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Coherence without additivity

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Abstract

The Dutch book argument is a coherence condition for the existence of subjective probabilities. This paper gives a general framework of analysis for this argument in a nonadditive probability setting. Particular cases are given by comonotonic and affinely related Dutch books that lead to Choquet expectations and Min expectations.

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1. Introduction

The Dutch book argument of [de Finetti \(1931\)](#) is a classic coherence condition for the existence and uniqueness of subjective probabilities. It also provides a standard justification for a model of choice based on subjective expectations. Its appeal and beauty rely on a concept of probability based on everyday life considerations like betting. [de Finetti \(1976\)](#) presented the Dutch book argument as follows:

... In English, a combination of bets devised in such a way that, profiting by an inconsistency in the odds given by the bookmaker, someone is certain to win *whatever happens* is called ‘Dutch Book’ (I don’t know why). However, if one wants to, this term could be used to express the condition of consistency that is the sole basis on which the whole theory of probability rests: suffice it to say that it consists in *allowing no chance of a Dutch Book occurring ...*

For a formalization of the meaning of Dutch book, see [Wakker \(1989\)](#). According to [Kyburg and Smokler \(1964, p. 11\)](#): “The restriction to coherence thus formulates a natural criterion of rationality in situations of uncertainty. Rationality is used in a normative sense

here; coherence formulates a criterion of how a person’s degrees of belief ought to be related.” The debate on the normative aspect of the argument is still going on.

In its original formulation, the argument is not immune to descriptive violations: its prescriptive bite has been challenged by experimental evidence (see, e.g., [Ellsberg, 1961](#); [Kahneman & Tversky, 1979](#)). Consider the following example.

Suppose Bruna decides to purchase an insurance contract for her country house. Clearly the insurance payments depend on some states of the world (for example, fire, flood, earthquake). Her preferences among contracts are the following: $(3, 3, 3) \succsim (11, 0, 0)$, $(3, 3, 3) \succ (0, 11, 0)$, and $(3, 3, 3) \succ (0, 0, 11)$. The contract $(11, 0, 0)$ means 11 thousand euros if fire, 0 if flood, and 0 if earthquake. The other contracts are defined similarly. These preferences are behaviorally plausible: in order to obtain a general purpose coverage, Bruna prefers to receive an equal and relatively small reimbursement in all states of the world rather than taking the risk of full reimbursement in one state and nothing in the others. Considering all the preferences together, and assuming that holding two contracts is equivalent to holding their statewise sum, yields $(9, 9, 9) < (11, 11, 11)$: a Dutch book. In fact, if combined, the declared preferences lead Bruna to a loss of 2 thousands of euros in each state of the world. This looks like an undesirable result: a set of good decisions, when

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taken together, should still be good, or, at least, it should not induce a violation of dominance.

This paper, while using Dutch book arguments, shows how to extend and generalize de Finetti's approach in order to accommodate these kinds of descriptive violations. The way to accommodate them is to invoke a weaker concept of coherence in a nonadditive environment. The first step is to allow Dutch books only when the involved gambles are not comonotonic (as in Bruna's country house example), thus extending to the infinite case Theorem 6 in Diecidue and Wakker (2002) and providing a general, Dutch-book-based characterization of the most popular rank-dependent model of Choquet expected utility (Schmeidler, 1989). From an empirical point of view, rank-dependent models have received a good deal of attention (see Birnbaum & McIntosh, 1996; Bleichrodt & Pinto, 2000; Gonzalez & Wu, 1999; Harless & Camerer, 1994; Tversky & Fox, 1995; Wakker, Erev, & Weber, 1994, Weber & Kirsner, 1996). In particular, many works focus on the evaluation of the rank-dependent probabilities (see Abdellaoui, 2000; Bleichrodt, van Rijn, & Johannesson, 1999; Luce 2000; Tversky & Kahneman, 1992).

On the other hand, in some situations, also the assumption of no Dutch books when the bets are comonotonic might be "reasonably" violated. Consider the following example.

John needs a new bike. He has found a second-hand one for 40 euros. He is now at the horse races trying to get this amount of money. He is evaluating alternative monetary gambles on a three-horses race and has the following preferences: $(40, 50, 60) \succeq (30, 80, 90)$ and $(40, 80, 90) \succeq (60, 60, 70)$. The gamble $(40, 50, 60)$ means that John will get 40 euros if the first horse wins the race, 50 if the second wins, and 60 if the third does. The other gambles are defined similarly. The first preference is motivated by the need of the bike: the second gamble involves the risk of not affording it. This time, the gambles are comonotonic, but considering all the preferences together gives $(80, 130, 150) < (90, 140, 160)$: a Dutch book.

This calls for an additional generalization, which is the main contribution of this paper. Allowing Dutch books only when the involved gambles are not affinely related, we provide a Dutch-book-based characterization of invariant biseparable preferences (see Ghirardato & Marinacci, 2001; Ghirardato, Maccheroni, & Marinacci, 2002). Moreover, with a further uncertainty aversion assumption, we get Min expected utility (Gilboa & Schmeidler, 1989), the generalized expected utility model most successful for finance applications (see Epstein & Wang, 1994; Epstein & Zin 1989).

Finally, considering the minimal requirement of no Dutch books existing when the involved gambles are sure prospects, we obtain a very general, still appealing concept of prevision.

To sum up, de Finetti, via the Dutch book argument, justified a model of choice based on expectations. We extend his approach in such a manner that the generalized argument can provide a new foundation for some very general nonexpected utility models of choice. These models are more and more popular and successful in economics and psychology; see for example Camerer (2001) and Starmer (2000). From an applied point of view, QALY evaluations of health policies in a rank-dependent spirit have received increasing attention (Bleichrodt, van Rijn, & Johannesson, 1999; Miyamoto, 1988, 1999).

In the next section we present the result, then conclusions follow.

2. Generalized Dutch books and coherence

2.1. The set-up

We consider a standard subjective setting; bets are simply "... wagers on any facts and for any amount ..." (de Finetti, 1976). Formally, let S be the set of *states of the world* and Σ be the *events* algebra (a nonempty family of subsets of S which is closed under finite unions and complementation). A *bet* is represented by a function $f: S \rightarrow \mathbb{R}$. The bettor wins $f(s)$ euros if state s obtains; we identify any bet with the function representing it. If the bettor holds two bets f and g , in state s she will receive $f(s) + g(s)$ euros: $f(s)$ from bet f and $g(s)$ from bet g . For this reason, the simultaneous holding of the two bets will be considered equivalent to the holding of the bet $f + g$. In particular, if $A \in \Sigma$, the indicator function $\mathbf{1}_A: S \rightarrow \mathbb{R}$, defined by

$$\mathbf{1}_A(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \in A^c, \end{cases}$$

is the bet paying 1 euro if A obtains and nothing otherwise. Analogously, the function $\alpha \mathbf{1}_A$ is the bet paying α euros if A obtains and nothing otherwise.

In general, if A_1, A_2, \dots, A_N form a partition of S in Σ and $\alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}$, the function $\alpha_1 \mathbf{1}_{A_1} + \alpha_2 \mathbf{1}_{A_2} + \dots + \alpha_N \mathbf{1}_{A_N} = \sum_{i=1}^N \alpha_i \mathbf{1}_{A_i}$ is the bet paying α_i euros if A_i obtains. We denote by B_0 the set of all functions of this kind and call them *simple bets*, and we denote by B the set of uniform limits of simple bets. Notice that, if Σ is a σ -algebra, B is the set of all bounded Σ -measurable functions; while, if $\Sigma = 2^S$, B is the set of all bounded functions on S . As usual we identify the real number α with the *constant* bet $\alpha \mathbf{1}_S$.

If $f(s) \geq g(s)$ for all $s \in S$, that is bet f yields at least as much as bet g in each state of the world, we write $f \geq g$; while $f > g$ means that $f(s) > g(s)$ for all $s \in S$.

Two bets f and g are *comonotonic* if $[f(s) - f(s')][g(s) - g(s')] \geq 0$ for all $s, s' \in S$ (see Schmeidler,

1986, 1989); while they are *affinely related* if there exist $a \geq 0$ and $b \in \mathbb{R}$ such that $f = ag + b$ or $g = af + b$ (see Gilboa & Schmeidler, 1989; Ghirardato, Klibanoff, & Marinacci, 1998). These concepts are now commonly used and their meanings well discussed in the literature, so we will not expound on them here.

Let \succsim be a binary relation representing the preferences of a decision maker among the bets in B ; as usual, \succ denotes the asymmetric part of \succsim , and \sim denotes the symmetric one. The crucial notion is the following.

Definition 1. A *Dutch book* consists of two arrays of simple bets $f^1, \dots, f^M, g^1, \dots, g^M \in B_0$ such that $f^j \succsim g^j$, for all $j = 1, \dots, M$, but $\sum_{j=1}^M f^j < \sum_{j=1}^M g^j$.

This notion represents something undesirable; therefore, *coherence* requires that no Dutch book is allowed. Whenever all involved bets are pairwise comonotonic we call the book a *comonotonic Dutch book*; if they are affinely related we call it *affine Dutch book*; finally, when they are constant, we call it *trivial Dutch book*.

In the sequel, we make use of the following properties of \succsim :

- *Weak Order*: For all f and g in B : $f \succsim g$ or $g \succsim f$. For all f, g , and h in B : if $f \succsim g$ and $g \succsim h$, then $f \succsim h$.
- *Monotonicity*: For all f and g in B : if $f \geq g$, then $f \succsim g$.
- *Fair Price*: For each f in B , there exists $\xi = \xi(f) \in \mathbb{R}$ such that $f \sim \xi$.¹
- *Uncertainty Aversion*: For all f and g in B : $f \sim g$ implies $\frac{1}{2}f + \frac{1}{2}g \succsim f$.²

Before stating the result a few more ingredients are needed.

A functional $V: B \rightarrow \mathbb{R}$ represents \succsim if $f \succsim g$ is equivalent to $V(f) \geq V(g)$; it is *monotonic* if $f \geq g$ implies $V(f) \geq V(g)$; it is a *prevision* if it is monotonic and $V(\beta \mathbf{1}_S) = \beta$ for all $\beta \in \mathbb{R}$; it is *constant-linear* if $V(af + b) = aV(f) + b$ for all $f \in B, a \geq 0$ and $b \in \mathbb{R}$.

A set function $C: \Sigma \rightarrow \mathbb{R}$ is a *capacity* if $C(\emptyset) = 0, C(S) = 1$, and $A \supseteq B$ implies $C(A) \geq C(B)$; it is a (*finitely additive*) *probability* if it is a capacity and $A \cap B = \emptyset$ implies $C(A \cup B) = C(A) + C(B)$; the set of set functions is endowed with the product topology. Finally, if C is a capacity, A_1, A_2, \dots, A_N form a partition of S in $\Sigma, \alpha_1 > \alpha_2 > \dots > \alpha_N \in \mathbb{R}$, and $f = \sum_{i=1}^N \alpha_i \mathbf{1}_{A_i}$ the (*Choquet*) *expectation* of f with respect to C is defined by

$$\int_S f dC = \sum_{i=1}^{N-1} (\alpha_i - \alpha_{i+1}) C\left(\bigcup_{j=1}^i A_j\right) + \alpha_N C(S),$$

the definition is extended to B by uniform approximation with simple bets. Notice that if C is a probability,

¹ $\xi(f)$ is also called *certainty equivalent* of f .

² *Uncertainty propension* and *uncertainty neutrality* are defined replacing \succsim with \lesssim and \sim , respectively.

then the expectation of f is simply

$$\int_S f dC = \sum_{i=1}^N \alpha_i C(A_i),$$

and its extension to B is the classic integral of a bounded function with respect to a finite, nonnegative, finitely additive set function (see, e.g., Aliprantis & Border, 1999, Chapter 11).

2.2. The result

We can now state the anticipated representation result.

Theorem 1. A binary relation \succsim on B is a monotonic weak order that satisfies the fair price property and allows no trivial Dutch books iff there exists a prevision $V: B \rightarrow \mathbb{R}$ representing \succsim . Moreover,

- (i) the relation \succsim allows no affine Dutch books iff the prevision V is constant-linear;
- (ii) the relation \succsim allows no comonotonic Dutch books iff there exists a capacity C on Σ such that $V(f) = \int_S f dC$ for all $f \in B$;
- (iii) the relation \succsim allows no Dutch books iff there exists a probability P on Σ such that $V(f) = \int_S f dP$ for all $f \in B$.

The prevision V is unique and for any bet it coincides with its fair price.

As stated in the Introduction, (i) characterizes invariant biseparable preferences, (ii) extending to the infinite case (Diecidue & Wakker, 2002) yields Choquet expectations, while (iii) is the famous de Finetti Dutch book Theorem. Among the many works referring to case (iii) of the theorem we mention: Berti, Regazzini, and Rigo (2002), Berti and Rigo (2000), Blackwell and Girshick (1954), Buehler (1976), Cifarelli and Regazzini (1996), Coletti (1990), Crisma, Gigante, and Millossovich (1997), Holzer (1985), Regazzini (1985), and Wakker (1989).

Proof. If $f \sim \xi \in \mathbb{R}$, set $V(f) = \xi$. V is well defined, in fact the fair price is unique. Assume the contrary: there exist $f \in B$ and $\xi > \gamma \in \mathbb{R}$ such that $f \sim \xi$ and $f \sim \gamma$, hence $\gamma \succ \xi$, but $\gamma < \xi$: a trivial Dutch book. It is easy to verify that V represents \succsim , in particular V is monotonic.

Assume \succsim allows no affine Dutch books. Next, we show that $V(f + \beta) = V(f) + \beta$ for all $f \in B_0$ and all $\beta \in \mathbb{R}$. Let $V(f + \beta) = \gamma, V(f) = \xi$ and, by contradiction $\gamma > \xi + \beta$. The bets $f + \beta, \xi, f$, and γ are affinely related, and $f + \beta \succ \gamma, \xi \succ f$, but $f + \xi + \beta < \gamma + f$: an affine Dutch book.

In the same manner, $V(nf) = nV(f)$ for all $f \in B_0$ and all $n \in \mathbb{N}$. Assume $V(nf) = \gamma$, $V(f) = \xi$ and $\gamma > n\xi$. The bets nf , ξ , f , and γ are affinely related, and $nf \succsim \gamma$, $\xi \succsim f$, but $nf + n\xi < nf + \gamma$: an affine Dutch book. Hence, $V(qf) = qV(f)$ for all $f \in B_0$ and all $q \in \mathbb{Q}_+$.

Let $f_n \in B_0$, and assume $f_n \rightarrow f \in B$ uniformly. There exists $\{\gamma_n\} \subseteq \mathbb{R}_+$ such that $\gamma_n \rightarrow 0$ and $f_n - \gamma_n \preceq f \preceq f_n + \gamma_n$, hence

$V(f_n) - \gamma_n = V(f_n - \gamma_n) \preceq V(f) \preceq V(f_n + \gamma_n) = V(f_n) + \gamma_n$, therefore $V(f_n) \rightarrow V(f)$. In particular, V is supnorm continuous in B_0 .

Let $\alpha > 0$, $f \in B_0$ and $\{q_n\} \subseteq \mathbb{Q}_+$ such that $q_n \rightarrow \alpha$. Hence $q_n f \rightarrow \alpha f$ uniformly, so $V(\alpha f) = \lim_n V(q_n f) = \lim_n q_n V(f) = \alpha V(f)$. In sum, V is constant-linear and supnorm continuous on B_0 . By uniform approximation with simple bets, it is easy to show that V is constant-linear and continuous on the whole B .

Next, we show that a preference represented by a constant-linear prevision allows no affine Dutch books. By contradiction, let $f^1 \succsim g^1, \dots, f^M \succsim g^M$ in B_0 be all affinely related, and $\sum_{j=1}^M f^j < \sum_{j=1}^M g^j$. Then $V(f^1) \geq V(g^1), \dots, V(f^M) \geq V(g^M)$, and by constant-linearity $V(\sum_{j=1}^M f^j) \geq V(\sum_{j=1}^M g^j)$. But $\sum_{j=1}^M f^j < \sum_{j=1}^M g^j$, constant linearity, and monotonicity imply $V(\sum_{j=1}^M f^j) < V(\sum_{j=1}^M g^j)$, which is absurd. This concludes the proof of (i).

Assume \succsim allows no comonotonic Dutch books. Next, we show that $V(f + g) = V(f) + V(g)$ for all comonotonic $f, g \in B_0$. Let $V(f + g) = \gamma$, $V(f) = \xi$ and $V(g) = \theta$ and $\gamma > \xi + \theta$. Then $f + g, \xi, \theta, \gamma, f, g$ are comonotonic, $f + g \succsim \gamma$, $\xi \succsim f$, $\theta \succsim g$, but $f + g + \xi + \theta < \gamma + f + g$: a Comonotonic Dutch book. By Schmeidler (1986) there exists a capacity C on Σ such that $V|_{B_0}(f) = \int_S f dC$ for all $f \in B_0$. V and $\int_S \cdot dC$ are both monotonic and constant-linear, hence continuous in B , and they coincide on B_0 which is dense in B , then $V = \int_S \cdot dC$. Which proves (ii), since the same argument we used to exclude affine Dutch books for preferences represented by a constant-linear prevision now permits to exclude comonotonic Dutch books for preferences represented by a Choquet integral. Clearly, the well known (iii) can be obtained in a similar way. \square

Next we consider the consequences of uncertainty aversion.

Corollary 2. *A binary relation \succsim on B is a uncertainty averse, monotonic weak order that satisfies the fair price property and allows no affine Dutch books iff there exists a (unique) compact and convex set \mathcal{C} of probabilities such that*

$$f \succsim g \Leftrightarrow \min_{P \in \mathcal{C}} \int_S f dP \geq \min_{P \in \mathcal{C}} \int_S g dP.$$

That is the prevision V representing \succsim is a minimum of expectations (see Gilboa & Schmeidler, 1989). Notice

that, in the special case in which \succsim allows no comonotonic Dutch books, this amounts to say that the capacity C representing \succsim , in the sense of point (ii) of Theorem 1, is convex.³ While the case in which no Dutch books are allowed is characterized by uncertainty neutrality.

Proof. Apply Theorem 1 to obtain a constant linear prevision V representing \succsim and then proceed like in Lemma 3.3 of Gilboa and Schmeidler (1989). \square

3. Conclusions

The Dutch book argument is now a classical coherence condition for the existence of subjective probabilities. In this paper we have investigated the possibility of using this type of argument to shed more light on some successful nonexpected utility models. The result is formally unifying and is still suggestive, even if it may lack some normative contents (which, however, face some criticism also in the original formulation; see, e.g., Schick, 1986). On the other hand, it is consistent with empirical data. As a matter of fact, it is possible to construct examples showing that Dutch books can pop up in real-life situations.

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³That is $C(A \cup B) + C(A \cap B) \geq C(A) + C(B)$ for all $A, B \in \Sigma$.

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