demonstrate that a logit model and a series of simple restrictions allows the researcher to find unique numerical solutions to a competitive system.

First, the specification needs a no-purchase option in the choice set. Second, demand enhancing marketing effort (such as advertising) can be included in the model as long as the cost of this effort is convex. The quadratic cost function is a straightforward approach to meet the condition of convexity and at the same time guarantee non-zero levels of marketing effort. Finally, we identify critical conditions (Propositions 3, 4 and 5) that the simulated prices need to satisfy. When these conditions are satisfied, then a simulated equilibrium satisfies the second order conditions and is unique. A numerical solution found outside the range derived in this paper may be an equilibrium; however, the usefulness of such analysis

Figure 1: Competitive Optimization Key Steps

<b>Key Steps</b>	
1	Define Problem (explanatory or normative)
2	Choose Price and Advertising parameters as a function of the assumed unit market share elasticities: <ul style="list-style-type: none"> <li>• <math>\beta = \frac{\frac{\epsilon_p}{p}}{1-MS}</math> where <math>\epsilon_p</math> is the price elasticity of market share and p is the average price of the brand under consideration with its market share MS.</li> <li>• <math>\gamma = \frac{\frac{\epsilon_x}{X}}{1-MS}</math> where <math>\epsilon_x</math> is the advertising elasticity of market share and X is the level of advertising. X can be thought of as the number of exposures in the target market or the number of GRP's targeted in the media plan. In general, the cost to obtain exposures in the target market or GRP's is a convex function of the level. The cost is assumed to be <math>X^2</math>.</li> </ul>
3	Choose intercepts - depending on desire to examine symmetric / asymmetric competition.
4	Derive first-order conditions for each firm
5	Carry out numerical search
6	Check the existence and uniqueness conditions

is limited since there are often multiple solutions outside the allowable range.

To ensure the existence of fixed points, we also identify implicit conditions for the price and marketing sensitivity parameters. Because the price and marketing parameters are exogenous and need to be chosen prior to conducting numerical analysis, researchers will find these conditions helpful.

Figure 1 summarizes the steps that a researcher can follow to construct a consistent system for competitive optimization. The first step is to define the problem (for example, is a study attempting to a) explain phenomena observed in a category or b) build a model to provide guidance for decision-makers). The second step is to set the exogenous parameters. This is done straightforwardly as shown in Figure 1 using reasonable estimates of price and advertising elasticities as inputs. The third step is to set the intercepts for each brand (this allows the researcher to analyze competition between firms that are similar in size or significantly different). The fourth step involves deriving the first order conditions (this can be done using a standard symbolic math software package). The fifth step entails using a numerical method to find a solution to the first order conditions (suitable methods include simplex, Newton-Raphson or quasi-Newton searches). Finally, the existence and uniqueness

conditions described in Propositions 4 and 5 need to be checked. Our hope is that these insights and clarifications will enable a richer analysis of competitive marketing strategies using a well-known market share model.

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# Appendix

## A Problems In Attraction Models

### A.1 Proof of Proposition 1

In a simple model with price as the only decision variable of a single-product firm competing among  $N$  firms (products), the attraction of a product (or firm)  $i$  is  $p_i^{\beta_i}$  where  $\beta_i$  is the price elasticity. This implies that market share is given by:

$$\text{MS}_i = \frac{p_i^{\beta_i}}{\sum_{j=1}^N p_j^{\beta_j}} \quad (\text{i})$$

The payoff function is  $\pi_i = \text{MS}_i (p_i - c_i)$ , hence the system of  $N$  equations of first-order conditions:

$$1 + \beta_i \frac{p_i - c_i}{p_i} (1 - \text{MS}_i) = 0 \quad (\text{ii})$$

Substitution for market share obtains the following system of equations:

$$-\frac{1}{-\beta} p_i^{\beta_i} + \left(1 - \frac{c_i}{p_i} - \frac{1}{-\beta}\right) \sum_{j \neq i}^N p_j^{\beta_j} = 0 \quad (\text{iii})$$

When marginal costs are zero, (iii) is a homogeneous system of linear<sup>1</sup> equations. The homogeneity of the system implies that when its determinant does not equal zero it has only the trivial solution. Alternatively there is an infinite number of solutions along with the trivial solution when the determinant is zero<sup>2</sup>. In case of non-zero marginal costs if competitors chose sufficiently high prices, such that  $\frac{c_i}{p_i} \rightarrow 0$ , then the system of first-order conditions would become a homogeneous one.

### A.2 Proof of Proposition 2

Assume that market share is given by:

$$\text{MS}_i = \frac{\alpha p_i^{\beta}}{1 + \sum_{j=1}^N \alpha p_j^{\beta}} \quad (\text{iv})$$

It follows from the first order conditions (ii) that when costs are zero<sup>3</sup>:

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<sup>1</sup>Linear relative to  $p_i^{\beta_i}$ .

<sup>2</sup>If  $p_i$  is a solution, then  $k^{-\beta_i} p_i$  is also a solution ( $k$  being any real number).

<sup>3</sup>In case of logit model it follows from first-order conditions that  $\text{MS}_i = 1 + \frac{1}{\beta p_i}$ , so, as opposed to MCI model, it cannot be concluded directly that  $p_i = p_j$ . Rather  $p_i = p_j$  would follow from the general symmetry of the problem.

$$MS_i = 1 + \frac{1}{\beta} \text{ and } p_i = p_j \quad (\text{v})$$

Solving for the price of  $i$ th brand, leads to the following expression for the equilibrium price:

$$p_i^{Equilibrium} = \left( \frac{1 + \frac{1}{\beta}}{\alpha \left( 1 - N \left( 1 + \frac{1}{\beta} \right) \right)} \right)^{\frac{1}{\beta}} \quad (\text{vi})$$

Feasibility implies that:  $\left( \frac{1 + \frac{1}{\beta}}{\alpha \left( 1 - N \left( 1 + \frac{1}{\beta} \right) \right)} \right) > 0$ . Because  $\beta < -1$  in most competitive markets, the numerator is positive. Therefore  $\alpha \left( 1 - N \left( 1 + \frac{1}{\beta} \right) \right) > 0$  which implies that  $\beta > -\frac{N}{-1+N}$ . Alternatively, when  $\beta \in [-1, 0)$ , the numerator is negative<sup>4</sup>. For the denominator to be negative  $\beta < -\frac{N}{-1+N} < -1$ , which is a contradiction.

### A.3 Logit Model - Linear vs. Quadratic Marketing Expense Function

Consider a model with two competing brands with price and marketing expenses as decision variables:

$$\pi_i = \frac{\exp(\alpha + \beta p_i + \gamma x_i)}{2} (p_i - c_i) - x_i \quad (\text{vii})$$

$$1 + \sum_{j=1} \exp(\alpha + \beta p_j + \gamma x_j)$$

The first derivative by marketing expense equals:

$$\frac{\partial \pi_i}{\partial x_i} = \gamma (p_i - c_i) MS_i (1 - MS_i) - 1 \quad (\text{viii})$$

Condition  $\frac{\partial \pi_i}{\partial x_i} < 0$  is equivalent to:

$$\gamma (p_i - c_i) MS_i (1 - MS_i) < 1 \quad (\text{ix})$$

Note that  $MS_i (1 - MS_i) \leq 0.25$  (the maximum of  $MS_i (1 - MS_i)$  occurs at  $MS_i = 0.5$ ).

As a result, when  $p_i - c_i < \frac{4}{\gamma}$ ,  $\frac{\partial \pi_i}{\partial x_i}$  is strictly negative. But for an internal maximum to exist, it is necessary that  $\frac{\partial \pi_i}{\partial x_i}$  be positive near  $x_i = 0$ . This means that in this model first-order conditions are not satisfied for small margins, i.e. for competitive industries.

However, if advertising costs are non-linear:

$$\pi_i = MS_i (p_i - c_i) - x_i^b \quad (\text{x})$$

with  $b > 1$ , the condition which corresponds to condition (ix) is:

$$\gamma (p_i - c_i) MS_i (1 - MS_i) - 2x_i^{b-1} > 0 \quad (\text{xi})$$

So when  $x_i$  is small enough the above statement is true (note that  $MS_i (x_i = 0) \geq 0$ ). However, since  $MS_i (1 - MS_i) \leq 0.25$ , the expression becomes negative with increasing  $x_i$ . This guarantees a internal advertising maximum for any margin.

<sup>4</sup>Note that  $p_i^{Equilibrium} = 0$  when  $\beta = -1$ .

## B Proof of Proposition 3

a)  $p_i < \infty$

$$\lim_{p_i \rightarrow \infty} \pi_i = \lim_{p_i \rightarrow \infty} (\text{MS}_i \times (p_i - c_i) - x_i^2) = \lim_{p_i \rightarrow \infty} (\text{MS}_i \times (p_i - c_i)) - x_i^2 \quad (\text{xii})$$

We can assume that non-variable costs, such as advertising costs are not (or should not be) taken into account when setting the prices. Hence we consider only the variable costs.

$$\lim_{p_i \rightarrow \infty} [\text{MS}_i \times (p_i - c_i)] = \lim_{p_i \rightarrow \infty} \frac{p_i - c_i}{\text{MS}_i^{-1}} \quad (\text{xiii})$$

According to l'Hôpital's rule,

$$\lim_{p_i \rightarrow \infty} \frac{p_i - c_i}{\text{MS}_i^{-1}} = \lim_{p_i \rightarrow \infty} \frac{1}{\frac{\partial(\text{MS}_i^{-1})}{\partial p_i}} \quad (\text{xiv})$$

$$\Rightarrow \frac{\partial(\text{MS}_i^{-1})}{\partial p_i} = \beta \left( \frac{1}{\text{MS}_i} - 1 \right) \quad (\text{xv})$$

In other words,  $\lim_{p_i \rightarrow \infty} \pi_i$  becomes  $\lim_{p_i \rightarrow \infty} \left( \beta \left( \frac{1}{\text{MS}_i} - 1 \right) \right)^{-1}$ . Since  $\lim_{p_i \rightarrow \infty} \text{MS}_i = 0$ , then  $\lim_{p_i \rightarrow \infty} \pi_i = 0$ . Given that the firms do not have an incentive to earn zero profits, they will not charge infinitely high prices. In other words,  $p_i < \infty$ . Note that the presence of  $x_i^2$  in the proof would reinforce the proposition.

b)  $c_i < p_i$

For any optimal price, i.e. price which satisfies the first order condition  $\frac{\partial \pi_i}{\partial p_i} = 0$

$$\frac{\partial \pi_i}{\partial p_i} = \text{MS}_i (1 + \beta (p_i - c_i) (1 - \text{MS}_i)) = 0 \quad (\text{xvi})$$

it follows that (see Besanko, Gupta and Jain 1998)

$$-\beta (p_i - c_i) = \frac{1}{(1 - \text{MS}_i)} > 1 \quad (\text{xvii})$$

and

$$p_i - c_i > \frac{1}{-\beta} > 0 \quad (\text{xviii})$$

hence  $p_i > c_i$ . **Q.E.D.**

## C Proof of Proposition 4

When  $\pi_i = f(p_i, x_i)$ , concavity condition (sufficient condition for the critical point(s) to be a maximum) is the negative definiteness of the Hessian:

$$\frac{\partial^2 \pi_i}{\partial p_i^2} < 0, \frac{\partial^2 \pi_i}{\partial x_i^2} < 0 \text{ and } \det(H_i) = \left( \frac{\partial^2 \pi_i}{\partial p_i^2} \right) \left( \frac{\partial^2 \pi_i}{\partial x_i^2} \right) - \left( \frac{\partial^2 \pi_i}{\partial x_i \partial p_i} \right)^2 > 0$$

Since  $\beta < 0$ ,

$$\frac{\partial^2 \pi_i}{\partial p_i^2} = \beta MS_i (1 - MS_i) (2 + \beta (p_i - c_i) (1 - 2 MS_i)) < 0 \quad (\text{xix})$$

if

$$2 + \beta (p_i - c_i) (1 - 2 MS_i) > 0 \quad (\text{xx})$$

This expression is satisfied if

$$p_i - c_i < \frac{2}{-\beta} \quad (\text{xxi})$$

Similarly to  $\frac{\partial^2 \pi_i}{\partial p_i^2}$ ,

$$\frac{\partial^2 \pi_i}{\partial x_i^2} = \gamma^2 (p_i - c_i) MS_i (1 - MS_i) (1 - 2 MS_i) - 2 < 0 \quad (\text{xxii})$$

as long as

$$\gamma^2 (p_i - c_i) MS_i (1 - MS_i) (1 - 2 MS_i) < 2$$

Note that  $MS_i (1 - MS_i) (1 - 2 MS_i) < \frac{1}{6\sqrt{3}}$  (the maximum of  $MS_i (1 - MS_i) (1 - 2 MS_i)$  at  $[0, 1]$  occurs at  $MS_i = \frac{1}{2} - \frac{\sqrt{3}}{6}$ ). Hence equation xxii is satisfied if  $\frac{1}{6\sqrt{3}} \gamma^2 (p_i - c_i) < 2$ . In other words,  $\frac{\partial^2 \pi_i}{\partial x_i^2} < 0$  when:

$$p_i - c_i < \frac{12\sqrt{3}}{\gamma^2} \quad (\text{xxiii})$$

We now consider  $\det(H_i)$ .

$$\begin{aligned} \det(H_i) &= \left( \frac{\partial^2 \pi_i}{\partial p_i^2} \right) \left( \frac{\partial^2 \pi_i}{\partial x_i^2} \right) - \left( \frac{\partial^2 \pi_i}{\partial x_i \partial p_i} \right)^2 = \\ &= MS_i (1 - MS_i) (-2\beta (2 + \beta (p_i - c_i) (1 - 2 MS_i)) - \gamma^2 MS_i (1 - MS_i)) \end{aligned}$$

If  $\det(H_i) > 0$  then:

$$-2\beta (2 + \beta (p_i - c_i) (1 - 2 MS_i)) - \gamma^2 MS_i (1 - MS_i) > 0$$

Since

$$-2\beta (2 + \beta (p_i - c_i) (1 - 2 MS_i)) - \gamma^2 MS_i (1 - MS_i) > -2\beta (2 + \beta (p_i - c_i) (1 - 2 MS_i)) - 0.25\gamma^2$$

$\det(H_i) > 0$  if

$$-2\beta (2 + \beta (p_i - c_i) (1 - 2 MS_i)) - \frac{1}{4}\gamma^2 > 0$$

In other words,  $\det(H_i) > 0$  implies a condition on the margin:

$$p_i - c_i < \frac{1}{-\beta} \left( 2 - \frac{1}{-8\beta} \gamma^2 \right) \quad (\text{xxiv})$$

Note that (xxiv) is stricter than (xxi). It also implies, that

$$\gamma < 4\sqrt{-\beta} \quad (\text{xxv})$$

It follows then, that  $\frac{12\sqrt{3}}{\gamma^2} > \frac{1}{-\beta} \left( 2 - \frac{1}{-8\beta} \gamma^2 \right)$ . Hence when condition (xxiv) is satisfied, conditions (xxi) and (xxiii) are also satisfied.

**Q.E.D.**

## D Proof of Proposition 5

In order to prove uniqueness, we use the contraction mapping formulation (Friedman 1986).

We need to prove that the best-reply price and advertising function is a contraction :  $\sum_{i=1}^2 \left| \frac{\partial R_j^*(x)}{\partial z_i} \right| \leq \lambda < 1$ , where  $R_j^* = (p_j^*, x_j^*)$  ( $j = 1, 2$ ) and  $z_i = (p_i, x_i)$  is the strategy combination of two players.

Given that the case under consideration is symmetric for the two brands, it is sufficient to prove the condition for one brand. Nevertheless, wherever appropriate, results are presented in general form.

Using the implicit function theorem obtain expressions for best reply functions:

$$\frac{\partial p_1^*}{\partial p_2} = - \frac{\left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial p_2 \partial p_1} \right) - \left( \frac{\partial^2 \pi_1}{\partial p_2 \partial x_1} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)}{\left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) - \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)^2} = \frac{N_1}{\det(H_1)} \quad (\text{xxvi})$$

$$\frac{\partial x_1^*}{\partial p_2} = \frac{- \left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial p_2 \partial x_1} \right) + \left( \frac{\partial^2 \pi_1}{\partial p_2 \partial p_1} \right) \left( \frac{\partial^2 \pi_1}{\partial p_1 \partial x_1} \right)}{\left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) - \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)^2} = \frac{N_2}{\det(H_1)} \quad (\text{xxvii})$$

$$\frac{\partial p_1^*}{\partial x_2} = - \frac{\left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_2 \partial p_1} \right) - \left( \frac{\partial^2 \pi_1}{\partial x_2 \partial x_1} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)}{\left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) - \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)^2} = \frac{N_3}{\det(H_1)} \quad (\text{xxviii})$$

$$\frac{\partial x_1^*}{\partial x_2} = \frac{- \left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_2 \partial x_1} \right) + \left( \frac{\partial^2 \pi_1}{\partial x_2 \partial p_1} \right) \left( \frac{\partial^2 \pi_1}{\partial p_1 \partial x_1} \right)}{\left( \frac{\partial^2 \pi_1}{\partial p_1^2} \right) \left( \frac{\partial^2 \pi_1}{\partial x_1^2} \right) - \left( \frac{\partial^2 \pi_1}{\partial x_1 \partial p_1} \right)^2} = \frac{N_4}{\det(H_1)} \quad (\text{xxix})$$

Note that the common denominator of above partials is the determinant of the Hessian which is positive under the existence conditions.

Consider the numerators of derivatives of the best-reply functions.

$$\begin{aligned} N_1 &= -2\beta MS_1 MS_2 (1 + \beta (p_1 - c_1) (1 - 2 MS_1)) \\ N_2 &= -\beta\gamma MS_1^2 MS_2 (1 - MS_1) \\ N_3 &= -2\gamma MS_1 MS_2 (1 + \beta (p_1 - c_1) (1 - 2 MS_1)) \\ N_4 &= -\gamma^2 MS_1^2 MS_2 (1 - MS_1) \end{aligned}$$

Since  $\det(H_1) > 0$  then the condition to prove can be rewritten as

$$\left| \frac{\partial p_1^*}{\partial p_2} \right| + \left| \frac{\partial x_1^*}{\partial p_2} \right| + \left| \frac{\partial p_1^*}{\partial x_2} \right| + \left| \frac{\partial x_1^*}{\partial x_2} \right| = \frac{|N_1| + |N_2| + |N_3| + |N_4|}{\det(H_1)} < 1 \quad (\text{xxx})$$

After simple transformations, contraction condition for brand 1 can be rewritten as

$$(-\beta + \gamma) \frac{2|1 + \beta (p_1 - c_1) (1 - 2 MS_1)| + \gamma MS_1 (1 - MS_1)}{-2\beta (2 + \beta (p_1 - c_1) (1 - 2 MS_1)) - \gamma^2 MS_1 (1 - MS_1)} < \frac{(1 - MS_1)}{MS_2} \quad (\text{xxxix})$$

It follows from the first-order conditions that in the neighbourhood of the equilibrium.

$$1 + \beta (p_i - c_i) (1 - 2 MS_i) > 0 \quad (\text{xxxixii})$$

hence

$$|1 + \beta (p_1 - c_1) (1 - 2 MS_1)| = (-\beta) (p_1 - c_1) MS_1 \quad (\text{xxxixiii})$$

Contraction condition (xxxix) becomes:

$$(-\beta + \gamma) \frac{2(-\beta) (p_1 - c_1) MS_1 + \gamma MS_1 (1 - MS_1)}{-2\beta (2 + \beta (p_1 - c_1) (1 - 2 MS_1)) - \gamma^2 MS_1 (1 - MS_1)} < \frac{(1 - MS_1)}{MS_2} \quad (\text{xxxixiv})$$

Note that  $\frac{1-MS_1}{MS_2}$  does not depend on  $p_1$  and  $x_1$  (and  $\frac{1-MS_2}{MS_1}$  does not depend on  $p_2$  and  $x_2$ ) and  $\frac{1-MS_1}{MS_2} \geq K_1$ , where  $K_1 = 1 + \frac{1}{\exp(\alpha + \beta c_2 + 0.25 \frac{\gamma^2}{-\beta})}$ . Taking into account that  $MS_1 (1 - MS_1) \leq 0.25$ , and the price first-order condition, it is sufficient to show that

$$(-\beta + \gamma) \frac{2(-\beta) (p_1 - c_1) + 0.25\gamma - 2}{2(-\beta) (-\beta) (p_1 - c_1) - 0.25\gamma^2} < K_1 \quad (\text{xxxixv})$$

When  $\gamma = 0$ , condition (xxxixv) is always true. Note that in (??) the numerator is a growing function of  $\gamma$  while the denominator, as well as  $K_1$  are diminishing functions of  $\gamma$ . Consequently there exists a positive  $\gamma^{\max}$  such that as long as  $\gamma < \gamma^{\max}$  condition (??) is satisfied.

The same is true for brand 2.

**Q.E.D.**