

# The Opportunity Cost and OCBA Selection Procedures in Ordinal Optimization for a Fixed Number of Alternative Systems<sup>+</sup>

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## Abstract

Ordinal Optimization offers an efficient approach for simulation optimization by focusing on ranking and selecting a finite set of good alternatives. Because simulation replications only give estimates of the performance of each alternative, there is a potential for incorrect selection. Two measures of selection quality are the alignment probability or the probability of correct selection ( $P\{CS\}$ ), and the expected opportunity cost,  $E[OC]$ , of a potentially incorrect selection. Traditional ordinal optimization approaches focus on the former case. This paper extends the optimal computing budget allocation (OCBA) approach of [2], which allocated replications to improve  $P\{CS\}$ , to provide the first OCBA-like procedure that optimizes  $E[OC]$  in some sense. The procedure performs efficiently in numerical experiments.

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## I. Introduction

Discrete-event systems (DES) simulation is a popular tool for analyzing systems and evaluating decision problems since real situations, more often than not, do not satisfy the assumptions of analytical models. A sample application is to choose the best inventory policy from several alternatives. DES simulation has many advantages for modeling complex systems and has been studied in several cases (cf. [8]). However, efficiency is still a significant concern when conducting simulation experiments [17]. Good estimates of the mean performance of a system (expressed as a confidence interval) typically have errors of size  $O(1/\sqrt{N})$ , the result of averaging independent and identically distributed (i.i.d.) noise, where  $N$  is the number of simulation samples, or replications. *Ordinal Optimization* has emerged as an effective approach in DES when the primary objective is to select the design with the best mean performance, *relative* to the others, rather than accurately estimating the mean performance of each alternative [12]. The error rate for identifying the best system decays exponentially in some cases, as opposed to the square root law for estimating means [9]. This idea has been successfully applied to several problems (e.g., [11], [13], [18], [22]).

While ordinal optimization could significantly reduce the computational cost for DES simulation, there is potential to further improve its performance by intelligently controlling the simulation experiments, i.e., by determining the best number of simulation samples among different designs as simulation proceeds. The main theme of this paper is to further enhance the efficiency of ordinal optimization in simulation experiments. The efficiency of ordinal optimization approaches relies on the effectiveness of the underlining selection procedure. Selection procedures are statistical sampling techniques used to efficiently identify the best or some top ones of a finite set of simulated alternatives, where ‘best’ is defined with respect to the mean performance of each alternative. There are two main measures of selection quality, and three main approaches to framing the problem of selecting the best system. We describe the measures of selection quality before describing the main approaches. The alignment probability or the probability of correct selection ( $P\{CS\}$ ) is the most commonly studied measure of performance. A second important measure of selection quality is the expected opportunity cost ( $E[OC]$ ), which penalizes particularly bad choices more than mildly bad choices. For example, it may be better to be wrong 99% of the time if the penalty for being wrong is \$1 (an expected opportunity cost of  $0.99 \times \$1 = \$0.99$ ) rather than being wrong only 1% of the time if the penalty is \$1,000 (an expected opportunity cost of  $0.01 \times \$1,000 = \$10$ ) [5]. Quantiles are another measure, but is beyond the scope of this paper.

The indifference-zone approach mainly focuses on guaranteeing a prespecified probability of correct selection ( $P\{CS\}$ ), the probability taken over repeated applications to any problem instance taken from a given class of problems. In the simulation context, this usually means all problems so that the mean performance of the “best” system is at least  $\delta$  units better than each of the alternatives, where  $\delta$  is the

minimum difference worth detecting (e.g., [1], [20]). [7] present a two-stage indifference-zone procedure that provides an ‘unexpected’ expected opportunity cost guarantee, and show that some existing indifference-zone selection procedures that were originally developed to provide  $P\{CS\}$  guarantees can be adapted to provide  $E[OC]$  guarantees. A second approach to selection procedures is the Bayesian, expected value of information approach of [6-7], that can allocate samples in two stages or sequentially to improve either  $P\{CS\}$  or  $E[OC]$ .

The third approach, and the approach espoused in this paper, is the OCBA [2], which allocates samples sequentially in order to maximize an approximation to the Bayesian posterior  $P\{CS\}$ . This is a nonlinear optimization problem. [3] introduce a greedy heuristic to iteratively determine which design appears to be the most promising for further simulation. OCBA procedures have been shown to be efficient empirically (e.g., [15]), but so far variations only exist to improve  $P\{CS\}$ .

In this paper, we develop a new greedy selection procedure to reduce the  $E[OC]$  of a potentially incorrect selection by using the OCBA approach to selection. The main idea is to allocate samples sequentially so that an approximation to  $E[OC]$  can be minimized when the simulations are finished. In addition to presenting a new efficient simulation budget allocation, this paper i) compares several different budget allocation procedures through a series of numerical experiments; ii) demonstrates that our budget allocation approaches are much more efficient than procedures that allocate samples evenly between systems or proportional to the variance of each system; and iii) shows that our approach is robust in different settings. Section II formulates the allocation problem. Since our approach is based on a Bayesian model, a brief discussion of that model is included. Sections III and IV present a greedy heuristic allocation rule for improving  $E[OC]$  using the OCBA approach. The performance of the technique is illustrated with a series of numerical examples in Section V. Section VI summarizes the conclusions of the paper – the new OCBA-EOC procedure compares quite favorably with several other procedures.

## II. Problem Statement

This section recalls a formal description of the selection problem from a Bayesian perspective, and then discusses the expected opportunity cost, ( $E[OC]$ ). The goal of the selection procedure is to identify the best of  $k$  simulated alternatives, where best is defined as the system with the smallest mean (the largest mean would be handled similarly), which is unknown and to be inferred from simulation. The selection procedure is developed with the assumption that the output is normally distributed with an unknown mean and a known variance. The normality assumption is generally not a problem, because typical simulation output is obtained from an average performance or batch means, so that Central Limit Theorem effects usually hold. Numerical experiments show that the new procedure works well even when the variance is estimated, and cases tested where the distribution is not normally distributed. The output of each system (design) is presumed to be independent in this paper. Common random numbers are not allowed in this paper. Denote by

$\mu_i$  : the unknown mean of design  $i$ ,

$\mathbf{s}_i^2$  : the known variance of design  $i$ . In practice,  $\mathbf{s}_i^2$  is unknown beforehand

and so it is approximated by the sample variance.

$\bar{X}_i$  : the sample mean of design  $i$ ,

$S_i^2$  : the sample variance of design  $i$ ,

$N_i$  : the number of simulation replications (so far) for design  $i$ .

The sample statistics are updated continually as simulations are run. We assume a noninformative prior distribution for the unknown mean, so that the posterior distribution for the unknown mean of system  $i$ , given  $N_i$  simulation replications so far, is ([4], [10], [14]):

$$\mathbf{m}_i \sim N(\bar{X}_i, \frac{\mathbf{s}_i^2}{N_i}), \quad i=1, \dots, k \quad (1)$$

After the simulation is performed,  $\bar{X}_i$  can be calculated according to the system outputs, and  $\mathbf{s}_i^2$  is approximated by the corresponding sample variance.

#### A. Opportunity Cost (OC)

The opportunity cost is “the cost of any activity measured in terms of the best alternative forgone.” [21]. For the simulation selection problem, the opportunity cost is the difference between the unknown mean of the selected system  $b$  (defined below) and the unknown mean of the actual best system. Set

$$\tilde{J}_{i,j} = \mathbf{m}_i - \mathbf{m}_j.$$

Because both  $\mathbf{m}_i$  and  $\mathbf{m}_j$  have a normal posterior distribution, the random variable

$\tilde{J}_{i,j}$  is normally distributed when  $i \neq j$ ,

$$\tilde{J}_{i,j} \sim N(\bar{X}_i - \bar{X}_j, \frac{\mathbf{s}_i^2}{N_i} + \frac{\mathbf{s}_j^2}{N_j}).$$

Denote by

$b$  : the design with the smallest sample mean so far, so  $b = \arg \min_i \{ \bar{X}_1, \bar{X}_2, \dots,$

$\bar{X}_k \}$  is the design we would select if no more replications were run,

$i^*$  : the true best design,  $i^* = \arg \min_i \{ \mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k \}$ ,

$f_{i,j}(x)$  : the probability density function (PDF) of the normally distributed

random variable,  $\tilde{J}_{i,j}$ , evaluated at  $x$ , for  $i \neq j$ .

The opportunity cost is therefore

$$\tilde{J}_{b,i^*} = \mathbf{m}_b - \mathbf{m}_{i^*} = \begin{cases} = 0 & \text{if } b = i^* \\ \geq 0 & \text{if } b \neq i^*. \end{cases}$$

If the best system is correctly selected, then the opportunity cost is zero.

### B. The Expected Opportunity Cost E[OC]

The expected opportunity cost (E[OC]) is defined as follows

$$E[\text{OC}] = E[\tilde{J}_{b,i^*}] = E[\mathbf{m}_b - \mathbf{m}_{i^*}] = \sum_{i=1, i \neq b}^k P(i = i^*) E[\mathbf{m}_b - \mathbf{m}_i | i = i^*].$$

The expectation is taken with respect to the posterior distribution of the  $\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k$  given all simulation seen so far. According to the definition of the conditional expectation [23],

$$E(X|A) = \frac{E[X \cdot I(A)]}{E[I(A)]} = \frac{E[X \cdot I(A)]}{P(A)},$$

where  $X$  is a random variable, and for an event  $A$  we define the *indicator function* to be  $I(A) = \begin{cases} 1 & \text{if } A \text{ does occur.} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$ . Therefore,

$$E[\text{OC}] = \sum_{i=1, i \neq b}^k E[I(i = i^*) \times (\mathbf{m}_b - \mathbf{m}_i)].$$

A convenient closed-form expression for E[OC] is unknown for all but a few special cases. We therefore present an approximation of E[OC] that is key for the new selection procedure. Denote the vector of means by  $\boldsymbol{\mu} = \{\mathbf{m}_1, \mathbf{m}_2, \dots, \mathbf{m}_k\}$ , denote the set of configurations where  $i$  is best by  $W_i \equiv \{\boldsymbol{\mu} : i^* = i\} = \{\boldsymbol{\mu} : \mathbf{m}_i < \mathbf{m}_j, \text{ for all } j \neq i\}$ , and denote the set of configurations where the true mean of system  $i$  is better than the true mean of the system whose sample mean is currently best by  $\Omega_i \equiv \{\boldsymbol{\mu} : \mathbf{m}_i < \mathbf{m}_b\}$ . Since  $W_i \subseteq \Omega_i$ ,

$$P(i = i^*) = P(\mathbf{m}_i < \mathbf{m}_b) * P(\mathbf{m}_i < \mathbf{m}_j \text{ for all } j \in \{1, b\} | \mathbf{m}_i < \mathbf{m}_b) \leq P(\mathbf{m}_i < \mathbf{m}_b).$$

We establish an upper bound,

$$E[I(i = i^*) \times (\mathbf{m}_b - \mathbf{m}_i)] \leq E[I(\mathbf{m}_i < \mathbf{m}_b) \times (\mathbf{m}_b - \mathbf{m}_i)].$$

Thus,

$$E[\text{OC}] \leq \sum_{i=1, i \neq b}^k E[I(\mathbf{m}_i < \mathbf{m}_b) \times (\mathbf{m}_b - \mathbf{m}_i)]$$

$$= \sum_{i=1, i \neq b}^k P(\bar{\mathbf{m}} < \bar{\mathbf{m}}_b) E[\bar{\mathbf{m}}_b - \bar{\mathbf{m}}_i | \bar{\mathbf{m}}_i < \bar{\mathbf{m}}_b] = EEOC,$$

and we refer to this upper bound of the expected opportunity cost as the *Estimated Expected Opportunity Cost (EEOC)*. Since system  $b$  has the best (smallest) sample mean so far, ( $\bar{X}_b < \bar{X}_i$  for all  $i$ ), the conditional probability,  $P(\bar{\mathbf{m}} < \bar{\mathbf{m}}_j$  for all  $j \notin \{i, b\} | \bar{\mathbf{m}} < \bar{\mathbf{m}}_b)$ , will often be close to 1, so that  $P(\bar{\mathbf{m}} < \bar{\mathbf{m}}_b)$  is often a good approximation to  $P(i = i^*)$ . Similarly we hope  $E[(\bar{\mathbf{m}}_i < \bar{\mathbf{m}}_b) * (\bar{\mathbf{m}}_b - \bar{\mathbf{m}}_i)]$  would be a reasonable approximation to  $E[I(i = i^*) * (\bar{\mathbf{m}}_b - \bar{\mathbf{m}}_i)]$ . Numerical tests below show that using *EEOC* to approximate  $E[OC]$  leads to a highly efficient procedure. According to the definition of *EEOC* and definition of  $\tilde{J}_{i,j}$  in Section II.A,

$$\begin{aligned} EEOC &= \sum_{i=1, i \neq b}^k P(\tilde{J}_{b,i} > 0) E[\tilde{J}_{b,i} | \tilde{J}_{b,i} > 0] \\ &= \sum_{i=1, i \neq b}^k \int_0^{+\infty} x f_{b,i}(x) dx, \end{aligned}$$

where  $f_{b,i}(x)$  is the PDF of  $\tilde{J}_{b,i}$ . Define  $s_{b,i}^2 = \frac{\mathbf{s}_i^2}{N_i} + \frac{\mathbf{s}_b^2}{N_b}$  and  $d_{b,i} = \bar{X}_b - \bar{X}_i$ , and let

$z_{b,i} = -d_{b,i}/s_{b,i}$  be the standardized statistic for the difference in means for systems  $b$  and  $i$ . Using integration by parts, one can show that

$$\int_0^{+\infty} x f_{b,i}(x) dx = s_{b,i} f(z_{b,i}) + d_{b,i} * \Phi(-z_{b,i}),$$

where  $f(x)$  and  $\Phi(x)$  are the PDF and CDF of standard normal distribution respectively.

Then we get

$$\begin{aligned} EEOC &= \sum_{i=1, i \neq b}^k \int_0^{+\infty} x f_{b,i}(x) dx \\ &= \sum_{i=1, i \neq b}^k \{ s_{b,i} f(z_{b,i}) + d_{b,i} * \Phi(-z_{b,i}) \}. \quad \text{--- (2)} \end{aligned}$$

This is easily and stably computable by many mathematical packages. We therefore use *EEOC* in the above equation to approximate  $E[OC]$  in our selection procedure.

The term *EEOC* can be interpreted as the sum of expected opportunity costs for each of the  $k-1$  pairwise comparisons between the best and each alternative, given all of the sampling information that is available. This term therefore resembles the well-known Bonferroni bound for probabilities, which states that the probability that the best system is not correctly selected, is less than the sum of the probabilities that the best is not correctly selected in the  $k-1$  pairwise comparisons of the best with each alternative. The Bonferroni bound is tight when  $k=2$ , and is known to become less tight as  $k$  increases. If only a few competing alternatives have a similar performance, then the bound can be relatively tight. Plotting a few numerical examples shows that the same is true for the *EEOC* bound.

### III. Approximation to the Estimated Expected Opportunity Cost (EEOC)

Ideally we would choose  $N_1, N_2, \dots, N_k$  to minimize  $E[OC]$ , given a limited computing budget. Since it can be very expensive to compute  $E[OC]$ , this paper sequentially minimizes *EEOC* by using (2). Using the basic ideas of the OCBA approach, as they were applied for  $P\{CS\}$ , we now allocate a few replications at each stage of a sequential procedure to reduce *EEOC* iteratively.

A critical component in the proposed procedure is to estimate how *EEOC* changes as  $N_i$  changes. Let  $\Delta_i$  be a nonnegative integer denoting the number of additional simulation replications allocated to design  $i$  in the next stage of sampling. We are interested in assessing how *EEOC* would be affected if design  $i$  were simulated for  $\Delta_i$  additional replications. In other words, we are interested in assessing how *promising* it is to simulate design  $i$ . This assessment must be made *before* actually conducting  $\Delta_i$  simulation replications. According to (1) in Section 2, if we conduct  $\Delta_i$  additional replications of design  $i$ , given a finite  $N_i$ , the posterior distribution for the unknown mean of design  $i$  is approximately

$$N\left(\frac{1}{N_i + \Delta_i} \sum_{j=1}^{N_i + \Delta_i} X_{ij}, \frac{\mathbf{s}_i^2}{N_i + \Delta_i}\right),$$

where  $X_{ij}$  is the  $j$ -th sample of design  $i$ . When  $\Delta_i$  is relatively small compared to  $N_i$ , the

difference between  $\frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij}$  and  $\frac{1}{N_i + \Delta_i} \sum_{j=1}^{N_i + \Delta_i} X_{ij}$  will be small. A heuristic approach

to the approximation of the predictive posterior distribution yields

$$\mathbf{m} \sim N\left(\frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij}, \frac{\mathbf{s}_i^2}{N_i + \Delta_i}\right) \quad \text{for design } i. \quad \text{--- (3)}$$

We therefore approximate the distribution of the unknown mean, given that  $\Delta_i$  is

small, by a  $N\left(\frac{1}{N_i} \sum_{j=1}^{N_i} X_{ij}, \frac{\mathbf{s}_i^2}{N_i + \Delta_i}\right)$  distribution.

The *EEOC* can then be calculated by plugging (3) into the *EEOC* formula in (2). In particular, if we allocate  $\Delta N_i$  additional samples for system  $i$  (for any  $i$ , including

system  $b$ ), then the corresponding estimated expected opportunity cost, denoted as  $EEOC_i$ , is determined by using the original distributions for the unknown means of systems other than  $i$ , and using the modified distribution immediately above for system  $i$ , to obtain

$$EEOC_i = \sum_{j=1, j \neq b}^k \int_0^{+\infty} x f_{b,j,i}^*(x) dx \quad \text{for } i = 1, \dots, k;$$

where  $f_{b,j,i}^*(x)$  is the PDF of the difference between system  $b$  and system  $j$ , given that  $\Delta N_i$  more replications are given to system  $i$ , and none is allocated to the others, for any  $i$  and  $j \neq b$ . If  $i=b$ , then  $f_{b,j,i}^*(x)$  is the PDF of a  $N(\bar{X}_b - \bar{X}_j, \frac{\mathbf{s}_b^2}{N_b + \Delta N_b} + \frac{\mathbf{s}_j^2}{N_j})$  random variable. If  $i=j$ , then  $f_{b,j,i}^*(x)$  is the PDF of a  $N(\bar{X}_b - \bar{X}_j, \frac{\mathbf{s}_b^2}{N_b} + \frac{\mathbf{s}_j^2}{N_j + \Delta N_j})$  random variable. If  $i$  is neither  $j$  nor  $b$ , then no new information is available to distinguish systems  $j$  and  $b$ , and  $f_{b,j,i}^*(x) = f_{b,j}(x)$ .

$EEOC_i$  is the estimated expected opportunity cost before an additional  $\Delta N_i$  replications are applied for system  $i$ , for any  $i$ , and  $EEOC_i$  can be computed easily.

#### IV. A Selection Procedure

Our goal is to minimize the E[OC] of a potentially incorrect selection. Doing so optimally and in full generality is a challenging optimization problem with an unknown solution. Inspired by the OCBA approach (which provides a heuristic that seeks to minimize a bound on the probability of correct selection, and where that bound is defined in a manner that is similar in spirit to the bounds and approximations in Section III above), we reduce E[OC] by sequentially minimizing the upper bound in Equation (2),  $EEOC$ , for E[OC]. At each stage, we allocate additional samples in a manner that greedily reduces the  $EEOC$  as much as possible. That improvement is based upon a heuristic measure of the improvement in  $EEOC$  at each step that is justified by the development in Section III. The effect of running an additional few replications on the expected opportunity cost,  $EEOC$ , is estimated by:

$$\begin{aligned} D_i &= EEOC_i - EEOC \\ &= \int_0^{+\infty} x f_{b,j,i}^*(x) dx - \int_0^{+\infty} x f_{b,i}(x) dx \leq 0, \quad i \neq b, \end{aligned}$$

and

$$\begin{aligned} D_b &= EEOC_b - EEOC \\ &= \sum_{i=1, i \neq b}^k \left\{ \int_0^{+\infty} x f_{b,i,b}^*(x) dx - \int_0^{+\infty} x f_{b,i}(x) dx \right\} \leq 0, \end{aligned}$$

where  $D_i$  and  $D_b$  represent the reduction of  $EEOC_i$  and  $EEOC_b$  respectively at each



step. These inequalities can be verified with a bit of algebra. They imply that more information leads to smaller losses, on average.

Note that before conducting the simulation, neither the *APCS* nor a good way to allocate the simulation budget is known. Therefore, all designs are initially simulated with  $n_0$  replications in the first stage. We then sequentially allocate replications to the systems that provide the greatest reduction in the *EEOC*. Note the  $D_i$  and  $D_b$  are less than zero. This is consistent with our intuition that the expected opportunity cost will decrease as more samples are observed. With this approximation, the  $\Delta N_i$ ,  $i = 1, \dots, k$ , should not be too large. In summary, we have the following new selection procedure.

### **OCBA-EOC Allocation Procedure**

- Step 0.** Choose a number of systems per stage to simulate,  $m$ , and a (small) total number of simulation replications per stage,  $\Delta$  such that  $t = \Delta/m$  is an integer, and a total number of replications  $T$  to run (assuming  $T - kn_0$  is a multiple of  $\Delta$ , where  $k$  is the total number of designs and  $n_0$  is number of simulation replications for each design that are run in an initial stage).
- Step 1.** Perform  $n_0$  simulation replications for each of the  $k$  designs.
- Step 2.** Select system  $b = \arg \min_i \{ \bar{X}_1, \bar{X}_2, \dots, \bar{X}_k \}$  as the best so far.
- Step 3.** For  $i=1, \dots, k$ , approximate the improvement in *EEOC* by:  

$$D_i \equiv EEOC_i - EEOC.$$
- Step 4.** Find the set  $S(m) \equiv \{ i : D_i \text{ is among the } m \text{ lowest values} \}$ .
- Step 5.**  $\Delta N_i = t$ , for all  $i \in S(m)$ ; otherwise,  $\Delta N_i = 0$ .
- Step 6.** Update all statistics and repeat Steps 2-6 until a total of  $T$  replications are observed.

Since this selection procedure is derived using the framework presented in [4], we named it OCBA-EOC to be consistent with the previous allocation procedure ‘Optimal Computing Budget Allocation (OCBA)’. The original OCBA is  $P\{CS\}$  oriented, whereas our new procedure in this paper is focused on reducing  $E[OC]$  by sequentially minimizing the approximation to *EEOC*, from Section III above, in a greedy manner at each iteration. To distinguish these two procedures, we denote the original OCBA by the name OCBA-PCS in later sections.

## **V. Numerical Experiments**

### **A. Summary of procedures**

We compared the new OCBA-EOC procedure with five other procedures to compare their relative effectiveness at identifying the best system. They are measured with respect to two measures of the quality of a correct selection: (i) the empirical probability of correct selection,  $P\{CS\}$ , and (ii) expected opportunity cost of a potentially incorrect selection,  $E[OC]$ , as a function of the total number of

replications taken during the simulation. Both  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are frequentist measures, and are estimated by averaging over many applications of a procedure to a given selection problem.

### **Equal Allocation**

This is the simplest way to conduct simulation experiments and has been widely applied. The simulation budget is equally allocated to all designs, so that all designs are simulated equally often. Such a procedure is equivalent to the sole use of ordinal optimization. The ordinal optimization can ensure that  $P\{CS\}_{iz}$  converges to 1.0 exponentially fast, even with equal allocation. The performance of equal allocation will serve as a benchmark for comparison.

### **Proportional to Variance (PTV) Allocation**

The two-stage procedure of Rinott [20] has been widely applied in the simulation literature. In the first stage, all designs are simulated for  $n_0$  samples. The number of additional simulation samples for each design in the second stage is proportional to the sample variance ( $S_i^2$ ) from the first stage of sampling,

$$n_i = \max(0, \lceil (S_i^2 h^2 / d^2) - n_0 \rceil), \text{ for } i = 1, 2, \dots, k, \quad \text{--- (4)}$$

where  $\lceil \bullet \rceil$  is the integer "round-up" function,  $d$  is minimum difference worth detecting,  $h$  is a constant that solves Rinott's integral (see [1], [24]). To get a sequential procedure, we modify Rinott's procedure to allocate samples so that the total number of replications grows in proportion to the sample variance. While we do not claim that the indifference-zone  $P\{CS\}_{iz}$  guarantee is maintained, this maintains the spirit of allocating replications proportional to the sample variance.

### **LL(S) and 0-1(S) Allocation**

The  $LL(S)$  and  $0-1(S)$  procedures [5] are sequential versions of two-stage procedures that were developed to improve the expected value of information of additional samples for the opportunity cost ( $LL$  stands for linear loss, another name for opportunity cost) and 0-1 loss functions (probability of incorrect selection). Both procedures function well in numerical experiments, although  $0-1(S)$  performs less well than  $LL(S)$  due to an extra asymptotic approximation in its derivation [15].

### **OCBA-PCS by Chen et al. (2000)**

The OCBA-PCS procedure is derived with asymptotic approximations, together with an approximation of the posterior probability of correct selection. There are several variations on the OCBA-PCS that are based upon various approximations. Numerical testing demonstrates that the following variation of the OCBA-PCS is very efficient and can dramatically reduce the number of simulation replications.

- $\frac{N_i}{N_j} = \left( \frac{\mathbf{s}_i/d_{b,i}}{\mathbf{s}_j/d_{b,j}} \right)^2, i, j \in \{1, 2, \dots, k\}, \text{ and } i \neq j \neq b,$  --- (5)

- $N_b = \mathbf{s}_b \sqrt{\sum_{i=1, i \neq b}^k \frac{N_i^2}{\mathbf{s}_i^2}}.$  --- (6)

This allocation is a function with respect to the difference of sample means and the variances. The variances are approximated by sample variances in practice.

### **OCBA-EOC Procedure**

The OCBA-EOC procedure is introduced in this paper. The details of the procedure are described in Section IV.

In summary, the key structural feature of the OCBA-EOC and *LL* procedures is that their greedy optimization steps focus on reducing the probability of particularly bad errors. That is, they weight each potential error by the amount worse that a potentially incorrect selection is worse than the true best. The OCBA-PCS and *0-1* procedures focus on reducing errors, but equally weight all potential errors, rather than treating potentially large errors differently than small errors. The PTV and Equal allocations are easy to describe and to implement, but do not appear to have any easy-to-describe weighting scheme for the relative size of errors.

### **B. Numerical Results**

In order to compare the performance of the six procedures in previous subsection, we tested them empirically for several typical selection problems. In all of the following figures, *T* represents the total computing budget. For all experiments in this paper, *m*=1. A summary is presented after presenting the results of each example.

#### **■ Example 1 : Monotone means and constant variances**

The first example assumed that there are 10 systems, and that the alternatives have  $N(i, \sigma^2)$  distribution, for system  $i = 1, 2, \dots, 10$ . We implemented  $n_0 = 10$  first-stage replications per system, and  $\Delta = 10$  replications per stage until a total of  $T = 5000$  replications are run per application of each procedure. Estimates of  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are based on 100,000 applications of each procedure to the problem.

The LL, OCBA-EOC, and OCBA-PCS procedures each do best for both the  $P\{CS\}_{iz}$  and the  $E[OC]_{iz}$  criteria. The Equal allocation is relatively much less effective.

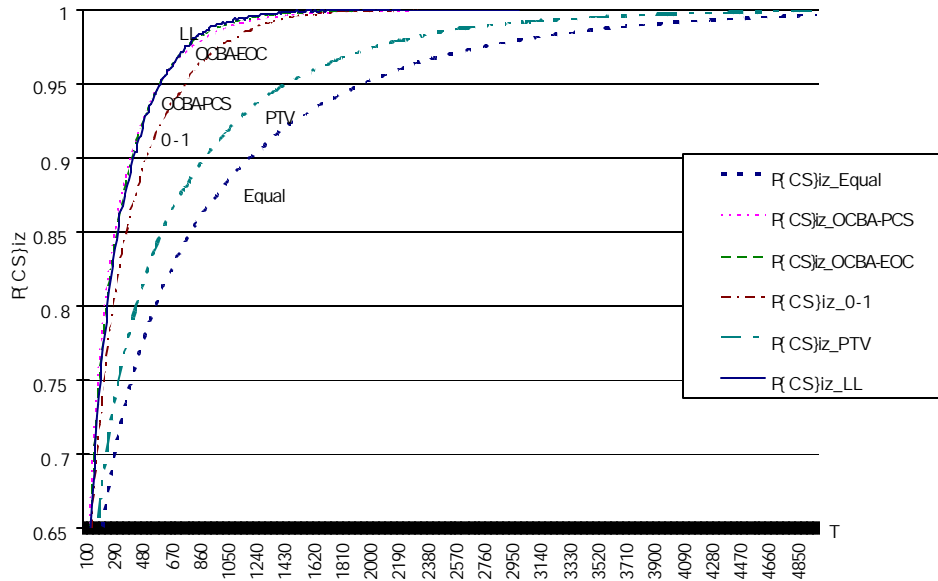


Figure 1a. Performance comparison of  $P\{CS\}z$  in Example 1

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	3930	2880	990	1060	1020	1280

Table 1a. Computation costs in order to attain  $P\{CS\}z = 99\%$  for Example 1.

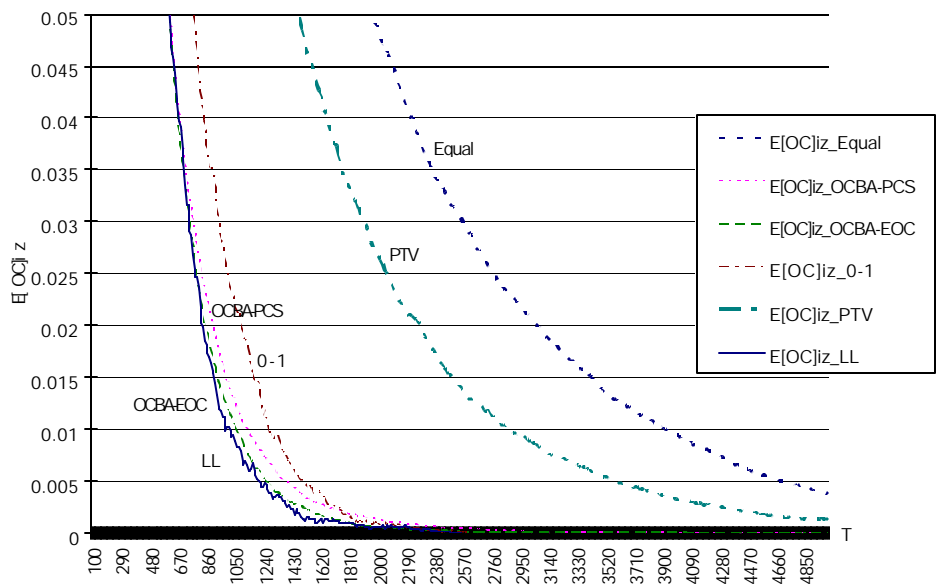


Figure 1b Performance comparison of  $E[OC]z$  in Example 1

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	3420	2500	990	930	910	1180

Table 1b. Computation costs in order to attain  $E[OC]_{iz} = 0.015$  for Example 1.

■ **Example 2: Monotone means and increasing variances**

The second example is a variant of example 1 with increasing variances. This example assumed that there are 10 systems, and that the alternatives have  $N(i, i^4)$  distribution, for system  $i = 1, 2, \dots, 10$ . We implemented  $n_0 = 10$  first stage replications per system, and  $\Delta = 10$  replications per stage until a total of  $T = 5000$  replications are run per application of each procedure. Estimates of  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are based on 100,000 applications of each procedure to the problem.

The 0-1 procedure does best for low levels of evidence for correct selection, but becomes worse as higher probabilities of correct selection are desired. The OCBA-PCS and OCBA-EOC, with the LL procedure, are best as more samples are taken.

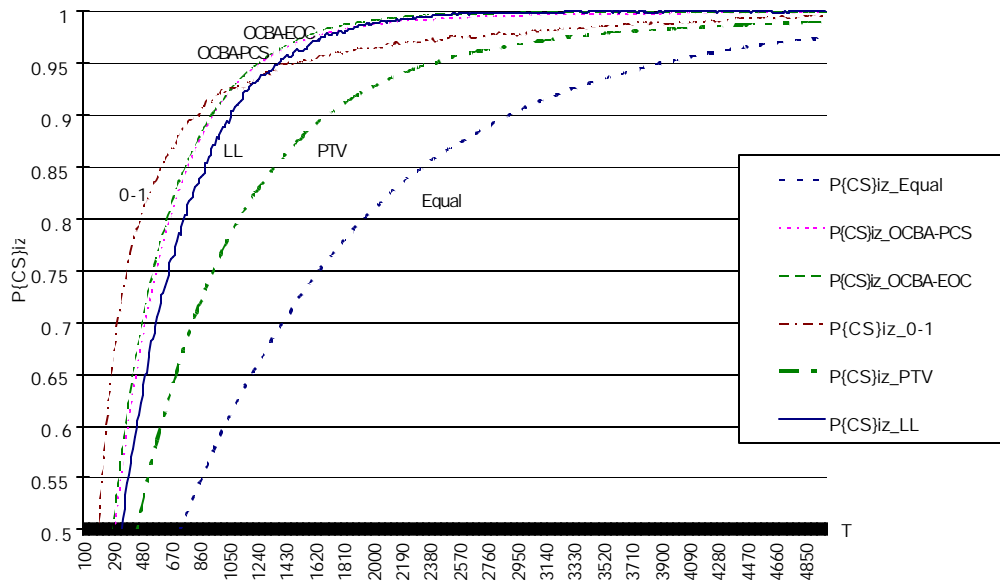


Figure 2a Performance comparison of  $P\{CS\}_{iz}$  in Example 2

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	>5000	4860	2150	1920	2060	4120

Table 2a. Computation costs in order to attain  $P\{CS\}_{iz} = 99\%$  for Example 2

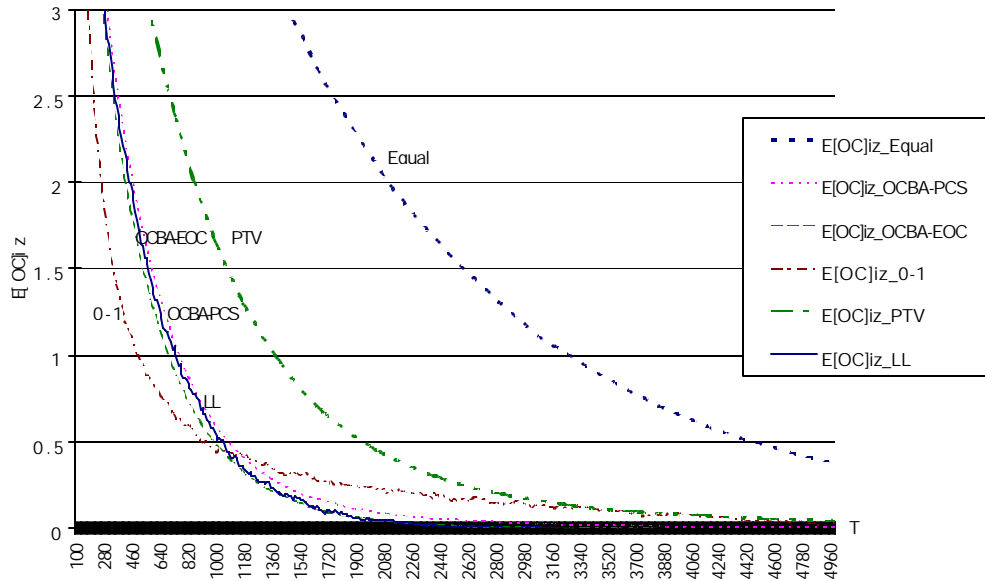


Figure 2b Performance comparison of  $E[OC]_{iz}$  in Example 2

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	>5000	4310	2300	1920	1900	4210

Table 2b. Computation costs in order to attain  $E[OC]_{iz} = 0.065$  for Example 2.

■ **Example 3: Monotone means and decreasing variances**

The third example assumed that there are 10 systems, and that the alternatives have  $N(i, (11-i)^4)$  distribution, for system  $i = 1, 2, \dots, 10$ . We implemented  $n_0 = 500$  first stage replications per system, and  $\Delta = 10$  replications per stage until a total of  $T = 5000$  replications are run. Estimates of  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are based on 100,000 applications of each procedure, except  $LL$  and  $0-1$  procedures. Since we ran  $LL$  and  $0-1$  under Matlab, 100,000 applications would take long time to run, so we reduced the number of applications of  $LL$  and  $0-1$  to 5000. The Matlab implementations run about the same speed for all procedures (OCBA-PCS, OCBA-EOC,  $LL(S)$ ,  $0-1(S)$ , etc.).

Because of the structure of the problem, including a larger variance than for the previous examples, many more samples are required to achieve a similar level of evidence for correct selection. PTV performs as well as the better procedures, unlike in other experiments, apparently due to the special relationship of the means and variances that are specific to this problem.

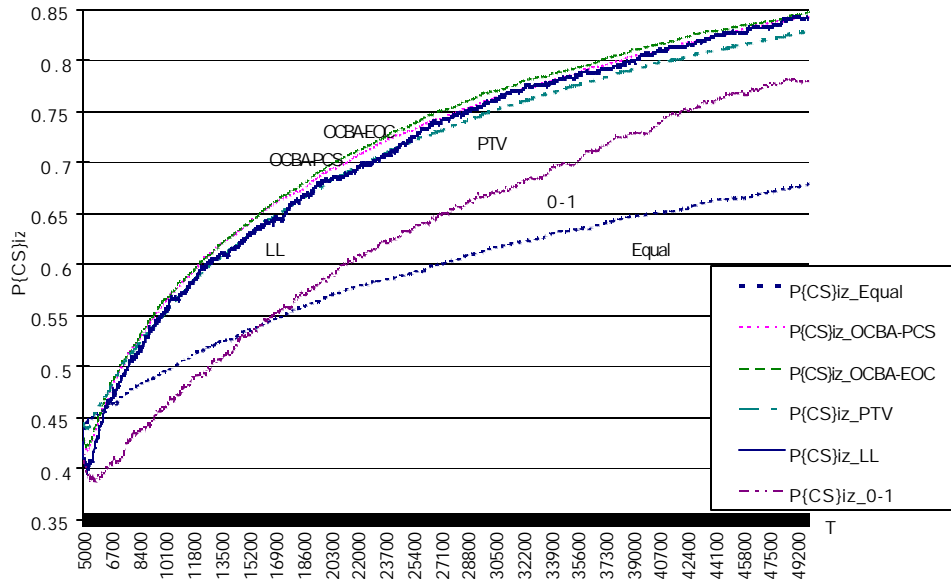


Figure 3a Performance comparison of  $P\{CS\}jz$  in Example 3

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	>50000	41450	38350	37040	39120	>50000

Table 3a. Computation costs in order to attain  $P\{CS\}jz = 80\%$  for Example 3.

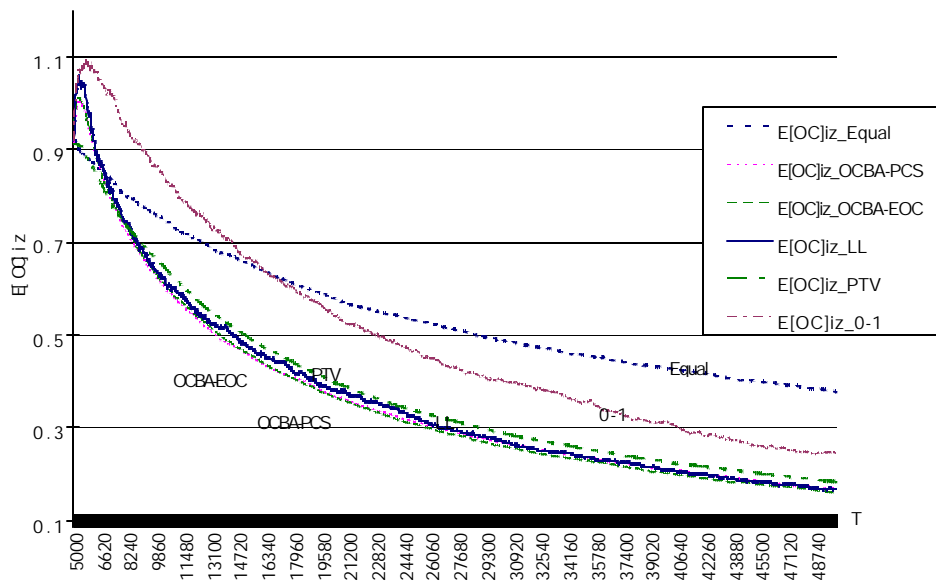


Figure 3b Performance comparison of  $E[OC]jz$  in Example 3

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	>50000	41520	38310	36840	38330	>50000

Table 3b. Computation costs in order to attain  $E[OC]_{iz} = 0.22$  for Example 3.

■ **Example 4: Slippage Configuration Problem with constant variances**

The slippage configuration (SC) assumed that there are 10 systems, and that the alternatives have  $N(m_i, 5^2)$  distribution, where  $m_1 = 0$  and  $m_i = 1$  when  $i = 2, \dots, 10$ .

We implemented  $n_0 = 200$  first stage replications per system, and  $\Delta = 10$  replications per stage until a total of  $T = 7000$  replications are run per application of each procedure. Estimates of  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are based on 100,000 applications of each procedure to the problem.

The OCBA-PCS, OCBA-EOC and LL procedures are again the best for both figures of merit.

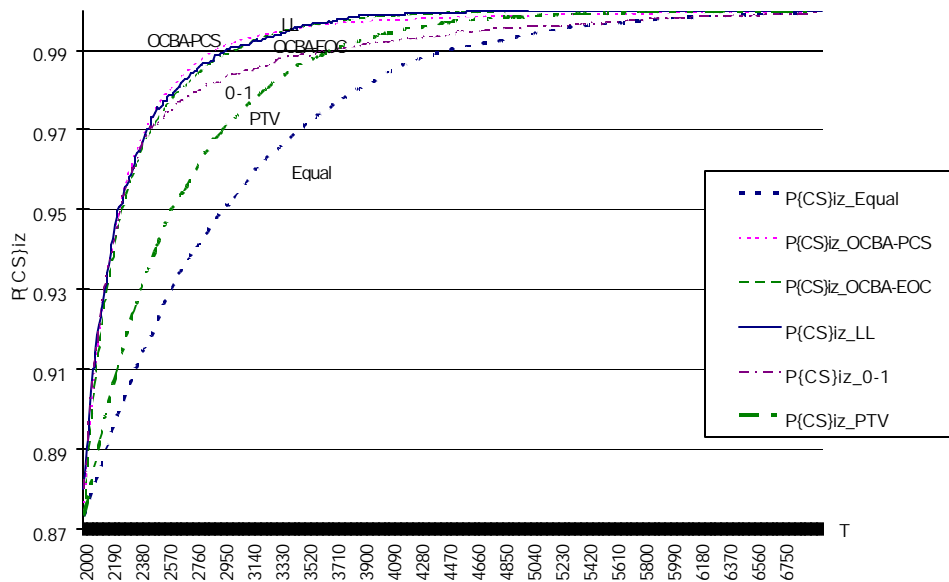


Figure 4a Performance comparison of  $P\{CS\}_{iz}$  in Example 4

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	4450	3710	2890	3020	2970	3720

Table 4a. Computation costs in order to attain  $P\{CS\}_{iz} = 99\%$  for Example 4



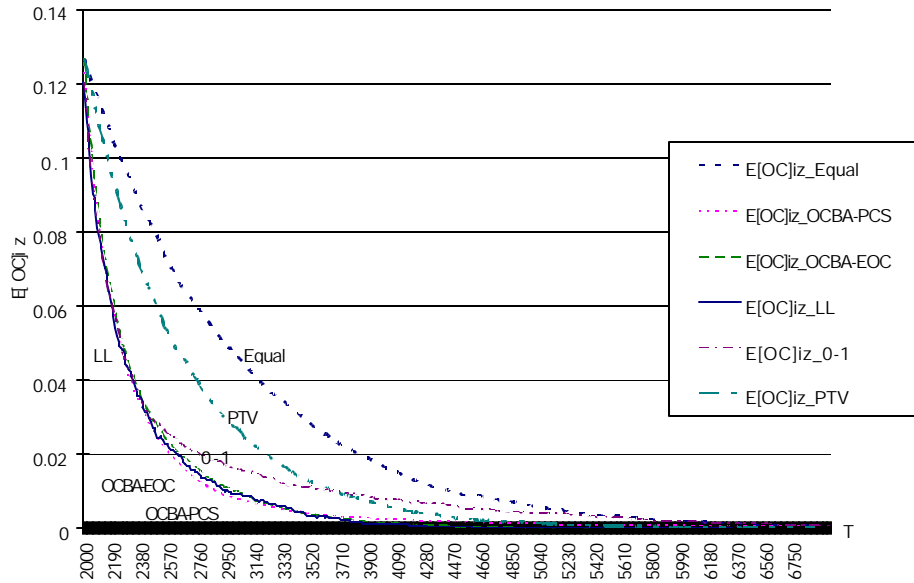


Figure 4b Performance comparison of  $E[OC]_{iz}$  in Example 4

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	4450	3680	2890	3010	2970	3710

Table 4b. Computation costs in order to attain  $E[OC]_{iz} = 0.01$  for Example 4

■ **Example 5 : Inventory Problem.**

The fifth example is an  $(s, S)$  inventory policy problem introduced by [16], and that was later analyzed by [19]. When random demand brings the inventory of system  $i$  on hand down to  $s$  units on hand, inventory is reordered so that the total inventory is  $S_i$ , for  $i = 1, 2, \dots, k$ . This example assumed that there are 5 systems, and that the alternative designs are defined by the parameters  $s = (s_1, s_2, \dots, s_5) = (20, 20, 40, 40, 60)$  and  $S = (40, 80, 60, 100, 100)$ , respectively. The second system has the smallest mean, which means the second system has the best policy. We implemented  $n_0 = 10$  first-stage replications per system, and  $\Delta = 10$  replications per stage until a total of  $T = 300$  replications are run. Estimates of  $P\{CS\}_{iz}$  and  $E[OC]_{iz}$  are based on 10,000 applications of each procedure to the problem.

For this configuration, procedure 0-1 performs about as well as procedures LL, OCBA-PCS and OCBA-EOC.

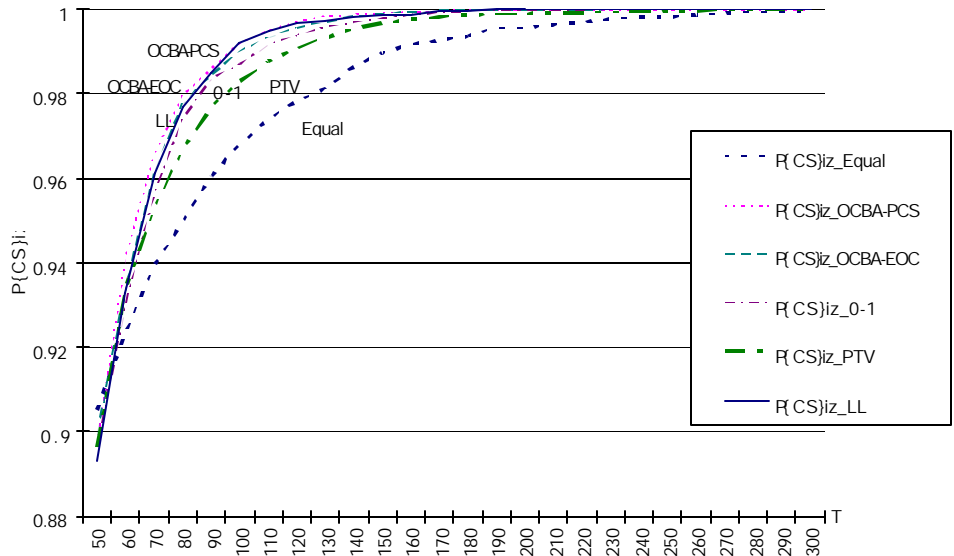


Figure 5a Performance comparison of  $P\{CS\}_{iz}$  in Example 5

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	160	120	100	100	100	110

Table 5a. Computation costs in order to attain  $P\{CS\}_{iz} = 99\%$  for Example 5

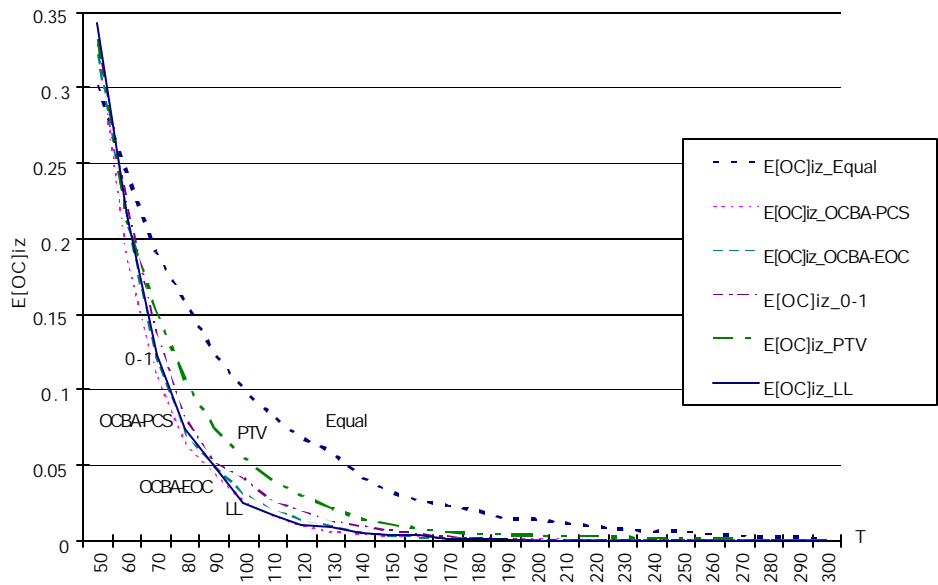


Figure 5a Performance comparison of  $E[OC]_{iz}$  in Example 5

	Equal	PTV	OCBA-PCS	OCBA-EOC	LL	0-1
$T$	>500	260	190	180	180	190

Table 5b. Computation costs in order to attain  $E[OC]_{iz} = 0.001$  for Example 5.

### C. Discussion

In these five examples, the OCBA-PCS, OCBA-EOC and *LL* procedures converge efficiently and these three procedures are similar in performance. While the *0-1* and PTV selection procedures can perform about as well as those three procedures in some specific examples, they perform clearly worse in other examples. In the third example, because the best design has the largest variance, PTV allocates most of the budget to the best design and it is similar to the allocation pattern of the OCBA-PCS, OCBA-EOC and *LL* procedures. The performance of PTV is therefore closer to the performance of OCBA-PCS, OCBA-EOC and *LL* in the third example, but this is due to the special structure of the specific selection problem. The equal allocation (which is equivalent to crude ordinal optimization) is the worst in all five examples. The *0-1* allocation performs less well than the three best procedures, for several experiments, due to the extra asymptotic approximation in its derivation [5].

## VI. Conclusions

Traditional ordinal optimization approaches focus on the alignment probability, or the probability of correctly selecting the best design. The use of the opportunity cost to guide a selection procedure's sampling allocations has proven to be efficient in the expected value of information context. The opportunity cost differs from previous approaches, which tend to focus on the probability of correct selection, in that it penalizes particularly bad choices more than slightly incorrect selections. This is particularly useful when the performance of each alternative is measured in financial terms (economic value) as opposed to other engineering measures (speed, etc.). This paper shows that the OCBA approach to sampling allocations can be adapted to account for opportunity costs in a computationally tractable way, and that the resulting selection procedure is consistently numerically efficient for each of the five empirical examples in this paper. This is important because the expected opportunity cost is often more important than the probability of correct selection when sampling allocations reflect the economic value of each simulated alternative. Our OCBA approach was shown to further enhance simulation efficiency over crude ordinal optimization in this context.

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