Parametric weighting functions ☆

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Abstract

This paper provides preference foundations for parametric weighting functions under rank-dependent utility. This is achieved by decomposing the independence axiom of expected utility into separate meaningful properties. These conditions allow us to characterize rank-dependent utility with power and exponential weighting functions. Moreover, by allowing probabilistic risk attitudes to vary within the probability interval, a preference foundation for rank-dependent utility with parametric inverse-*S* shaped weighting function is obtained.
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1. Introduction

Many empirical studies have shown that expected utility theory (EU), in particular its crucial independence axiom, does not provide an accurate description of people's actual choice behavior. This evidence has motivated researchers to develop alternative more flexible models. One

prominent class of these alternatives is rank-dependent utility (RDU), which was introduced by Quiggin [39,40], and which is the basis of prospect theory [45,31].[1]

Most derivations of RDU require some structural richness on the set of consequences because the proposed preference conditions focus on the derivation of continuous cardinal utility. In those derivations the weighting functions are obtained as a bonus. In this paper we follow the traditional approach put forward by von Neumann and Morgenstern [47] by focusing on the structure naturally offered by the probability interval, and we provide preference conditions that focus on the derivation of the probability weighting function. Typical for this approach is that cardinal utility is obtained as a bonus.

Axiomatizations of RDU for general outcomes have been provided by Nakamura [36] and more recently by Abdellaoui [2] and Zank [57] in the axiomatic framework of von Neumann and Morgenstern. In these approaches the weighting function is unrestricted. Empirical evidence, however, suggests a particular pattern for probability weighting: small probabilities are overweighted while large ones are underweighted. Specific parametric forms have been proposed in the literature to accommodate these features. Some involve a single parameter [25,26,41,17,45, 32,24,42,29], while others use two or more parameters [7,22,17,28,38]. A recent experimental investigation of various parametric weighting functions [43] favors the variant of Prelec [38].

Despite the large interest in parametric specifications for the weighting function under RDU, little research has been invested in the axiomatic foundation of testable preference conditions in the RDU framework with general lotteries. Several recent foundations restrict attention to binary lotteries where one consequence is the zero payoff (see, e.g., [29,37,30,4,6]). RDU for binary lotteries reduces to a simple and tractable multiplicative form which many decision models agree with (see [21] for a general axiomatic framework).

The preference foundations for parametric weighting functions presented in this paper apply to general sets of lotteries. Except for weak ordering and continuity, the properties that we propose are all implied by the independence axiom. For instance, we retain stochastic dominance and, in line with all rank-dependent theories, we assume comonotonic independence. By focusing on specific functional forms for the weighting functions, we obtain, in contrast to the afore mentioned parametric derivations, the separation of utility and the weighting function free of charge.

Specific behavioral implications of the independence axiom have been analyzed before. Machina [34] distinguished two properties, mixture separability and replacement separability, in an analysis of nonexpected utility models. We explore the implications of these separability conditions within our rank-dependent framework. It turns out that our restricted versions can be employed to characterize RDU with a power weighting function and RDU with a linear/exponential weighting function. Because these weighting functions each involve a single parameter, they cannot accommodate at the same time probabilistic risk seeking and probabilistic risk aversion within the probability interval. That is, they are incompatible with the inverse-S shaped form that received extensive empirical support, e.g., [11,54,44,23,1,10,27,3].

We relax the previous preference conditions further, namely to hold only on specific subsets of the probability interval. This way, we provide foundations for inverse-S shaped weighting functions under RDU, which are entirely based on testable preference conditions. Our analysis focuses on functional forms that may involve three parameters. One parameter describes the

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[1] Because prospect theory comes down to RDU if consequences are of the same sign (that is, either all consequences are gains or all are losses), the arguments presented in this paper apply to prospect theory as well.

probabilistic risk attitudes for small probabilities while a second one describes such attitudes for large probabilities. A third parameter indicates a probability where probabilistic risk attitudes may change from risk aversion to risk seeking. As it turns out, these parametric forms are in agreement with the interpretation of modeling sensitivity towards changes from impossibility and certainty, as proposed by Tversky and Kahneman [45] and formalized in Tversky and Wakker [46]. In particular, the first two parameters represent measures of the degrees of sensitivity (or curvature), and, together with the third parameter, they can be used to quantify the relative sensitivity between certainty and impossibility (or elevation).

The organization of the paper is as follows. In Section 2 general notation and preliminary results are presented. We then proceed with a separation of the independence axiom of EU into specific variants of the separability conditions proposed by Machina [34]. In Section 3 we analyze mixture separability restricted to worst consequences, and we analyze replacement separability restricted to best and worst consequences. In Section 4 we provide results for parametric inverse-*S* shaped probability weighting functions and Section 5 concludes. All proofs are deferred to Appendix A.

2. Preliminaries

This paper uses the von Neumann and Morgenstern setup as in [2] and [57]. Let X denote the set of consequences. For simplicity of exposition, we assume a finite set of consequences, such that $X = \{x_0, \ldots, x_n\}$ for $n \geqslant 3$, and further, that consequences are ordered from worst to best, i.e., $x_0 \prec \cdots \prec x_n$. A *lottery* is a finite probability distribution over the set X. It can be represented by $P = (\tilde{p}_0, x_0; \ldots; \tilde{p}_n, x_n)$ meaning that probability $\tilde{p}_j$ is assigned to consequence $x_j \in X$, for $j = 0, \ldots, n$. Another way of representing lotteries is in terms of decumulative probabilities, i.e., $P = (p_1, \ldots, p_n)$ where $p_j = \sum_{i=j}^n \tilde{p}_i$ denotes the likelihood of getting at least x_j, $j = 1, \ldots, n$. Here, we simplified the notation by suppressing the consequences and by noting that the worst consequence x_0 always has decumulative probability equal to 1. Let L denote the set of all lotteries, which we identify with the set $\{(p_1, \ldots, p_n): 1 \geqslant p_1 \geqslant \cdots \geqslant p_n \geqslant 0\}$. A preference relation $\succcurlyeq$ is assumed over L, and its restriction to subsets of L (e.g., all degenerate lotteries) is also denoted by $\succcurlyeq$. The symbol $\succ$ denotes strict preference while $\sim$ denotes indifference.

In what follows we provide preference conditions for $\succcurlyeq$ in order to *represent* the preference relation over L by a function V. That is, V is a mapping from L into the set of real numbers, $\mathbb{R}$, such that for all $P, Q \in L$,

$$P \succcurlyeq Q \quad \Leftrightarrow \quad V(P) \geqslant V(Q).$$

This necessarily implies that $\succcurlyeq$ must be a *weak order*, i.e. $\succcurlyeq$ is *complete* ($P \succcurlyeq Q$ or $P \preccurlyeq Q$ for all $P, Q \in L$) and *transitive* ($P \succcurlyeq Q$ and $Q \succcurlyeq R$ implies $P \succcurlyeq R$ for all $P, Q, R \in L$).

The preference relation $\succcurlyeq$ satisfies *monotonicity* if $P \succ Q$ whenever $p_j \geqslant q_j$ for all $j = 1, \ldots, n$ and $P \neq Q$. The preference relation $\succcurlyeq$ satisfies *Jensen-continuity* on the set of lotteries L if for all lotteries $P \succ Q$ and R there exist $\rho, \mu \in (0, 1)$ such that

$$\rho P + (1 - \rho) R \succ Q \quad \text{and} \quad P \succ \mu R + (1 - \mu) Q.$$

A monotonic weak order that satisfies Jensen-continuity on L also satisfies the stronger Euclidean-continuity on L (see [2, Lemma 18]). It then follows from Debreu [18] that there exists a continuous function $V : L \to \mathbb{R}$, strictly increasing in each decumulative probability, that represents $\succcurlyeq$. The function V is unique up to strictly increasing continuous transformations.

An additional condition is needed to show that the representing function V is additively separable [19,48,14]. To define this property we introduce some useful notation. For $i \in \{1, \ldots, n\}$, $P \in L$ and $\alpha \in [0, 1]$, we denote by $\alpha_i P$ the distribution that agrees with P except that p_i is replaced by α. Whenever this notation is used it is implicitly assumed that $p_{i-1} \geqslant \alpha \geqslant p_{i+1}$ (respectively, $\alpha \geqslant p_{i+1}$ if $i = 1$ and $p_{i-1} \geqslant \alpha$ if $i = n$) to ensure that $\alpha_i P \in L$. Similarly, for $I \subset \{1, \ldots, n\}$ we write $\alpha_I P$ for the distribution that agrees with P except that p_i is replaced by α for all $i \in I$, whenever the probabilities in $\alpha_I P$ are ranked from highest to lowest.

A preference relation $\succcurlyeq$ satisfies *comonotonic independence* if $\alpha_i P \succcurlyeq \alpha_i Q \Leftrightarrow \beta_i P \succcurlyeq \beta_i Q$ for all $\alpha_i P, \alpha_i Q, \beta_i P, \beta_i Q \in L$.

Deriving additive separability on rank-ordered sets is not trivially extended from Debreu's classical result, but invokes more complex mathematical tools. From [48] and [14] it follows that a preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence if and only if $\succcurlyeq$ can be represented by an additive function

$$V(P) = \sum_{j=1}^{n} V_j(p_j),$$

with continuous strictly monotonic functions $V_1, \ldots, V_n : [0, 1] \to \mathbb{R}$ which are bounded except maybe V_1 and V_n which could be infinite at extreme probabilities (i.e., at 0 or at 1). The functions $V_1, \ldots, V_n$ are jointly cardinal, that is, they are unique up to location and common scale.

In the next sections we provide preference foundations for specific rank-dependent utility models using as common point of departure the additive representation obtained above. Before proceeding we recall the general form of rank-dependent utility.

Rank-dependent utility (RDU) holds if the preference relation is represented by the function

$$V(P) = u(x_0) + \sum_{j=1}^{n} w(p_j)\big[u(x_j) - u(x_{j-1})\big], \tag{1}$$

where the utility function $u : X \to \mathbb{R}$ agrees with $\succcurlyeq$ on X, and the *weighting function* $w : [0, 1] \to [0, 1]$ is strictly increasing and continuous with $w(0) = 0$ and $w(1) = 1$. Under RDU utility is cardinal and the weighting function is uniquely determined. If the weighting function is linear then RDU reduces to expected utility (EU). A concave weighting function resembles probabilistic risk seeking behavior while a convex one resembles probabilistic risk aversion [49,50].

For completeness we recall the classical preference condition leading to EU. Recall that for $\alpha \in [0, 1]$ and $P, Q \in L$, the probability mixture $\alpha P + (1 - \alpha)Q$ is defined as the lottery $(\alpha p_1 + (1 - \alpha)q_1, \ldots, \alpha p_n + (1 - \alpha)q_n)$.

Axiom 1. A preference relation $\succcurlyeq$ satisfies *vNM-independence* (short for *von Neumann–Morgenstern independence*) if for all $P, Q, R \in L$ and all $\alpha \in (0, 1)$ it holds that

$$P \succcurlyeq Q \quad \Leftrightarrow \quad \alpha P + (1 - \alpha)R \succcurlyeq \alpha Q + (1 - \alpha)R.$$

Note that vNM-independence implies monotonicity and comonotonic independence for a weak order.

3. Common ratio and common consequence effects

One of the difficulties of EU is to accommodate preferences that exhibit the common ratio effect. Allais [5] compared the choice behavior for the following two decision problems. In problem 1 there is the choice between the following lotteries:

$$A_1 = (1, 1M) \quad \text{and} \quad B_1 = (0.2, 0M; 0.8, 5M),$$

where M denotes \$-millions. In problem 2 the choice is between

$$A_2 = (0.95, 0M; 0.05, 1M) \quad \text{and} \quad B_2 = (0.96, 0M; 0.04, 5M).$$

The literature has reported (e.g., [5,33,16,53]) that a significant majority of people exhibit a preference for A_1 in the first choice problem and a preference for B_2 in the second choice problem. Substituting EU immediately reveals that this leads to a conflicting relationship. Such preferences are, however, not in conflict with monotonicity and neither with comonotonic independence, the two implications of vNM-independence considered in the previous section. It is a different aspect of vNM-independence that is violated by such preferences, which will be termed common ratio invariance below.

A further difficulty for EU-preferences concerns the replacement of common consequences. The common consequence effect originates from observing behavior among the following pairs of choice problems. In problem 3 the choice is between

$$A_3 = (1, 1M) \quad \text{and} \quad B_3 = (0.01, 0M; 0.89, 1M; 0.1, 5M),$$

and in problem 4 the choice is between

$$A_4 = (0.89, 0M; 0.11, 1M) \quad \text{and} \quad B_4 = (0.9, 0M; 0.1, 5M).$$

It has been observed in experiments that a significant majority of people exhibit a preference for A_3 in the former choice problem and a preference for B_4 in the latter choice problem (e.g., [5,33,16,53], but see also related evidence in [51,9,8,35]). If one writes the previous lotteries as decumulative distributions over consequences 0, $1M$, and $5M$, then one can immediately see that $A_4 = (0.11, 0)$ and $A_3 = A_4 + (0.89, 0)$, and that $B_4 = (0.1, 0.1)$ and $B_3 = B_4 + (0.89, 0)$. Clearly, exhibiting initially $A_3 \succ B_3$ together with a second preference $A_4 \prec B_4$ directly violates vNM-independence but does not violate monotonicity and neither comonotonic independence.

In the next two subsections we identify precisely those two behavioral implications of vNM-independence which are violated by the corresponding Allais examples.

3.1. Common ratio invariance

In this subsection we weaken vNM-independence such that the common consequence effect can be accommodated. Below we use the notation $\alpha P\ (= (\alpha p_1, \ldots, \alpha p_n))$ as the short form for a mixture of P with the worst consequence x_0, i.e., $\alpha P + (1 - \alpha)(0, \ldots, 0)$. For simplicity, we demand our subsequent axioms to hold only on the set L_0, which defines the set of distributions P such that $p_1 < 1$ and $p_n > 0$ (hence, for lotteries where the worst and best consequence have positive probability).

Axiom 2. A preference relation $\succcurlyeq$ satisfies *common ratio invariance for decumulative distributions* if

$$P \sim Q \quad \Leftrightarrow \quad \alpha P \sim \alpha Q$$

for all $0 < \alpha < 1$, and $P, Q \in L_0$.

Common ratio invariance for decumulative distributions says that shifting proportionally probability mass from good consequences to the worst consequence (or doing the opposite) leaves preferences unaffected, which precisely rules out the common ratio effect. The property is a weak form of mixture separability [34]. The latter axiom, defined formally in Appendix A, demands that a preference between two lotteries is maintained if each of the lotteries is mixed with any common consequence. In contrast, common ratio invariance for decumulative distributions demands that such mixtures are only permitted if the common consequence is the worst. From a behavioral perspective, the condition means that probabilistic risk attitudes are invariant to proportional changes in decumulative probabilities.

The condition has also appeared in Safra and Segal [42], where it was called zero-independence, and where it has been used in the derivation of a specific version of the dual theory [55], namely RDU with linear utility and power weighting function. Luce [29, p. 84], working in a different framework, has also proposed a similar condition. The next result shows that the condition is powerful enough to yield RDU-preferences with power weighting without restricting the generality of the utility function.

Theorem 1. *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) *The preference relation $\succcurlyeq$ on L is represented by rank-dependent utility with a power weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^{n} p_j^a \big[u(x_j) - u(x_{j-1}) \big],$$

with $a > 0$, and monotonic utility function $u : X \to \mathbb{R}$.
(ii) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance for decumulative distributions.*

The function u is cardinal.

Proof. See Appendix A. $\square$

It has previously been documented that preferences exhibiting the common ratio effect exclude RDU preferences with power weighting. Our result above demonstrates that it is precisely this class of RDU-preferences with power weighting, including EU-preferences, that cannot accommodate common ratio effect preferences. That the result is very general can also be inferred from the fact that, except for monotonicity, no further restrictions apply to utility.

3.2. Extreme replacement separability

We now consider preferences that can accommodate the common ratio effect.

Axiom 3. A preference relation $\succcurlyeq$ satisfies *extreme replacement separability* if

$$(p_1, \ldots, p_n) \sim (q_1, \ldots, q_n) \quad \Leftrightarrow \quad (p_1 + \alpha, \ldots, p_n + \alpha) \sim (q_1 + \alpha, \ldots, q_n + \alpha),$$

whenever $(p_1, \ldots, p_n), (q_1, \ldots, q_n), (p_1 + \alpha, \ldots, p_n + \alpha), (q_1 + \alpha, \ldots, q_n + \alpha) \in L_0$.

From a behavioral perspective, extreme replacement separability means that probabilistic risk attitudes are invariant with respect to common absolute changes in decumulative probabilities. A similar condition has been termed replacement separability in [34]. We define the property formally in Appendix A.

The following theorem shows that for RDU-preferences the only weighting functions that are able to accommodate extreme replacement separability are linear or exponential ones.

Theorem 2. *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) *The preference relation $\succcurlyeq$ on L is either represented by expected utility, or it is represented by rank-dependent utility with an exponential weighting function, i.e.,*

$$V(P) = u(x_0) + \sum_{j=1}^{n} \frac{e^{cp_j} - 1}{e^c - 1} \big[u(x_j) - u(x_{j-1}) \big],$$

with $c \neq 0$, and monotonic utility function $u : X \to \mathbb{R}$.

(ii) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and extreme replacement separability.*

The function u is cardinal.

Proof. See Appendix A. □

Note that RDU-preferences satisfying both common ratio invariance for decumulative distributions and extreme replacement separability can only be represented by EU. This follows immediately by observing that the only possible weighting function that is common in Theorems 1 and 2 is the linear weighting function $w(p) = p$.

The properties considered in this section can easily be formulated for cumulative distributions. Jensen-continuity, monotonicity, comonotonic independence, and also extreme replacement separability have mathematically equivalent counterparts which are obtained by simply replacing the decumulative distributions by the corresponding cumulative ones. However, doing the same for the afore mentioned common ratio invariance property leads to a different but analog property which is also implied by vNM-independence. Employing this common ratio invariance for cumulative instead of decumulative probabilities in Theorem 1 leads to a corresponding RDU-representation with a weighting function that is the dual of a power function, i.e., $w(p) = 1 - (1-p)^b$, $b > 0$.

4. Inverse-S shaped weighting functions

The parametric forms derived in the previous section are too rigid for modeling empirically observed probabilistic risk attitudes. Such risk attitudes are reflected in the curvature of the probability weighting function [15,55,12,49,2,13]. The afore mentioned RDU-preferences either exhibit exclusively probabilistic risk aversion (i.e., the weighting function is convex) or exclusively probabilistic risk seeking (i.e., the weighting function is concave) throughout the probability interval. While there is theoretical interest in overall convex or overall concave probability weighting, empirical findings suggest that a combination of probabilistic risk seeking for small probabilities and probabilistic risk aversion for large probabilities is an appropriate way of

modeling sensitivity towards probabilities (see [50] for a review of empirical evidence). Because the concave region for small probabilities is followed smoothly by a convex region for larger probabilities (see [45,44,54,1]), such weighting functions are referred to as inverse-S shaped.

A few parametric forms have been proposed for inverse-S shaped weighting functions [25,26, 22,17,28,45,38], and their parameters have been estimated in many empirical studies [11,44,54, 23,1,10,27,20,3]. Most of these parametric forms lack an appropriate axiomatic underpinning. This is problematic because it is unclear what kind of preference condition must be assumed to generate such weighting functions, and therefore, it is unclear what kind of behavioral properties are captured when using such weighting functions.

Axiomatizations have been proposed for the class of weighting functions introduced by Prelec [38] (see also [30,4]). The class introduced by Goldstein and Einhorn [22] has been discussed in [23], where necessary preference conditions have been proposed. A restrictive aspect of these axiomatizations is that a representing functional, where the continuous utility is already separated from probability weighting, must be assumed prior to invoking the additional invariance property that generates the required parametric form. An open and from an empirical point of view important question is whether, on their own, those characterizing properties are powerful enough to induce such a separation once additive separability, as done in this paper, has been derived.

Recall that the results presented in the previous section are free of restrictions on the richness of the set of consequences, and also free of additional separability conditions that ensure RDU to hold prior to invoking the invariance properties. But note at the same time that these preference conditions do not allow inverse-S shaped probability weighting functions under RDU. We would like to have both preference conditions for general consequences and also axiomatizations that allow for inverse-S shaped weighting functions under RDU. In what follows we propose such a preference condition, and show that it leads to a new family of parametric weighting functions.

To derive RDU with inverse-S shaped weighting functions we restrict the preference conditions presented in Section 3 to hold for a restricted set of probabilities. An analogous approach for general, non-parametric weighting functions and capacities was pursued by Tversky and Wakker [46] and Wakker [50]. This seems to be a reasonable compromise because, as we show below, these conditions are still powerful enough to separate utility from probability weighting if additive separability holds.

4.1. Switch-power weighting functions

The results presented in this subsection focus on the class of weighting functions which are power functions for probabilities below some $\hat{p} \in (0, 1)$, and dual power functions above $\hat{p}$, i.e.,

$$w(p) = \begin{cases} cp^a, & \text{if } p \leqslant \hat{p}, \\ 1 - d(1-p)^b, & \text{if } p > \hat{p}, \end{cases}$$

with the parameters involved as discussed below. We call these functions *switch-power weighting functions*.

We presented the function above with five parameters a, b, c, d and $\hat{p}$. However, these reduce to three, first because of continuity of w at $\hat{p}$, and second by assuming differentiability at $\hat{p}$, which seems plausible in this context. Continuity and monotonicity imply that $a, c, b, d > 0$. Continuity and differentiability at $\hat{p}$ relates c and d to a, b and $\hat{p}$ through
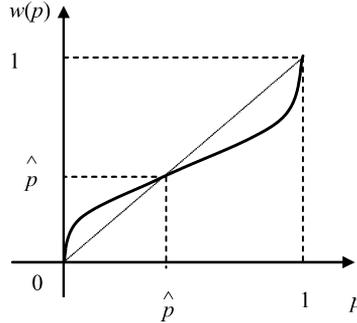
Fig. 1. A two-parameter switch-power weighting function.

$$c = \hat{p}^{-a}\left[\frac{b\hat{p}}{b\hat{p} + a(1 - \hat{p})}\right],$$

$$d = (1 - \hat{p})^{-b}\left[\frac{a(1 - \hat{p})}{b\hat{p} + a(1 - \hat{p})}\right].$$

If $0 < a \leqslant 1$ the probability weighting function is concave on $(0, \hat{p})$, and if $0 < b \leqslant 1$ it is convex on $(\hat{p}, 1)$, hence, has an inverse-S shape. For $a, b \geqslant 1$ we have an S-shaped probability weighting function.[2]

When $\hat{p}$ approaches 1 or 0, the weighting function reduces to a power weighting function or a dual power weighting function, respectively. Moreover, substitution of $\hat{p}$ into w gives

$$w(\hat{p}) = \frac{b\hat{p}}{b\hat{p} + a(1 - \hat{p})}$$

$$= 1 - \frac{a(1 - \hat{p})}{b\hat{p} + a(1 - \hat{p})},$$

from which one can easily derive the relationship

$$w(\hat{p}) \lessgtr \hat{p} \quad \Leftrightarrow \quad b \lessgtr a.$$

In particular, this shows that whenever $a = b$ the weighting function intersects the 45° line precisely at $\hat{p}$ (see Fig. 1). It is worthwhile noting that in this case the derivative of w at $\hat{p}$ equals a, and therefore this parameter controls for the curvature of the weighting function. The parameter $\hat{p}$, however, indicates whether the interval for overweighting of probabilities is larger than the interval for underweighting, and therefore controls for the elevation of the weighting function (see also [23] for a similar interpretation of the parameters in the "linear in log-odds" weighting function of Goldstein and Einhorn [22]).

In general, when $a \neq b$, both parameters control for curvature. In that case $\hat{p}$ need not demarcate the regions of over and underweighting because it may not lie on the 45° line. Nevertheless, $\hat{p}$ will still influence elevation, however, whether there is more overweighting relative to underweighting now also depends on the relationship between the magnitudes of the parameters a

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[2] Tversky and Wakker [46, Proposition 4.1] presented behavioral properties, which enforce the inverse-S shape for switch-power weighting functions. This follows from the fact that, for general weighting functions, these properties imply bounded *sub-additivity*, that is, for all $\varepsilon \geqslant 0$ and $\varepsilon' \geqslant 0$, w satisfies bounded sub-additivity with respect to ε and ε' if $w(q) \geqslant w(p + q) - w(p)$ whenever $p + q \leqslant 1 - \varepsilon$ and $1 - w(1 - q) \geqslant w(p + q) - w(p)$ whenever $p \geqslant \varepsilon'$.
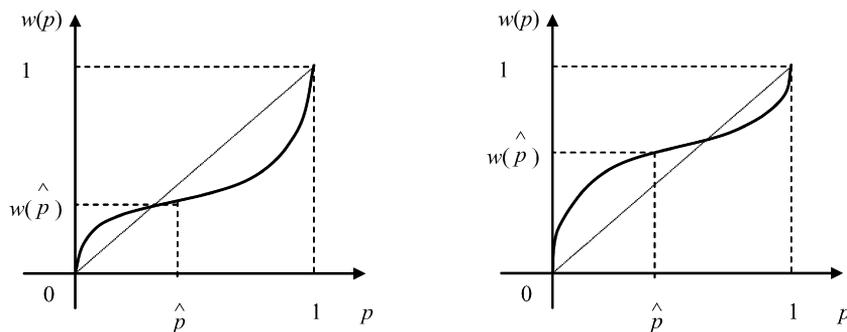
Fig. 2. A 3-parameter function with underweighting respectively overweighting at $\hat{p}$.

and b. Fig. 2 depicts, for the case of an inverse-S shaped weighting function, the two scenarios of underweighting ($0 < b < a < 1$), respectively, overweighting ($0 < a < b < 1$) at $\hat{p}$.

As it turns out, it is more appropriate to interpret these parameters as was initially proposed by Tversky and Kahneman [45]. All parameters may influence elevation, however, the main role of $\hat{p}$ is to demarcate the interval of probabilistic risk aversion from the interval of probabilistic risk seeking. The parameter a indicates diminishing (or increasing) sensitivity to changes from impossibility to possibility. It can be inferred (by inspecting the derivative of w for probabilities in the range $(0, \min\{\hat{p}, 1 - \hat{p}\})$) that sensitivity increases if $a > 1$, decreases if $a < 1$, and for $a = 1$ sensitivity is constant. Similarly, for changes away from 1, sensitivity increases if $b > 1$ and decreases if $b < 1$, and is constant for $b = 1$.

Recall that the objective of the one-parameter weighting function employed in [45] was to account for curvature. By contrast, the two-parameter forms [22,38] allow for a distinction between curvature and elevation. In the psychological literature the parameters have been interpreted accordingly, e.g. [23], although formal measures for these concepts have not yet been identified. One can give a similar psychological interpretation to the parameters of the switch-power weighting function. This holds in particular for the case of a switch-power weighting function where the powers a and b are equal. The switch-power weighting functions are motivated by probabilistic risk behavior which is a meaningful and well-established concept in both economics and psychology [54,50,2,23]. Accordingly, the powers a and b, while proxies for curvature, are interpreted as indexes of relative risk aversion in the probabilistic sense in much analogy to the Arrow–Pratt index of relative risk aversion developed for utility. That this index may be different for changes in probabilities close to 0 compared to changes in probabilities close to 1 is a well-established empirical phenomenon, which shows that people treat probabilities of good consequences markedly different than the similar probabilities for bad consequences. It is precisely this behavioral aspect of risk attitude which motivates the preference conditions employed in the next section.

4.2. Preference foundation

Before formulating the preference condition that is necessary for RDU with (inverse) S-shaped switch-power weighting function, we note that if a lottery is written as a decumulative distribution $P = (p_1, \ldots, p_n)$ then writing the same lottery as a cumulative distribution results in $\tilde{P} = (1 - p_1, \ldots, 1 - p_n)$. The difference in the latter notation lies in the interpretation of the cumulative probability $1 - p_i$, which now refers the likelihood of getting at most x_{i-1}, $i = 1, \ldots, n$, whereas the decumulative probability p_i was associated with the con-

sequences x_i, $i = 1, \ldots, n$. We denote by $\tilde{L}$ the set of cumulative distributions. The notation $\alpha \tilde{P}$ ($= (\alpha(1 - p_1), \ldots, \alpha(1 - p_n)))$ and the set $\tilde{L}_0$ are defined analogously to the notation used in Section 3.1.

Axiom 4. A preference relation satisfies *common ratio invariance at tails* if for each $p \in (0, 1)$ we have

$$P \sim Q \quad \Leftrightarrow \quad \alpha P \sim \alpha Q, \tag{2}$$

whenever all $P, Q, \alpha P, \alpha Q \in L_p := \{R \in L_0 : r_1 \leqslant p\}$ or

$$\tilde{P} \sim \tilde{Q} \quad \Leftrightarrow \quad \beta \tilde{P} \sim \beta \tilde{Q}, \tag{3}$$

whenever $\tilde{P}, \tilde{Q}, \beta \tilde{P}, \beta \tilde{Q} \in \tilde{L}_p := \{R \in \tilde{L}_0 : 1 - r_n \leqslant 1 - p\}$.

Clearly common ratio invariance at tails requires preferences to be immune to common proportional changes in decumulative probabilities whenever these are all below p or immunity of preferences to common proportional changes in cumulative probabilities if these are all below $1 - p$ for any $p \in (0, 1)$.

Recall, that in the common ratio effect, the good lotteries A_1 and B_1 were mixed with the worst consequence (i.e., with 0) which induced a reversal of preferences. Our condition respects this pattern of behavior since mixture separability is demanded only if good lotteries are mixed with best consequences (i.e., the indifference (2) holds), or it is demanded only if bad lotteries are mixed with the worst consequence (i.e., the indifference (3) holds). The additional flexibility here is that risk attitudes are allowed to vary within the probability interval which is in line with the empirical evidence on inverse-S shaped probability weighting functions.

As the result below shows, replacing common ratio invariance for decumulative distributions in Theorem 1 with common ratio invariance at tails does give RDU with a switch-power weighting function.

Theorem 3. *The following two statements are equivalent for a preference relation $\succcurlyeq$ on L:*

(i) *The preference relation $\succcurlyeq$ on L is represented by RDU with a switch-power weighting function*

$$w(p) = \begin{cases} cp^a, & \text{if } p \leqslant \hat{p}, \\ 1 - d(1 - p)^b, & \text{if } p > \hat{p}, \end{cases}$$

for some $\hat{p} \in (0, 1)$ with $a, b, c, d > 0$, and $c = 1/\hat{p}^a - d(1 - \hat{p})^b/\hat{p}^a$.

(ii) *The preference relation $\succcurlyeq$ is a Jensen-continuous monotonic weak order that satisfies comonotonic independence and common ratio invariance at tails.*

The parameters $\hat{p}, a, b, d$ are uniquely determined. Further, the utility function u is cardinal.

Proof. See Appendix A. $\quad \Box$

It should be remarked that in Theorem 3 the parameters a and b are not constrained so as to imply an inverse-S shaped weighting function. Additional conditions are needed, such as those proposed in Tversky and Wakker [46, Proposition 4.1], to identify the latter shape.

5. Summary

Our main objective in this paper has been to provide preference foundations for parametric weighting functions in a general RDU framework where the set of consequences is arbitrary. Inevitably, these preference foundations have to employ conditions that exploit the mathematical structure offered by the probability interval. Initially, we have derived RDU-forms with a single parameter for probability weighting. In all these derivations cardinal utility is obtained as a bonus in addition to the specific parametric form (power, exponential, or dual power) of the weighting functions.

Building on mixture separability and replacement separability [34], we have identified behavioral preference conditions that characterize RDU with power, linear, and exponential weighting function. The central point of departure is, once more, the vNM-independence axiom. The power weighting function is directly related to the common ratio pattern of preferences and the exponential weighting function is directly related to the common consequence pattern of preferences [5], a somewhat surprising connection that has not been mentioned before in the literature. The dual power weighting function has no documented EU-paradox to be linked to, but we think that a dual analog of the common ratio paradox of Allais can easily be constructed.

These nonlinear weighting functions are still inflexible to accommodate empirically observed risk behavior. In particular it is not possible to separate sensitivity to changes in small probabilities from the sensitivity to changes in large probabilities because there is a single parameter that has to govern risk behavior at both ends of the probability scale. By separating risk behavior according to probability mixtures of good lotteries with best consequences and probability mixtures of bad lotteries with worst consequences, we have proposed a further relaxation of vNM-independence and obtained parametric inverse-S shaped weighting functions under RDU.

Following the line of arguments presented in Section 4, one can also extend these ideas to provide axiomatic characterizations for RDU with an analog to the switch-power weighting function that is first a dual power weighting function followed by a power weighting function. The extreme replacement invariance property can also be weakened to permit a switch in behavior as done in the case of common ratio invariance at tails. This property can then be used to derive RDU with "switch-exponential weighting function." Although these families of weighting functions are motivated by probabilistic risk attitudes in much analogy to the well-known concepts of relative and absolute risk aversion for utility functions, the obtained parameters can also be interpreted as indicators for curvature and elevation, which have been identified as important components of the psychophysics of probability weighting in the psychology literature. Our approach has focused on the theoretical aspects of probability weighting, and empirical studies are now required to put down first estimates for the parameter values.

Appendix A

Axiom 2′. A preference relation $\succcurlyeq$ satisfies *mixture separability* if for any $i \in \{0, \ldots, n\}$ it holds that

$$P \sim Q \quad \Leftrightarrow \quad \alpha P + (1-\alpha)1_{\{1,\ldots,i\}}(0,\ldots,0) \sim \alpha Q + (1-\alpha)1_{\{1,\ldots,i\}}(0,\ldots,0)$$

for all $\alpha \in (0,1)$ such that $P, Q, (\alpha p_1 + (1-\alpha), \ldots, \alpha p_i + (1-\alpha), \alpha p_{i+1}, \ldots, \alpha p_n)$, and $(\alpha q_1 + (1-\alpha), \ldots, \alpha q_i + (1-\alpha), \alpha q_{i+1}, \ldots, \alpha q_n) \in L_0$.[3]

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[3] Note that for $i = 0$ the mixture $\alpha P + (1-\alpha)1_{\{1,\ldots,i\}}(0,\ldots,0)$ equals $(\alpha p_1, \ldots, \alpha p_n)$.

Axiom 3′. A preference relation $\succeq$ satisfies *replacement separability* if for any $i \in \{1, \ldots, n\}$ it holds that

$$P \sim Q \quad \Leftrightarrow \quad (p_1 + \alpha, \ldots, p_i + \alpha, p_{i+1}, \ldots, p_n) \sim (q_1 + \alpha, \ldots, q_i + \alpha, q_{i+1}, \ldots, q_n),$$

whenever $P, Q, (p_1 + \alpha, \ldots, p_i + \alpha, p_{i+1}, \ldots, p_n), (q_1 + \alpha, \ldots, q_i + \alpha, q_{i+1}, \ldots, q_n) \in L_0$.

Proof of Theorem 1. That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Common ratio invariance for decumulative distributions follows from substitution of the RDU-functional with power weighting function.

Next we prove that statement (ii) implies statement (i). We know that weak ordering, Jensen-continuity, monotonicity and comonotonic independence imply the existence of an additively separable functional representing the preference $\succeq$, as outlined in Section 2. We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n. To show that our additive functional in fact is a RDU form with power weighting function we use results presented in Wakker and Zank [52]. Wakker and Zank did not have the restrictions that $p_1 < 1$ and $p_n > 0$ but permitted any non-negative rank-ordered real numbers x_i, $i = 1, \ldots, n$, because they worked in a setup with monetary outcomes instead of decumulative probabilities as we do here. But their results apply to our framework with minor modifications, in particular the restriction $p_1 \leqslant 1$ is not posing any difficulty. In their Lemma A2 they derived a similar additive representation as we have here, and then in their Lemma A3, using the analog of common ratio invariance for decumulative distributions, they showed that their additive representation in fact is a RDU form with common positive power function as "utility" and increasing "weighting function." To apply their results we just need to interchange the roles of utility and weighting function. Further, because the functions V_j, $j = 1, \ldots, n$, are proportional they can continuously be extended to 0 and 1 (this follows from [48, Proposition 3.5]). Hence, we can conclude that there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = a + cp^b$, for some real a, b, c. Monotonicity and continuity imply that b, c are positive, and requiring further that $w(0) = 0$ and $w(1) = 1$ shows that $a = 0$ and $c = 1$. Hence, $w(p) = p^b$ is established. We define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, $V_j(p_j) = w(p_j)s_j = p_j^b[u(x_j) - u(x_{j-1})]$ for $j = 1, \ldots, n$ with strictly monotonic utility u. Hence, the preference is represented by RDU with a power weighting function and monotonic utility. Therefore statement (i) has been derived.

Uniqueness results follow from the joint cardinality of the functions V_j, and the fact that they are proportional. These properties translate into the weighting function being unique because it assigns 0 to impossibility and 1 to certainty, and the utility being cardinal. This concludes the proof of Theorem 1. $\quad \square$

Proof of Theorem 2. That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. Extreme replacement separability follows from substitution of the RDU-functional with linear/exponential weighting function.

Next we prove that statement (ii) implies statement (i). As in the proof of Theorem 1, statement (ii) implies that there exists an additively separable functional, $V(P) = \sum_{j=1}^{n} V_j(p_j)$, representing the preference $\succeq$. Attention is initially restricted to the case that $p_1 < 1$ and $p_n > 0$

to exclude unbounded V_1 and V_n. To show that this additive functional is RDU with an exponential weighting function we use results presented in Zank [56]. Zank did allow for non-negative vectors with rank-ordered monetary outcomes in his Lemma 7 instead of probabilities as we have here. However, those results apply to the case considered here if we interchange the roles of utility and decision weights. Hence, we can conclude that in the additive representation the functions V_j are increasing exponential functions, i.e.,

$$V_j(p) = s_j\big[a\exp(cp) + b\big],$$

with $ac > 0$ and $s_j > 0$, and real b (or they are linear $V_j(p) = s_j[ap + b]$ with $a > 0$). As the functions are proportional, we can extend them continuously to all of $[0, 1]$ by [48, Proposition 3.5]. We fix scale and location of the otherwise jointly cardinal V_j, i.e., $V_j(0) = 0$, $V_j(1) = 1$. Hence,

$$V_j(p) = s_j\left[\frac{e^{cp_j} - 1}{e^c - 1}\right],$$

with $c \neq 0$ (or $V_j(p) = s_j p$). We use the positive s_j's to define utility as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + s_j$ for $j = 1, \ldots, n$. Therefore, the V_j's are exponential or linear for $j = 1, \ldots, n$ and u is strictly monotonic. Hence, statement (i) has been derived.

Uniqueness results follow by similar arguments as in the proof of Theorem 1. This concludes the proof of Theorem 2. $\quad\square$

Proof of Theorem 3. That statement (i) implies statement (ii) follows from the specific form of the representing functional. Jensen-continuity, weak order, and comonotonic independence as well as monotonicity follow immediate. For $\succcurlyeq$ restricted to $L_{\hat{p}}$ ($\tilde{L}_{\hat{p}}$), common ratio invariance at tails comes down to common ratio invariance for decumulative (cumulative) distributions and can easily be derived by substitution of the corresponding RDU functional.

Recall that our conditions in statement (ii) imply the existence of an additively separable functional representing the preference $\succcurlyeq$, as discussed in Section 2. We restrict the attention to the case that $p_1 < 1$ and $p_n > 0$ to avoid the problem of dealing with unbounded V_1, V_n.

Next we prove that statement (ii) implies statement (i). There are three cases that need to be considered. First, suppose that there exist no probability q such that the indifference (3) in the common ratio at tails condition holds. Then, the indifference (2) must hold for all $p \in (0, 1)$, or equivalently common ratio for decumulative probabilities holds. We can then apply Theorem 1 to obtain the corresponding version of statement (i).

The second case is analogous to the first one. Suppose that there exist no probability p such that the indifference (2) in the common ratio at tails condition holds. Then, the indifference (3) must hold for all $q \in (0, 1)$, or equivalently "common ratio for cumulative probabilities" holds. This condition, replacing common ratio invariance for decumulative probabilities in Theorem 1 gives RDU with a dual of a power weighting function as indicated at the end of Section 3. Hence, we can obtain the corresponding version of statement (i).

We can now consider the third case. Suppose, that the indifference (2) holds for some probability $r \in (0, 1)$ and that indifference (3) holds for some probability $q \in (0, 1)$. Then common ratio invariance for decumulative probabilities holds on L_p for all $0 < p \leqslant r$. Similarly, common ratio invariance for cumulative probabilities holds on $\tilde{L}_q$, and it follows that common ratio invariance for cumulative probabilities holds on $\tilde{L}_p$ for all probabilities $q \leqslant p < 1$. In addition, for any $0 < p < 1$, one of the invariance properties must hold on either L_p or $\tilde{L}_p$. This then implies

that there exists a unique $\hat{p} \in (0, 1)$ separating the probability interval such that common ratio invariance for decumulative probabilities holds on $L_{\hat{p}}$ and common ratio invariance for cumulative probabilities holds on $\tilde{L}_{\hat{p}}$.

Similarly to the proof of Theorem 1, we can now use the results of Wakker and Zank [52]. The arguments used in the proof of Theorem 1 remain valid if we restrict the analysis to probability distributions in $L_{\hat{p}}$. We can conclude that the functions V_j in our additive representation are proportional power functions for decumulative probabilities not exceeding $\hat{p}$. Therefore, by [48, Proposition 3.5], we can extend them continuously at 0. Hence, there exist positive numbers s_j such that

$$V_j(p_j) = s_j w(p_j),$$

with $w(p) = p^a + k$, for some positive a and real k.

Similarly, if we restrict the analysis to probability distributions in $\tilde{L}_{\hat{p}}$ we can conclude that the V_j's are proportional dual power functions for cumulative probabilities not exceeding $1 - \hat{p}$. Therefore, by [48, Proposition 3.5], we can extend them continuously at 1. Hence, there exist positive numbers $\hat{s}_j$ such that

$$V_j(p_j) = \hat{s}_j w(p_j),$$

with $w(p) = \hat{k} - (1 - p)^b$, for some positive b and $\hat{k} > 0$.

Continuity at $\hat{p}$ implies that the parameters are related through $s_j/\hat{s}_j = [\hat{k} - (1 - \hat{p})^b]/[\hat{p}^a + k]$, for $j = 1, \ldots, n$. We can choose $k = 0$ to fix the location for the functions $V_j(0) = 0$, for $j = 1, \ldots, n$, and fix the scale for the additive representation such that $\sum_{j=1}^{n} V_j(1) = 1$. Hence, $\sum_{j=1}^{n} \hat{s}_j = 1/\hat{k} =: d$ follows. Thus, for $p \in [0, 1]$ we can write

$$V_j(p) = (\hat{s}_j/d) c p^a, \quad \text{if } p \leqslant \hat{p}, \quad \text{and}$$
$$V_j(p) = (\hat{s}_j/d)\big[1 - d(1 - p)^b\big], \quad \text{if } p > \hat{p},$$

with $c = [1 - d(1 - \hat{p})^b]/\hat{p}^a$.

We define utility iteratively as $u(x_0) = 0$ and $u(x_j) = u(x_{j-1}) + \hat{s}_j/d$ for $j = 1, \ldots, n$. Hence, statement (i) of the theorem has been obtained.

Uniqueness results follow from the joint cardinality of the functions V_j, and the fact that they are proportional. This concludes the proof of Theorem 3.   □

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