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Between First and Second-Order Stochastic Dominance

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Keywords: (1 + y) Stochastic Dominance; Transfers; Indirect Utility

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Between First and Second-Order Stochastic Dominance

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Abstract

We develop a continuum of stochastic dominance rules, covering preferences from first to second-order stochastic dominance. The motivation for such a continuum is that while decision makers have preference for "more is better," they are mostly risk averse but cannot assert that they would dislike any risk. For example, situations with targets, aspiration levels, and local convexities in induced utility functions in sequential decision problems may lead to preferences for some risks. We relate our continuum of stochastic dominance rules to utility classes, the corresponding integral conditions, and probability transfers, and discuss the usefulness of these interpretations. Several examples involving, e.g., finite-crossing cumulative

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distribution functions, location-scale families, and induced utility illustrate the implementation of the framework developed here. Finally, we extend our results to a combined order including convex (risk-taking) stochastic dominance.

Keywords: $(1+\gamma)$ stochastic dominance; transfers; indirect utility

1 Introduction

With respect to preferences for money, it is almost universally assumed that more is preferred to less, implying in an expected utility context that the utility function is increasing. The stronger assumption that most people are risk averse and that the utility function is therefore concave is almost as ubiquitous, to the point where some might believe erroneously that concavity is a normative axiom of utility theory. The implications for stochastic dominance (SD) follow, with first-order stochastic dominance (FSD) requiring increasing utility and second-order stochastic dominance (SSD) requiring that utility be increasing and concave. In practice, FSD can be used to screen out some weak actions, but that typically leaves actions that cannot be ranked by FSD. We can often rank some of these actions by SSD, but to do so we have to make the stronger assumption of risk aversion. The jump from FSD to SSD is substantial, and it would be useful to think about preference relations falling in between them, thereby having a continuum of stochastic dominance rules rather than a discrete set with jumps from FSD to SSD, then to third-order SD, and so on. In this paper, we develop such a continuum between FSD and SSD.

Consider, for example, an individual whose preferences are risk averse for many decisions, but whose utility function is not concave everywhere. In an early paper, Friedman and Savage (1948) note that a utility function that is concave everywhere is inconsistent with some frequently observed actions. For example, many people buy insurance (a risk-averse choice) and also gamble (a risk-loving choice). Friedman and Savage propose a utility function for wealth with three segments: concave for low wealth, convex for medium wealth, and concave for high wealth. Markowitz (1952) argues for a fourth segment, a convex segment for the lowest wealth levels.

Convex segments in otherwise concave utility functions can be related to a variety of factors. One such factor is an aspiration level that is important to an individual and causes a sharp increase in the utility function in the neighborhood of the aspiration level. This is an example of a reference-point effect, something studied and modeled in the behavioral and non-expected utility literatures, e.g., Kőszegi and Rabin (2006, 2007), Delquié and Cillo (2006), Cillo and Delquié (2014). The reference point can be related to the status quo, the situation (e.g., a target that will yield extra rewards if met or exceeded, as in Gaba, Tsetlin, and Winkler (2004)), to factors such as regret or disappointment as in Bell (1982, 1985), or simply to the goals and desires of the decision maker (Heath, Larrick, and Wu, 1999). In the corporate world, typical targets are the previous period's profits, analysts' expectations, and preset levels chosen internally, leading to earnings management (Degeorge, Patel, and Zeckhauser, 1999).

Even when an individual's utility function for wealth is concave everywhere, the nature of some decision-making problems will lead to induced utility that does not share that attribute. Sequential decisions provide good examples of this phenomenon. McCardle and Winkler (1992) consider a situation with repeated gambles and concave utility for terminal wealth at the end of the sequence of gambles. When the decision is being made for any gamble except the last one (after the previous gambles are made and settled), the induced utility for the wealth after that gamble is settled is not concave, with lower and upper concave segments corresponding to losing and winning the gamble, respectively. As a result, it can be optimal for a risk-averse decision maker to take an unfavorable risk in the hope of getting on the upper segment of the induced utility. It is common in sequential decision making for favorable outcomes to make it more likely to achieve additional favorable outcomes later, along the lines of "the rich get richer." A related example is a decision maker who is willing to take an unfavorable risk in an attempt to obtain sufficient funds to participate in an attractive opportunity.

The idea of interpolating between integer degree dominance relations is not new. Fishburn (1976, 1980) uses fractional integration to define a continuum of stochastic dominance relations. We compare his interpolation to ours in Section 2.2.

In this paper, we develop stochastic dominance of order $1+\gamma$ for $0 \leq \gamma \leq 1$, a continuum from order 1 (FSD) to order 2 (SSD). The basic definitions and results are presented in § 2, followed by some examples. An example involving induced utility is given in § 3. In § 4 we develop a combined order that incorporates the consideration of preference relations between FSD and convex (risk-loving) SSD in addition to those between FSD and concave (risk-avoiding) SSD and we show connections with reference-dependent utility and loss aversion. Concluding remarks are presented in § 5.

2 Between First and Second-Order Stochastic Dominance

We develop stochastic dominance of order $1+\gamma$ in this section, presenting basic definitions and results in § 2.1 and illustrating these results with some examples in § 2.4.

2.1 Stochastic Dominance of Order $1 + \gamma$

We begin by defining some notation. The random variables X and Y have cumulative distribution functions (cdfs) F and G, respectively, and means μ_X and μ_Y or μ_F and μ_G. Similarly, for any order relation $\leq_*$, we will write $X \leq_* Y$ or $F \leq_* G$ interchangeably. For expected utilities, $\mathbb{E}[u(X)] = \mathbb{E}_F(u)$ and $\mathbb{E}[u(Y)] = \mathbb{E}_G(u)$. We assume that all distributions have a finite mean and that all expected utilities are finite.

Definition 2.1. For $0 \leq \gamma \leq 1$ let $\mathscr{U}_\gamma$ be the class of continuously differentiable functions u such that

$$0 \leq \gamma u'(y) \leq u'(x) \quad \text{for all } x \leq y. \tag{2.1}$$

Definition 2.2. For $0 \leq \gamma \leq 1$, Y dominates X by $(1+\gamma)$-SD, denoted $X \leq_{(1+\gamma)\text{-SD}} Y$, if

$$\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)] \quad \text{for all functions } u \in \mathscr{U}_\gamma.$$

Note that $u \in \mathscr{U}_0$ if and only if u is increasing, and $u \in \mathscr{U}_1$ if and only if u is increasing and concave. Thus, $\gamma = 0$ corresponds to FSD and $\gamma = 1$ corresponds to SSD, with $0 < \gamma < 1$ corresponding to preference relations falling between FSD and SSD. The parameter γ provides a bound on how much marginal utility $u'(x)$ can decrease as x decreases, and its reciprocal $1/\gamma$ gives a bound on how much marginal utility can increase as x increases.

The differentiability condition in Definition 2.1 is not critical, and we relax it by considering $\mathscr{U}_\gamma^*$ defined below. This will be helpful in proofs as well as in applications where functions with kinks occur.

Definition 2.3. Let $\mathscr{U}_\gamma^*$ be the class of increasing functions u that satisfy

$$0 \leq \gamma \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \right) \leq \frac{u(x_2) - u(x_1)}{x_2 - x_1} \quad \text{for all } x_1 < x_2 < x_3 < x_4.$$

This class of functions has been considered before in the context of generalizations of expected utility. Chateauneuf, Cohen, and Meilijson (2005) define the *index of greediness* (or non-concavity) of a utility function as

$$\Gamma_u = \sup_{x_1 < x_2 \leq x_3 < x_4} \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \Big/ \frac{u(x_2) - u(x_1)}{x_2 - x_1} \right). \tag{2.2}$$

Thus $\mathscr{U}_\gamma^*$ is the class of utility functions u with index of greediness $\Gamma_u \leq 1/\gamma$.

We will prove an integral condition for $(1+\gamma)$-SD. In particular we will show that $F \leq_{(1+\gamma)\text{-SD}} G$ if and only if

$$\int_{-\infty}^{t} (G(x) - F(x))_+ \, \mathrm{d}x \leq \gamma \int_{-\infty}^{t} (F(x) - G(x))_+ \, \mathrm{d}x \quad \text{for all } t \in \mathbb{R}. \tag{2.3}$$

Theorem 2.4. *The following three conditions are equivalent:*

(a) $F \leq_{(1+\gamma)\text{-SD}} G$,

(b) $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ *for all functions* $u \in \mathscr{U}_\gamma^*$,

(c) $\int_{-\infty}^{t} (G(x) - F(x))_+ \, \mathrm{d}x \leq \gamma \int_{-\infty}^{t} (F(x) - G(x))_+ \, \mathrm{d}x$ *for all* $t \in \mathbb{R}$.

The integral condition given in Theorem 2.4 is simplified if F and G are single-crossing, as indicated in Corollary 2.5. Instead of comparing the positive and negative areas between F and G up to t for all values of t, we need only to make one such comparison.

Corollary 2.5. *Suppose there exists x_1 such that $F(x) \geq (\leq) G(x)$ for $x < (>) x_1$, so that F and G are single-crossing at x_1. Denote*

$$A = \int_{-\infty}^{x_1} (F(x) - G(x)) \, \mathrm{d}x \quad \text{and} \quad B = \int_{x_1}^{\infty} (G(x) - F(x)) \, \mathrm{d}x. \tag{2.4}$$

Then $F \leq_{(1+\gamma)\text{-SD}} G$ if and only if

$$\gamma \geq \frac{B}{A}.$$

Note that

$$\mu_G - \mu_F = \int_{-\infty}^{\infty} (F(x) - G(x)) \, \mathrm{d}x = \int_{-\infty}^{x_1} (F(x) - G(x)) \, \mathrm{d}x - \int_{x_1}^{\infty} (G(x) - F(x)) \, \mathrm{d}x = A - B.$$

When $\mu_G = \mu_F$, then $A = B$, the condition holds for $\gamma = 1$, and $F \leq_{\text{SSD}} G$. If $B = 0$, which means that $F(x) \geq G(x)$ everywhere, then F and G never cross, the condition holds for all $\gamma \geq 0$, and $F \leq_{\text{FSD}} G$. In between these two extremes, when $\mu_G > \mu_F$ but $F(x) < G(x)$ for some x, the condition only holds for some $0 < \gamma < 1$ and $F \leq_{(1+\gamma)\text{-SD}} G$ only holds for some γ.

If for given F and G second order stochastic dominance holds, but first order stochastic dominance does not, then we can always determine from Theorem 2.4(c) the smallest $\gamma = \gamma(F, G)$ for which $F \leq_{(1+\gamma)\text{-SD}} G$ holds. This $\gamma(F, G)$ can be interpreted as an index of risk aversion that makes it necessary for a decision maker to prefer G to F. From the two previous theorems it follows that in all practically relevant cases, $\gamma(F,G)$ can be determined easily. In the case of a single crossing of the distribution functions, Corollary 2.5 implies $\gamma(F,G) = B/A$, with A and B being the areas as defined in the theorem. Similarly, in the case of a finite number of crossings of the distributions functions Corollary B.1 in Appendix B implies

$$\gamma(F,G) = \max_{j\in\{1,\dots,M\}} \frac{\sum_{i=1}^{j+1} B_i}{\sum_{i=1}^{j+1} A_i}.$$

2.2 A related class of stochastic dominances

Fishburn (1976, 1980) considers the utility functions

$$u_{\alpha,t}(x) = -(t-x)_+^\alpha$$

and defines a continuum of stochastic dominance rules, which we denote

$$X \leq_{(1+\alpha)\text{-FD}} Y \quad \text{iff} \quad \mathbb{E}[u_{\alpha,t}(X)] \leq \mathbb{E}[u_{\alpha,t}(Y)] \quad \text{for all } t \in \mathbb{R}.$$

For $\alpha = 0$ and $\alpha = 1$, $(1+\alpha)$-FD corresponds to FSD and SSD, but for $0 < \alpha < 1$ the situation is different. In particular, $u_{\alpha,t}(x)$ has infinite left derivative at t. Furthermore, there seems to be no simple method to check this condition. We will investigate now the relation between Fishburn's idea and our concept by considering the case of 50:50 lotteries. Fix $w < z$. We want to check for which value of a a lottery with outcomes w and z is preferred to a lottery with outcomes $w - a$ and $z + 1$. Consider two random variables X and Y such that

$$\mathbb{P}(X = w - a) = \mathbb{P}(X = z + 1) = \frac{1}{2},$$
$$\mathbb{P}(Y = w) = \mathbb{P}(Y = z) = \frac{1}{2}.$$

It is clear that, if $a \geq 1$, then SSD holds but FSD does not. We easily get that $X \leq_{(1+\alpha)\text{-FD}} Y$ if and only if $a \geq 1/\gamma$, independently of w and z. To get Fishburn's dominance, however, the condition on a depends on the distance $z - w$. Checking $(1+\alpha)$-FD is much more involved. We have to show that for all t

$$\zeta(t) := \mathbb{E}[u_{\alpha,t}(Y)] - \mathbb{E}[u_{\alpha,t}(X)] \geq 0.$$

A critical value for the function ζ is $t = z + 1$. Solving

$$\mathbb{E}[u_{\alpha,z+1}(Y)] - \mathbb{E}[u_{\alpha,z+1}(X)] \geq 0$$

we get

$$a^* = (z - w + 1)^\alpha + 1)^{1/\alpha} - (z - w + 1)$$

and thus the necessary condition that $a \geq a^*$. We now prove that this is also sufficient by showing that for $a = a^*$

$$\mathbb{E}[u_{\alpha,t}(Y)] - \mathbb{E}[u_{\alpha,t}(X)] \geq \mathbb{E}[u_{\alpha,z+1}(Y)] - \mathbb{E}[u_{\alpha,z+1}(X)] = 0 \quad \text{for all } t \in \mathbb{R}.$$

For $t < z+1$ this is quite straightforward. For $t > z+1$ we can use the fact that on $(-\infty, z+1]$ the function $u_{\alpha,t}$ is less convex than $u_{\alpha,z+1}$, i.e., $u_{\alpha,t}(\cdot) = g(u_{\alpha,z+1}(\cdot))$ for some concave g and therefore

$$\mathbb{E}[u_{\alpha,z+1}(Y)] - \mathbb{E}[u_{\alpha,z+1}(X)] = 0$$

implies

$$u_{\alpha,z+1}(z+1) - u_{\alpha,z+1}(z) = u_{\alpha,z+1}(w) - u_{\alpha,z+1}(w-a),$$

and concavity of g therefore yields

$$u_{\alpha,t}(z+1) - u_{\alpha,t}(z) \leq u_{\alpha,t}(w) - u_{\alpha,t}(w-a),$$

hence $\mathbb{E}[u_{\alpha,t}(X)] \leq \mathbb{E}[u_{\alpha,t}(Y)]$. Thus $X \leq_{(1+\alpha)\text{-FD}} Y$ if and only if

$$a \geq a^* = ((z - w + 1)^\alpha + 1)^{1/\alpha} - (z - w + 1).$$

Considering the special case $\alpha = 1/2$ we get

$$a^* = 1 + 2\sqrt{z - w + 1},$$

which varies between 3 and infinity depending on the distance $z - w$, whereas in our concept $a^* = 1/\gamma$ is independent of the distance $z - w$ and thus shares this invariance property for lotteries with FSD and SSD.

This shows that the concepts of $(1+\alpha)$-FD in the sense of Fishburn and our concept of $(1+\gamma)$-SD are not comparable.

Our concept is easier to check due to the representation in terms of an integral condition for the cdfs. For the concept of Fishburn such an integral condition seems not to be known.

2.3 Transfers

Integral conditions are commonly used to characterize stochastic dominance relations. The integral condition in (2.3) characterizes $(1+\gamma)$-SD, which for $0 < \gamma < 1$ serves to fill in the gap between FSD and SSD by representing preference relations in that gap. Stochastic dominance relations can also be characterized in terms of transfers of probability, as is done in Rothschild and Stiglitz (1970). In this section, we show that $(1+\gamma)$-SD can also be related to probability transfers.

Definition 2.6. Consider two discrete cdfs F and G with respective mass functions f and g.

(a) We say that G is obtained from F via an *increasing transfer* if there exist $x_1 < x_2$ and $\eta > 0$ such that

$$\begin{aligned} g(x_1) &= f(x_1) - \eta, \\ g(x_2) &= f(x_2) + \eta, \\ g(z) &= f(z) \quad \text{for all other values } z. \end{aligned}$$

(b) We say that G is obtained from F via a *γ-transfer* if there exist $x_1 < x_2 < x_3 < x_4$ and $\eta_1, \eta_2 > 0$ with $\eta_2(x_4 - x_3) = \gamma\eta_1(x_2 - x_1)$ such that

$$\begin{aligned} g(x_1) &= f(x_1) - \eta_1, \\ g(x_2) &= f(x_2) + \eta_1, \\ g(x_3) &= f(x_3) + \eta_2, \\ g(x_4) &= f(x_4) - \eta_2, \\ g(z) &= f(z) \quad \text{for all other values } z. \end{aligned}$$

(c) A γ-transfer with $\gamma = 1$ is called a *concave transfer*.

Our concave transfer is a mean-preserving contraction, which is the reverse of a mean preserving spread à la Rothschild and Stiglitz (1970). In terms of F and G, Definition 2.6 implies that for an increasing transfer (which is a γ-transfer with $\gamma = 0$), the difference $F - G$ is a positive constant between x_1 and x_2, as illustrated in Figure 1. For the more general γ-transfer, $F-G$ is a positive constant between x_1 and x_2, a negative constant between x_3 and x_4, and zero otherwise in such a way that the area of the rectangle between the graph of $F-G$ in $[x_3, x_4]$ and the x-axis is γ times the area of the rectangle between the graph of $F-G$ in $[x_1, x_2]$ and the x-axis, as in Figure 2. For a concave transfer ($\gamma = 1$) the areas of these two rectangles are equal, as in Figure 3.

FIGURES 1, 2, 3 ABOUT HERE

We now show that a preference for γ-transfers is consistent with $(1+\gamma)$-SD.

Theorem 2.7. *Suppose that X and Y assume only a finite number of values. Then $X \leq_{(1+\gamma)\text{-SD}} Y$ if and only if G can be obtained from F via a finite sequence of γ-transfers and increasing transfers.*

For continuous random variables we need to use convergence in distribution. First, we show that the order $\leq_{(1+\gamma)\text{-SD}}$ behaves well with respect to such convergence. The notation $X_n \Rightarrow X$ indicates that X_n converges to X in distribution and $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

Theorem 2.8. *Assume that X and Y are random variables with finite means. Then $X \leq_{(1+\gamma)\text{-SD}} Y$ if and only if there exist two sequences $\{X_n\}$ and $\{Y_n\}$ with finite supports such that*

$$X_n \Rightarrow X, \quad Y_n \Rightarrow Y, \text{ and } X_n \leq_{(1+\gamma)\text{-SD}} Y_n \quad \text{for all } n \in \mathbb{N}.$$

Combining Theorem 2.7 and Theorem 2.8 yields the following corollary.

Corollary 2.9. *Given two random variables X and Y, $X \leq_{(1+\gamma)\text{-SD}} Y$ if and only if there exist two sequences $\{X_n\}$ and $\{Y_n\}$ such that for all $n \in \mathbb{N}$ the distribution of Y_n can be obtained from the distribution of X_n via a finite sequence of γ-transfers and increasing transfers, $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$.*

Corollary 2.9 connects $(1+\gamma)$-SD with γ-transfers. A decision maker's preferences satisfy $(1+\gamma)$-SD if and only if any γ-transfer and any increasing transfer are acceptable. For $\gamma = 1$, that becomes a preference for mean-preserving contractions (concave transfers) and increasing transfers. If a decision maker asserts that she likes any γ'-transfer with $\gamma' \leq \gamma$, her preferences satisfy $(1+\gamma)$-SD. We return to this discussion in the next section, in the context of concave and convex transfers.

2.4 Examples of $(1+\gamma)$-SD

Here the results given in § 2.1 are illustrated with some examples.

Example 2.10 (Single crossing)**.** For a simple example with single-crossing F and G, let G be degenerate, corresponding to a sure gain of c. If $c = \mu_F$, then $F \leq_{\text{SSD}} G$. If $c > \mu_F$, then intuitively, the dominance of G over F is stronger than just SSD. In this case $A = \int_{-\infty}^{c}(F(x)-0)\,\mathrm{d}x$, $B = \int_{c}^{\infty}(1-F(x))\,\mathrm{d}x$, and $F \leq_{\text{FSD}} G$ if and only if $F(c) = 1$ (i.e., the support of F is below c, so that $B = 0$ and $\gamma = 0$). Overall, γ decreases (i.e., the dominance of G over F gets stronger) as $c - \mu_F$ goes up and B goes down.

Note that $B = \int_{c}^{\infty}(1-F(x))\,\mathrm{d}x = \mathbb{E}\left[(X-c)_+\right]$ can be interpreted as an upside of drawing from the distribution F, compared to a sure gain c. Similarly, $A = \int_{-\infty}^{c}(F(x)-0)\,\mathrm{d}x = \mathbb{E}\left[(c-X)_+\right]$ can be viewed as a downside. If the upside is small relative to the downside (i.e., $B << A$), we are close to FSD. If the upside is almost the same as the downside (i.e., $A - B > 0$ is small), then we are close to SSD. We have $X \leq_{(1+\gamma)\text{-SD}} c$ if

$$\gamma \geq \frac{\mathbb{E}\left[(X-c)_+\right]}{\mathbb{E}\left[(c-X)_+\right]},$$

which is the ratio of the upside to the downside. This ratio is well known in the literature on performance measurement of financial investments as the so called *Omega ratio* (see Shadwick and Keating, 2002). Thus any decision maker with a utility function in $\mathscr{U}_\gamma$ prefers a fixed return of c to an uncertain return with an Omega ratio below γ, if $\gamma < 1$. Notice that in portfolio optimization it is usually optimal to choose a portfolio with an uncertain return whose Omega ratio is, however, typically much larger than one. We return to the Omega ratio in the context of convex stochastic dominance in § 4.

Example 2.11 (Location-scale families)**.** Let F and G be cdfs from the same location-scale family, namely,

$$F(x) = H\left(\frac{x-\mu_F}{\sigma_F}\right) \quad \text{and} \quad G(x) = H\left(\frac{x-\mu_G}{\sigma_G}\right),$$

where H is the cdf of a random variable with mean zero and standard deviation one. Then F and G have means μ_F and μ_G and standard deviations σ_F and σ_G. A necessary and sufficient condition for $F \leq_{\text{SSD}} G$ is $\mu_G \geq \mu_F$ and $\sigma_G \leq \sigma_F$. The cdfs are single-crossing at

$$x_1 = \frac{\mu_G\sigma_F - \mu_F\sigma_G}{\sigma_F - \sigma_G}.$$

By Corollary 2.5 (details are in Appendix A), $F \leq_{(1+\gamma)\text{-SD}} G$ with

$$\gamma = \frac{\int_{\frac{\Delta\mu}{\Delta\sigma}}^{\infty} (1 - H(z))\, \mathrm{d}z}{\int_{\frac{\Delta\mu}{\Delta\sigma}}^{\infty} (1 - H(z))\, \mathrm{d}z + \frac{\Delta\mu}{\Delta\sigma}}.$$

As we can see, γ decreases (i.e., the dominance gets stronger) as $\Delta\mu/\Delta\sigma$ increases.

As an illustration with a location-scale family, let $H = \Phi$ represent the standard normal cdf. Denote $V(t) = \int_t^{\infty}(1 - \Phi(z))\, \mathrm{d}z$, and note that

$$\int \Phi(z)\ \mathrm{d}z = z\Phi(z) + \phi(z) \quad \text{and} \quad \int (1 - \Phi(z))\, \mathrm{d}z = z(1 - \Phi(z)) - \phi(z).$$

Therefore,

$$V(t) = (\phi(t) - t\Phi(-t)) \quad \text{and} \quad \gamma\left(\frac{\Delta\mu}{\Delta\sigma}\right) = \frac{V\left(\frac{\Delta\mu}{\Delta\sigma}\right)}{V\left(\frac{\Delta\mu}{\Delta\sigma}\right) + \frac{\Delta\mu}{\Delta\sigma}}.$$

FIGURE 4 ABOUT HERE

Figure 4 plots γ as a function of $\Delta\mu/\Delta\sigma$ for the case of normal distributions. As $\Delta\mu/\Delta\sigma \to 0, \gamma \to 1$, so that we approach SSD. At the other extreme, we approach FSD as $\Delta\mu/\Delta\sigma$ increases, with a relatively rapid drop as $\Delta\mu/\Delta\sigma$ goes from 0 to 1, where $\gamma = 0.077$. At $\Delta\mu/\Delta\sigma = 2$, $\gamma = 0.004$, which is very close to zero, so that for practical purposes, $F \leq_{\text{FSD}} G$.

3 Local Convexities and Preferences Satisfying $(1+\gamma)$-SD

In this section we discuss the fact that local convexities can exist in a utility function, either because they directly reflect preferences or because the nature of a decision-making situation can lead to an induced utility function with local convexities even when the decision maker's utility function for wealth or changes in wealth is concave. We then illustrate the latter case involving induced utility to demonstrate the connection with $(1+\gamma)$-SD.

Arguments for the existence of local convexities that directly reflect preferences have been made in the utility literature, dating back to early papers of Friedman and Savage (1948) and Markowitz

(1952). For example, Friedman and Savage show that risk-averse decisions for some gambles (e.g., buying insurance) and risk-taking decisions for other gambles (e.g., purchasing lottery tickets) can be explained by a concave-convex-concave utility function such as that shown in Figure 5. Many types of insurance involve making a small investment to avoid a small chance of a possible large loss (preferring a sure thing to a risk with a higher expected value), whereas buying lottery tickets involves making a small investment to have a very small chance of a very large gain (eschewing the sure thing of current wealth in order to take a risk with a smaller expected value).

FIGURE 5 ABOUT HERE

The convex portion of a concave-convex-concave utility function for wealth is generally viewed as being near the current wealth level. Other factors may explain local convexities that reflect preferences, such as a strong desire to attain a particular wealth level, perhaps for the status it would convey. This is an example of a reference-point effect.

There are many cases where preferences are consistent with a utility function for wealth that is concave everywhere, but the situation causes local convexities. Perhaps the most common type of such situations involves sequential decision making under uncertainty. When we have a sequence of decisions over time, during each time period we may have a change in our wealth due to the decision made for that period, and also a change in probabilities as we learn from what occurred in that period. Folding back the decision tree using dynamic programming, we start with the final-period decision and work our way back through the tree to find an overall strategy that indicates decisions that should be made at each period conditional on what has happened in earlier periods. The dynamic-programming value function for a given period depends on the future uncertainties and opportunities and can lead to induced utility with local convexities. In a multi-round contest setting, Tsetlin, Gaba, and Winkler (2004) show that the optimal level of variability depends on the position relative to other contestants and on the number of remaining rounds.

We now illustrate how local convexities can arise in induced utility. Building on an example from Bell (1988), we consider a decision maker who has initial wealth x and is given an opportunity to undertake a project with a payoff Y that is normally distributed with mean μ and standard deviation σ. The decision maker's utility for wealth w is linear plus exponential, $u(w) = w - b\exp\{-rw\}$, with $b > 0$ and $r > 0$. This utility function is concave and exhibits decreasing absolute risk aversion $R(w)$ as wealth increases:

$$R(w) = \frac{r^2 b}{rb + \exp\{rw\}}.$$

The expected utility of the project is $\mathbb{E}\left[u(x+Y)\right] = x + \mu - b\exp\{-r(x+\mu-(r\sigma^2/2))\}$. The project should be undertaken if and only if $\mathbb{E}\left[u(x+Y)\right] \geq u(x) = x - b\exp\{-rx\}$, which simplifies to

$$\mu \geq b\exp\{-rx\}\left(\exp\left\{-r\left(\mu - \frac{r\sigma^2}{2}\right)\right\} - 1\right).$$

If $\mu \geq r\sigma^2/2$, then it is optimal to undertake the project at any value of x, and if $\mu \leq 0$, it is never optimal to undertake the project. We now consider the case where $0 < \mu < r\sigma^2/2$, or equivalently, $\mu > 0$ and $\sigma^2 > 2\mu/r$.

For $0 < \mu < r\sigma^2/2$, the project should (should not) be undertaken if $x > (<) x_0$, where

$$x_0 = \frac{1}{r}\ln\left(\frac{b}{\mu}\left(\exp\left\{-r\left(\mu - \frac{r\sigma^2}{2}\right)\right\} - 1\right)\right).$$

Therefore, the induced utility for x is

$$\varphi(x) = \max\{u(x), \mathbb{E}[u(x+Y)]\} = \begin{cases} x - b\exp\{-rx\} & \text{if } x \leq x_0, \\ x + \mu - b\exp\left\{-r\left(x + \mu - \frac{r\sigma^2}{2}\right)\right\} & \text{if } x \geq x_0. \end{cases}$$

This function is increasing everywhere and is concave on $(-\infty, x_0]$ and $[x_0, \infty)$. But it has a kink at x_0, with the left-hand derivative equal to $u'(x_0) = 1 + rb\exp\{-rx_0\}$ and the right-hand derivative equal to $\mathbb{E}\left[u'(x_0+Y)\right] = 1 + rb\exp\{-r\left(x_0 + \mu - r\sigma^2/2\right)\}$. Therefore, if $0 < \mu < r\sigma^2/2$, the induced utility $\varphi(x)$ belongs to $\mathscr{U}^*_\gamma$ from Definition 2.3 with

$$\gamma = \frac{u'(x_0)}{\mathbb{E}\left[u'(x_0+Y)\right]} = \frac{1 + rb\exp\{-rx_0\}}{1 + rb\exp\{-rx_0\}\exp\{-r(\mu - r\sigma^2/2)\}}.$$

Using $\exp\{-rx_0\} = \mu/(b\left(\exp\{-r(\mu - r\sigma^2/2)\} - 1\right))$ and simplifying, we get

$$\gamma = \frac{1 + (r\mu - 1)\exp\{r(\mu - r\sigma^2/2)\}}{1 + r\mu - \exp\{r(\mu - r\sigma^2/2)\}},$$

which is decreasing in σ^2, ranging from 1 when $\sigma^2 = 2\mu/r$ to $1/(1+r\mu)$ as $\sigma^2 \to \infty$. If $\mu \leq 0$ or $\mu \geq r\sigma^2/2$, then induced utility is linear plus exponential and thus belongs to $\mathscr{U}_1$.

We now illustrate this setting with monetary amounts expressed in millions of dollars and $u(x) = -100e^{-0.01x} + x$, so that $r = 0.01$ and $b = 100$.

FIGURE 6 ABOUT HERE

Figure 6 plots the induced utility $\max\{u(x), \mathbb{E}[u(x+Y)]\}$ for $-100 \le x \le 200$ (corresponding to $0.0073 \le R(x) \le 0.2012$) when $\sigma = 300$ and μ varies between 150 and 400. Kinks in the induced utility functions associated with $\mu = 200, 250, 300$, and 350 can be seen at their break-even values $x_0 = 172, 94, 15$, and -71, respectively. A higher μ allows a lower x_0 for undertaking the project and its higher associated level of risk aversion. For $\mu = 150$, the kink does not appear in Figure 6 because $x_0 = 254$, and the decision maker should not undertake the project. For $\mu = 400$, the kink again does not appear because $x_0 = -182$, and the decision maker is better off undertaking the project.

FIGURE 7 ABOUT HERE

Figure 7 plots γ as a function of σ for different values of μ. As derived above, γ is decreasing in σ, starting from 1 (for $\sigma \le \sqrt{2\mu/r} = \sqrt{200\mu}$, when undertaking the project dominates not undertaking the project by SSD for any x) and approaching $1/(1+r\mu) = 1/(1+0.01\mu)$ in the limit. The larger μ is, the longer γ remains at 1 as σ increases and the lower it eventually goes once it drops from 1. For example, when $\mu = 10$, γ starts decreasing at $\sigma = 44.7$ and approaches 0.909; when $\mu = 1000$, γ starts decreasing at $\sigma = 447.2$ and approaches 0.091. For larger μ, we get closer to FSD (because γ becomes smaller) as σ increases.

Figure 6 shows how kinks in an induced utility function can create local convexities. These kinks occur when the optimal decision changes, resulting in the utility depending on an upper envelope of utility functions associated with different actions that can be taken. As a result, SSD does not apply because the induced utility function is not concave, and being able to consider a continuum of SD rules between FSD and SSD as we do with (1+γ)-SD for $0 < \gamma < 1$ can be useful.

4 Combined Order: Concave and Convex

4.1 Concave and convex stochastic dominance of order $(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})$

In Definition 2.1, γ provides a bound on how much marginal utility can decrease as x decreases and $1/\gamma$ gives a bound on how much marginal utility can increase as x increases. This section explores the implications of incorporating another parameter that gives bounds on how much marginal utility can decrease as x increases and increase as x decreases. This enables us to extend our (1+γ)-SD, which is concerned with preference relations between FSD and concave (risk-averse) SSD, by incorporating the consideration of preference relations between FSD and convex (risk-loving) SSD. Risk-prone preferences have received some attention recently (Crainich, Eeckhoudt, and Trannoy, 2013, Ebert, 2013) and a multivariate version of convex stochastic dominance is discussed in Denuit, Eeckhoudt, Tsetlin, and Winkler (2013).

Definition 4.1 below extends Definition 2.1 to a utility class that corresponds to a combined order, including concave and convex $(1+\gamma)$-SD as special cases.

Definition 4.1. Let $\mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ be the class of continuously differentiable functions u such that

$$0 \le \gamma_{\text{cv}} u'(y) \le u'(x) \le \frac{1}{\gamma_{\text{cx}}} u'(y) \quad \text{for all } x \le y. \tag{4.1}$$

Corresponding to γ in Definition 2.1, γ_{cv} puts a limit on how much marginal utility can decrease as x decreases (with $\mathscr{U}_{\gamma_{\text{cv}},0} = \mathscr{U}_{\gamma_{\text{cv}}}$ if $\gamma_{\text{cx}} = 0$), while the new parameter γ_{cx} puts a limit on how much marginal utility can decrease as x increases.

Definition 4.2. For $0 < \gamma_{\text{cv}}, \gamma_{\text{cx}} < 1, G$ dominates F by $(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})$-SD, denoted

$$F \le_{(1+\gamma_{\text{cv}},1+\gamma_{\text{cx}})\text{-SD}} G,$$

if $\mathbb{E}_G(u) \ge \mathbb{E}_F(u)$ for all functions $u \in \mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$.

For some values of γ_{cv} and γ_{cx}, we get "standard" (previously studied) cases, namely:

- The case $\gamma_{\text{cv}} = 0$, $\gamma_{\text{cx}} = 0$ is FSD, where $\mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ is the class of increasing functions.
- The case $\gamma_{\text{cv}} = 1$, $\gamma_{\text{cx}} = 0$ is SSD, where $\mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ is the class of increasing concave functions.
- The case $\gamma_{\text{cv}} = 0$, $\gamma_{\text{cx}} = 1$ is risk-loving SSD, where $\mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ is the class of increasing convex functions.
- The case $\gamma_{\text{cv}} = \gamma$, $\gamma_{\text{cx}} = 0$ is concave $(1+\gamma)$-SD from Definition 2.2.
- The case $\gamma_{\text{cv}} = 0$, $\gamma_{\text{cx}} = \gamma$ is convex $(1+\gamma)$-SD.
- The case $\gamma_{\text{cv}} = \gamma_{\text{cx}} = \gamma = \varepsilon/(1-\varepsilon)$ is ε-almost first-degree stochastic dominance (ε-AFSD, Leshno and Levy (2002)) with $\varepsilon = \gamma/(1+\gamma)$.
- The case $\gamma_{\text{cv}} = \gamma_{\text{cx}} = 1$ corresponds to affine utility functions, and therefore induces a complete order based solely on the comparison of expectations.

Theorem 4.3 provides an integral condition for $(1+\gamma_{\mathrm{cv}}, 1+\gamma_{\mathrm{cx}})$-SD.

Theorem 4.3. *For* $0 \le \gamma_{\mathrm{cv}} \le 1$, $0 \le \gamma_{\mathrm{cx}} \le 1$, $F \le_{(1+\gamma_{\mathrm{cv}},1+\gamma_{\mathrm{cx}})\text{-SD}} G$ *if and only if*

$$\begin{aligned} &\min\left\{1, \frac{\gamma_{\mathrm{cv}}}{\gamma_{\mathrm{cx}}}\right\} \int_{-\infty}^{t} (G(x)-F(x))_+ \,\mathrm{d}x + \min\left\{1, \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right\} \int_{t}^{\infty} (G(x)-F(x))_+ \,\mathrm{d}x \\ &\quad\le \gamma_{\mathrm{cv}} \int_{-\infty}^{t} (F(x)-G(x))_+ \,\mathrm{d}x + \gamma_{\mathrm{cx}} \int_{t}^{\infty} (F(x)-G(x))_+ \,\mathrm{d}x \quad \textit{for every } t \in \mathbb{R}. \end{aligned} \tag{4.2}$$

For example, if $\gamma_{\mathrm{cx}} = 0$, then the integral condition in (4.2) reduces to

$$\int_{-\infty}^{t} (G(x)-F(x))_+ \,\mathrm{d}x \le \gamma_{\mathrm{cv}} \int_{-\infty}^{t} (F(x)-G(x))_+ \,\mathrm{d}x,$$

which corresponds to (2.3) with $\gamma = \gamma_{\mathrm{cv}}$ and thus is an integral condition for concave $(1+\gamma_{\mathrm{cv}})$-SD. If $\gamma_{\mathrm{cv}} = \gamma_{\mathrm{cx}} = \gamma$, then (4.2) reduces to

$$\int_{-\infty}^{\infty} (G(x)-F(x))_+ \,\mathrm{d}x \le \gamma \int_{-\infty}^{\infty} (F(x)-G(x))_+ \,\mathrm{d}x,$$

which, for $\gamma = \varepsilon/(1-\varepsilon)$, corresponds to the integral condition for ε-AFSD from Leshno and Levy (2002). The same condition holds for generalized AFSD, or GAFSD, as considered in Tsetlin, Winkler, Huang, and Tzeng (2015), although GASD differs from ASD for higher degrees. This shows that AFSD (with $\varepsilon = \gamma/(1+\gamma)$) is weaker than the $(1+\gamma)$-SD order. Finally, if $\gamma_{\mathrm{cv}} = 0$, then (4.2) reduces to

$$\int_{t}^{\infty} (G(x)-F(x))_+ \,\mathrm{d}x \le \gamma_{\mathrm{cx}} \int_{t}^{\infty} (F(x)-G(x))_+ \,\mathrm{d}x,$$

which is an integral condition for convex $(1+\gamma_{\mathrm{cx}})$-SD. For instance, in the context of Example 2.10, a random return X dominates a sure gain of c via convex $(1+\gamma_{\mathrm{cx}})$-SD if

$$\frac{\mathbb{E}[(X-c)_+]}{\mathbb{E}[(c-X)_+]} \ge \frac{1}{\gamma_{\mathrm{cx}}},$$

i.e., if the Omega ratio is greater than $1/\gamma_{\mathrm{cx}}$.

Overall, γ_{cv} can be interpreted as a limit on convexity (the smaller γ_{cv} is, the more u' can increase as x increases) and γ_{cx} as a limit on concavity (the smaller γ_{cx} is, the more u' can decrease as x increases). Then their ratio, $\gamma_{\mathrm{cx}}/\gamma_{\mathrm{cv}}$, tells us whether $(1+\gamma_{\mathrm{cv}}, 1+\gamma_{\mathrm{cx}})$-SD is closer to convex or concave SSD. For example, if $\gamma_{\rm cx} > \gamma_{\rm cv}$, then the limit on convexity ($\gamma_{\rm cv}$) is smaller, and we are closer to the convex case. If $\gamma_{\rm cv} = 1$, then convexity is prohibited, and $\gamma_{\rm cv} = 1, \gamma_{\rm cx} = 0$ corresponds to standard (concave) SSD. Similarly, as $\gamma_{\rm cx}$ increases, less and less concavity is allowed, with $\gamma_{\rm cv} = 0, \gamma_{\rm cx} = 1$ corresponding to convex SSD. If $\gamma_{\rm cv} = \gamma_{\rm cx} = 1$, both concavity and convexity are prohibited, so this case corresponds to ranking the distributions by their expectations. Finally, $\gamma_{\rm cv} = \gamma_{\rm cx} = \gamma$ corresponds to ε-AFSD with $\varepsilon = \gamma/(1+\gamma)$, which is "equidistant" from concave and convex SSD. This suggests that the ratio of positive to negative probability mass transfers, leading to improvement in the sense of $(1+\gamma_{\rm cv}, 1+\gamma_{\rm cx})$-SD, would depend on whether the total transfer is inward or outward (i.e., whether it is a "decreasing risk" or an "increasing risk" transfer). Below we develop these ideas more formally, via concave and convex γ-transfers.

Definition 2.6 defines a (concave) γ-transfer. A convex γ-transfer is defined similary in Definition 4.4, and then Theorem 2.7 and Corollary 2.9 are extended to $(1+\gamma_{\rm cv}, 1+\gamma_{\rm cx})$-SD.

Definition 4.4. Consider two discrete cdfs F and G with respective mass functions f and g. We say that G is obtained from F via a convex γ-transfer if there exist $x_1 < x_2 < x_3 < x_4$ and $\eta_1, \eta_2 > 0$ with $\eta_1(x_2 - x_1) = \gamma\eta_2(x_4 - x_3)$ such that

$$\begin{aligned} g(x_1) &= f(x_1) + \eta_1, \\ g(x_2) &= f(x_2) - \eta_1, \\ g(x_3) &= f(x_3) - \eta_2, \\ g(x_4) &= f(x_4) + \eta_2, \\ g(z) &= f(z) \quad \text{for all other values } z. \end{aligned}$$

Theorem 4.5. *Suppose that X and Y assume only a finite number of values. Then $X \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y$ if and only if G can be obtained from F via a finite sequence of $\gamma_{\rm cv}$-transfers, convex $\gamma_{\rm cx}$-transfers, and increasing transfers.*

Theorem 4.5 is a consequence of Müller (2013, Theorem 2.4.1).

Corollary 4.6. *Given two random variables X and Y, $X \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y$ if and only if there exist two sequences $\{X_n\}$ and $\{Y_n\}$ such that for all $n \in \mathbb{N}$ the distribution of Y_n can be obtained from the distribution of X_n via a finite sequence of $\gamma_{\rm cv}$-transfers, convex $\gamma_{\rm cx}$-transfers, and increasing transfers, $X_n \Rightarrow X$, and $Y_n \Rightarrow Y$.*

To relax the differentiability assumption, the class $\mathscr{U}_{\gamma_{\rm cv},\gamma_{\rm cx}}$ can be extended to the class $\mathscr{U}^*_{\gamma_{\rm cv},\gamma_{\rm cx}}$, defined as follows.

Definition 4.7. Let $\mathscr{U}^*_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ be the class of functions u such that

$$0 \le \gamma_{\text{cv}} \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \right) \le \frac{u(x_2) - u(x_1)}{x_2 - x_1} \le \frac{1}{\gamma_{\text{cx}}} \left(\frac{u(x_4) - u(x_3)}{x_4 - x_3} \right) \quad \text{for all } x_1 < x_2 \le x_3 < x_4.$$

It is interesting to observe that the utility class $\mathscr{U}^*_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$, which defines $(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})$-SD, is directly related to indices of greediness and thriftiness from Chateauneuf et al. (2005). Indeed, for a strictly increasing function u, Chateauneuf et al. (2005) define the *index of greediness* (non-concavity) (2.2) and the *index of thriftiness* (or non-convexity)

$$T_u = \sup_{x_1 < x_2 \le x_3 < x_4} \left(\frac{u(x_2) - u(x_1)}{x_2 - x_1} \Big/ \frac{u(x_4) - u(x_3)}{x_4 - x_3} \right).$$

Thus, $\mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ is the set of all continuously differentiable increasing functions with index of greediness at most $1/\gamma_{\text{cv}}$ and index of thriftiness at most $1/\gamma_{\text{cx}}$, and $(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})$-SD relates to preferences of decision makers with such utility functions.

Similar to the proof of Theorem 2.1 in Denuit and Müller (2002), $\mathbb{E}_G(u) \ge \mathbb{E}_F(u)$ for all functions $u \in \mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ implies $\mathbb{E}_G(u) \ge \mathbb{E}_F(u)$ for all $u \in \mathscr{U}^*_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$, and thus Theorem 4.3 provides an integral condition for all $u \in \mathscr{U}^*_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$.

Overall, then, we can think about γ from Definition 2.1 and $(1+\gamma)$-SD, or γ_{cv} and γ_{cx} from Definition 4.1 and $(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})$-SD, in three different ways: in terms of integral conditions involving areas between cdfs, in terms of limits on ratios of marginal utilities, and in terms of transfers. These three interpretations are useful in different ways. Integral conditions are helpful to compare distributions in practice, even though they may be hard to think about in trying to elicit γ or γ_{cv} and γ_{cx}. Similarly, the ratio of marginal utilities may be understandable in general but not easy to think about in numerical terms. Perhaps viewing these ratios in terms of indices of greediness and thriftiness and thus relating to $1/\gamma_{\text{cv}}$ and $1/\gamma_{\text{cx}}$ can be helpful. The notion of transfers is probably the easiest to connect to one's preferences and to think about in attempting to elicit γ or γ_{cv} and γ_{cx}. By way of analogy, it is easier to explain mean-preserving contractions and mean-preserving spreads, interpreting risk aversion in terms of liking the former and risk loving in terms of liking the latter, than to think in terms of concavity or convexity of utility or of decreasing or increasing marginal utility. Of the three interpretations, working with transfers is closer in spirit to working with comparisons of lotteries in utility elicitation. Moreover, the transfers do not rely on the expected utility paradigm and can be used to consider monotonicity of preferences in a non-expected utility framework.

We conclude this section by considering transformations, applied to either utility functions or the random variables themselves.

Theorem 4.8. *If* $X \le_{(1+\gamma_{\rm cv}^{(1)}\times\gamma_{\rm cv}^{(2)},1+\gamma_{\rm cx}^{(1)}\times\gamma_{\rm cx}^{(2)})\text{-SD}} Y$ *and* $u_2 \in \mathscr{U}_{\gamma_{\rm cv}^{(1)},\gamma_{\rm cx}^{(2)}}$*, then* $u_2(X) \le_{(1+\gamma_{\rm cv}^{(1)},1+\gamma_{\rm cx}^{(1)})\text{-SD}} u_2(Y)$*.*

Theorem 4.8 implies, in particular, the invariance of the dominance rule under location and scale transformations: if $X \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y$, then $aX+b \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} aY+b$ for all $a>0$ and $b\in\mathbb{R}$. Another important case of Theorem 4.8 is when u_2 is increasing and concave and thus $u_2 \in \mathscr{U}_{1,0}$. Then $X \le_{(1+\gamma)\text{-SD}} Y$ implies $u_2(X) \le_{(1+\gamma)\text{-SD}} u_2(Y)$.

The following theorems show that the order $(1+\gamma_{\rm cv}, 1+\gamma_{\rm cx})$-SD is closed under mixture and convolution.

Theorem 4.9. *If the random variables* X, Y, Θ *are such that* $[X \mid \Theta=\theta] \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} [Y \mid \Theta=\theta]$*, for all* θ *in the support of* Θ*, then* $X \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y$*.*

Theorem 4.10. *Let* Z *be independent of* X *and* Y*. If* $X \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y$*, then* $X+Z \le_{(1+\gamma_{\rm cv},1+\gamma_{\rm cx})\text{-SD}} Y+Z$*.*

4.2 Reference dependent preferences and loss aversion

Some approaches to decision modeling other than expected utility include the concept of reference dependence, in which a decision maker considers outcomes relative to a reference point, with outcomes below the reference point viewed as losses and outcomes above the reference point as gains. The utility function is generally assumed to be risk-taking (convex) for losses and risk-avoiding (concave) for gains, resulting in an S-shaped utility function. An example is the commonly used prospect theory of Kahneman and Tversky (1979) with reference-dependent utility functions such as $u(x) = h_1(x-r)$ for $x>r$ and $h_2(x-r)$ for $x<r$, where r is the reference point. In this section we show connections of reference-dependent utility with $\mathscr{U}_{\gamma_{\rm cv},\gamma_{\rm cx}}$ and discuss the bounds on $\gamma_{\rm cv}$ and $\gamma_{\rm cx}$.

Following Köbberling and Wakker (2005), without loss of generality we take $r=0$ and consider utility of the form $u(x)$ for $x\ge 0$ and $\lambda u(x)$ for $x<0$,where $u(x)$ is increasing, concave for $x>0$, and convex for $x<0$, with $\lambda>0$, $u(0)=0$, and $u'(0)=1$. The parameter λ represents a loss aversion index, and $\lambda>1$ allows utility to be steeper for losses than for gains, reflecting the fact that people pay more attention to losses (Köbberling and Wakker, 2005, p. 121). The scaling with $u'(0)=1$ is to avoid functional forms for u that have extreme derivatives of 0 or ∞ at $x=0$, which create difficulties (Wakker, 2010, pp. 267–271) and also seem unrealistic. For instance, power utility, $u(x) = x^\alpha$ for $x > 0$ and $-|x|^\beta$ for $x < 0$ with $\alpha, \beta \in (0, 1)$ results in infinite derivatives at $x = 0$. The scaling represented by $u'(0) = 1$ is not too restrictive, because it is consistent with the general feeling that people should be approximately risk-neutral for "small" gambles.

Let $[-d, d]$, $d > 0$, be an interval containing the supports of all lotteries that are being considered. Then, for $-d \le x \le d$ and $\lambda > 1$, we have $u \in \mathscr{U}_{\gamma_{\text{cv}},\gamma_{\text{cx}}}$ with $\gamma_{\text{cv}} = \lambda u'(-d)/\lambda u'(0) = u'(-d)$ and $\gamma_{\text{cx}} = u'(d)/\lambda u'(0) = u'(d)/\lambda$.

Typical conditions on overall utility are that for $x > 0$, $-\lambda u(-x) > u(x)$ and $\lambda u'(-x) > u'(x)$. This results in a kink in the S-shaped utility function at $x = 0$, just as there is a kink in the induced utility in §3. Also, for $x = d$, $\lambda u'(-d) > u'(d)$, which implies that $\gamma_{\text{cv}} > \gamma_{\text{cx}}$. That means that a) any choice that violates $(1 + \gamma_{\text{cv}}, 1 + \gamma_{\text{cx}})$-SD cannot be explained by location or change of the reference point, and b) that $(1 + \gamma_{\text{cv}}, 1 + \gamma_{\text{cx}})$-SD implied by reference-dependent utility is closer to concave (risk averse) SD than to convex (risk-loving) SD because $\gamma_{\text{cv}} > \gamma_{\text{cx}}$, which is due to the loss aversion.

For example, if u and d are such that $u'(-d) = u'(d) = 1/2$ and $\lambda = 2$, we get $\gamma_{\text{cv}} = 1/2$ and $\gamma_{\text{cx}} = 1/4$. Then all choices should obey $(1.5, 1.25)$-SD, and connecting it to $(1 + \gamma)$-SD from Section 2, all choices should obey 1.5-SD.

5 Concluding Remarks

Stochastic dominance partially ranks the distributions if it is known that preferences belong to a particular class. Preference for "more is better" is given by FSD, and preference for "less risk" is described by SSD, corresponding to increasing and increasing concave utility functions, respectively, in the expected utility framework.

The gap between "increasing" and "increasing concave" is substantial, leaving out decision makers who are mostly risk averse, but cannot assert that they would dislike any risk. For example, some risk appetite can occur because they foresee attractive future opportunities that might be available only at higher levels of wealth, as in the example in § 3. In this paper we develop SD for such decision makers, for decision makers who are mostly risk seeking, but can be risk averse to some risks at some wealth levels, and for decision makers with S-shaped reference-dependent utility functions.

There are three ways to look at such preferences: via integral conditions (Theorems 2.4 and 4.3), via limits on ratios of marginal utilities (Definitions 2.1 and 4.1), and via transfers (Corollaries 2.9 and 4.6). Integral conditions are useful for comparing distributions. Corollary 2.5 shows that this comparison is particularly simple if distributions are single-crossing. Limits on ratios of marginal utilities are useful for interpreting $(1+\gamma)$-SD preferences within the expected utility framework, and they can be connected with indices of greediness and thriftiness. Transfers are useful for explaining $(1+\gamma)$-SD to decision makers and for facilitating the elicitation of γ. Thus, our developments allow different interpretations of $(1+\gamma)$-SD preferences, checking whether such preferences apply, and then establishing non-dominated alternatives.

A topic for further work is the extension of our approach to dominance of higher orders, e.g., preferences falling between SSD and third-degree SD. Then these results can be useful for comparative statics, as illustrated in Tsetlin et al. (2015) for generalized ASD.

Also for further research is the use of the connections of our stochastic dominance with reference-dependent utility to add to the toolkit of testing for reference dependence and related preferences. This would be in the spirit of Baucells and Heukamp (2006) on second-order stochastic dominance for preferences represented by S-shaped value functions and loss aversion.

Another possible application of the concept of $(1+\gamma)$-SD is related to stochastic optimization. In a recent paper, Armbruster and Delage (2015) discuss the usefulness of ASD in handling stochastic dominance constraints, and the results developed here should be applicable in a similar way.

A Proofs

Proofs of Section 2

Proof of Theorem 2.4. Any $u \in \mathscr{U}_\gamma^*$ can be approximated by a sequence of functions in $\mathscr{U}_\gamma$ as in the proof of Theorem 2.1 in Denuit and Müller (2002), since $\mathscr{U}_\gamma^*$ is invariant under translations. From this result it follows that (a) implies (b).

We now show that (b) implies (c). We will frequently use the fact that for $u \in \mathscr{U}_\gamma^*$ with a right derivative u' and F, G with finite means the equality

$$\int u \, \mathrm{d}G - \int u \, \mathrm{d}F = \int u'(x)(F(x) - G(x)) \, \mathrm{d}x$$

holds due to integration by part.

We define for a fixed t the utility function v with the following right derivative:

$$v'(x) = \begin{cases} \gamma & \text{if } G(x) \le F(x) \text{ and } x \le t, \\ 1 & \text{if } F(x) < G(x) \text{ and } x \le t, \\ 0 & \text{if } t < x. \end{cases}$$

Then $v \in \mathscr{U}_\gamma^*$ and

$$\int v \, \mathrm{d}G - \int v \, \mathrm{d}F = \gamma \int_{-\infty}^{t} (G(x) - F(x))_- \, \mathrm{d}x - \int_{-\infty}^{t} (G(x) - F(x))_+ \, \mathrm{d}x,$$

hence $\int v \, \mathrm{d}F \le \int v \, \mathrm{d}G$ implies (2.3).

It remains to show that (c) implies (a). Let $u \in \mathscr{U}_\gamma$. Without loss of generality we can assume that $\sup_{x\in\mathbb{R}} u'(x) = 1$.

For a fixed $\varepsilon \in (0,1)$ call K the largest integer k for which

$$(1+\varepsilon)(1-\varepsilon)^k - \varepsilon \ge \inf_{x\in\mathbb{R}} u'(x),$$

and define a partition of the real line into intervals $[z_k, z_{k+1}]$ as follows: let $z_0 = -\infty$, $z_{K+1} = \infty$ and

$$z_k := \sup\{x : u'(x) \ge (1+\varepsilon)(1-\varepsilon)^k - \varepsilon\}, \quad k = 1, \dots, K.$$

Then we define

$$m_k := \sup\{u'(x) : z_{k-1} \le x \le z_k\} = (1+\varepsilon)(1-\varepsilon)^{k-1} - \varepsilon$$

and it follows from (2.1) that $\gamma m_{k+1} \le u'(x) \le m_k$, which is equivalent to

$$\gamma \left((m_k + \varepsilon)(1-\varepsilon) - \varepsilon \right) \le u'(x) \le m_k$$

for all $x \in [z_{k-1}, z_k]$ and $k = 1, \dots, K+1$.

This implies

$$\begin{aligned}
&\int_{z_{k-1}}^{z_k} u'(x)\left(F(x)-G(x)\right)\, \mathrm{d}x \\
&\qquad = \int_{z_{k-1}}^{z_k} u'(x)(G(x)-F(x))_-\, \mathrm{d}x - \int_{z_{k-1}}^{z_k} u'(x)(G(x)-F(x))_+\, \mathrm{d}x \\
&\qquad \geq \gamma\left((m_k+\varepsilon)(1-\varepsilon)-\varepsilon\right)\int_{z_{k-1}}^{z_k} (G(x)-F(x))_-\, \mathrm{d}x - m_k \int_{z_{k-1}}^{z_k} (G(x)-F(x))_+\, \mathrm{d}x \\
&\qquad = m_k T_k - \varepsilon c_k,
\end{aligned}$$

where

$$T_k := \gamma \int_{z_{k-1}}^{z_k} (G(x)-F(x))_-\, \mathrm{d}x - \int_{z_{k-1}}^{z_k} (G(x)-F(x))_+\, \mathrm{d}x$$

and

$$c_k := \gamma\left(m_k+\varepsilon\right)\int_{z_{k-1}}^{z_k} (G(x)-F(x))_-\, \mathrm{d}x.$$

The integral condition (2.3) implies $\sum_{i=1}^{k} T_i \geq 0$ for all $k \leq K+1$, which in turn implies $\sum_{k=1}^{K+1} m_k T_k \geq 0$ for all decreasing non-negative sequences m_k. Thus

$$\int u'(x)(F(x)-G(x))\, \mathrm{d}x \geq \sum_{k=1}^{K+1}(m_k T_k - \varepsilon c_k) \geq -\varepsilon \sum_{k=1}^{K+1} c_k \geq -\varepsilon\gamma\left(1+\varepsilon\right)\int (G(x)-F(x))_-\, \mathrm{d}x.$$

As this holds for arbitrary $\varepsilon > 0$, we get

$$\int u\, \mathrm{d}G - \int u\, \mathrm{d}F = \int u'(x)\left(F(x)-G(x)\right)\, \mathrm{d}x \geq 0. \qquad \square$$

Proof of Corollary 2.5. We define the function

$$\phi_\gamma(t) := \gamma \int_{-\infty}^{t} (G(x)-F(x))_-\, \mathrm{d}x - \int_{-\infty}^{t} (G(x)-F(x))_+\, \mathrm{d}x,\ t \in \mathbb{R}.$$

If the single crossing condition holds with the areas A and B as given in (2.4), this function is non-decreasing on $(-\infty, t_0)$ and non-increasing on (t_0, ∞) with

$$\lim_{t\to-\infty} \phi_\gamma(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \phi_\gamma(t) = \gamma A - B.$$

Thus ϕ_γ is non-negative if and only if $\gamma \geq B/A$. □

Intermediate steps in Example 2.11. If we define B as in Corollary 2.4, then

$$\begin{aligned} B &= \int_{x_1}^{\infty} (G(x) - F(x)) \, \mathrm{d}x \\ &= \int_{x_1}^{\infty} \left(H\left(\frac{x-\mu_G}{\sigma_G}\right) - H\left(\frac{x-\mu_F}{\sigma_F}\right)\right) \, \mathrm{d}x \\ &= \int_{x_1}^{\infty} \left(1 - H\left(\frac{x-\mu_F}{\sigma_F}\right)\right) \, \mathrm{d}x - \int_{x_1}^{\infty} \left(1 - H\left(\frac{x-\mu_G}{\sigma_G}\right)\right) \, \mathrm{d}x. \end{aligned}$$

In turn, using

$$z = \frac{x-\mu_G}{\sigma_G},$$

we get

$$\int_{x_1}^{\infty} \left(1 - H\left(\frac{x-\mu_G}{\sigma_G}\right)\right) \, \mathrm{d}x = \sigma_G \int_{\frac{x_1-\mu_G}{\sigma_G}}^{\infty} (1 - H(z) \, \mathrm{d}z.$$

If we let $\Delta\mu = \mu_G - \mu_F$ and $\Delta\sigma = \sigma_F - \sigma_G$, we have

$$\frac{x_1-\mu_G}{\sigma_G} = \frac{\Delta\mu}{\Delta\sigma}.$$

Then

$$\int_{x_1}^{\infty} \left(1 - H\left(\frac{x-\mu_G}{\sigma_G}\right)\right) \, \mathrm{d}x = \sigma_G \int_{\frac{\Delta\mu}{\Delta\sigma}}^{\infty} (1 - H(z)) \, \mathrm{d}z.$$

Similarly,

$$\int_{x_1}^{\infty} \left(1 - H\left(\frac{x-\mu_F}{\sigma_F}\right)\right) \, \mathrm{d}x = \sigma_F \int_{\frac{\Delta\mu}{\Delta\sigma}}^{\infty} (1 - H(z)) \, \mathrm{d}z$$

and

$$B = \Delta\sigma \int_{\frac{\Delta\mu}{\Delta\sigma}}^{\infty} (1 - H(z)) \, \mathrm{d}z.$$

Theorem 2.7 follows from Müller (2013, Theorem 2.4.1), but we prefer to give a constructive proof here, since the proof in Müller (2013) is not constructive.

For the proof of Theorem 2.7 we will need the following lemma.

Lemma A.1. *We have $F \leq_{(1+\gamma)\text{-SD}} G$ if and only if*

$$\int_0^p (G^{-1}(x) - F^{-1}(x))_- \, \mathrm{d}x \leq \gamma \int_0^p (G^{-1}(x) - F^{-1}(x))_+ \, \mathrm{d}x$$

for all $p \in (0,1)$.

Proof. Given random variables X, Y with distribution functions F, G then $X \le_{(1+\gamma)\text{-SD}} Y$ holds if and only if (2.3) holds, i.e., if the function

$$t \mapsto h(t) := \int_{-\infty}^{t} \gamma(G(x)-F(x))_- - (G(x)-F(x))_+ \,\mathrm{d}x \ge 0 \quad \text{for all } t.$$

The function h assumes local minima in the points b_k where the distribution functions F and G cross, going from $F \le G$ left of b_k to $F > G$ right of b_k. Therefore it is sufficient to check $h(b_k) \ge 0$ in these points. If we define $p_k := G(b_k)$ and

$$\tilde{h}(p) := \int_0^p \gamma(G^{-1}(x)-F^{-1}(x))_+ - (G^{-1}(x)-F^{-1}(x))_- \,\mathrm{d}x,$$

then we have $h(b_k) = \tilde{h}(p_k)$ and the function $\tilde{h}$ assumes its local minima in the points p_k. Therefore $h \ge 0$ if and only if $\tilde{h} \ge 0$. □

Proof of Theorem 2.7. We use an idea similar to Machina and Pratt (1997). Define

$$A^+(p) := \int_0^p (G^{-1}(x)-F^{-1}(x))_+ \,\mathrm{d}x \quad \text{and } A^-(p) := \int_0^p (G^{-1}(x)-F^{-1}(x))_- \,\mathrm{d}x.$$

Let $u(a)$ and $v(a)$ be the smallest probabilities that solve

$$A^+(u(a)) = a \quad \text{and } A^-(v(a)) = \gamma a, \quad 0 < a < A^-(1).$$

It follows from Lemma A.1 that $A^-(p) \le \gamma A^+(p)$, hence $u(a) \le v(a)$ for all $0 < a < A^-(1)$. We set

$$x_1(a) := F^{-1}(u(a)),\ x_2(a) := G^{-1}(u(a)),\ x_3(a) := G^{-1}(v(a)),\ x_4(a) := F^{-1}(v(a)),$$

$0 < a < A^-(1)$. If F and G assume only finitely many values, then there is a finite sequence $0 = a_1 < a_2 < \ldots < a_k \le A^-(1)$ such that the functions $a \mapsto x_1(a), \ldots, x_4(a)$ are constant on (a_{i-1}, a_i). We denote the corresponding values of these functions as

$$x_{\ell,i} := x_\ell(a) \quad \text{for } a \in (a_{i-1}, a_i),\ \ell = 1, \ldots, 4.$$

Moreover, in the points $x_{1,i}$ and $x_{4,i}$ the function F has jumps of sizes at least $\eta_{1,i}$ and $\eta_{2,i}$, and in the corresponding points $x_{2,i}$ and $x_{3,i}$ the function G has jumps of sizes at least $\eta_{1,i}$ and $\eta_{2,i}$, where

$\eta_{1,i}$ and $\eta_{2,i}$ are given by the equation

$$\eta_{1,i}(x_{2,i} - x_{1,i}) = \frac{1}{\gamma}\eta_{2,i}(x_{4,i} - x_{3,i}) = a_i - a_{i-1}, \quad i = 1, \dots, k.$$

For $x \geq x_{4,k}$ we have $F(x) \geq G(x)$. Thus G is obtained from F by a sequence of k γ-transfers described by the corresponding x's and η's above, plus a finite number of increasing transfers moving the mass from F to G right of $x_{4,k}$. □

Proof of Theorem 2.8. First note that for random variables X_n, X with distribution functions F_n, F the convergence $X_n \Rightarrow X$ mentioned in the theorem holds if and only if convergence in the so called Kantorovich metric holds, i.e.

$$\int_{-\infty}^{\infty} |F_n(x) - F(x)| \, \mathrm{d}x \to 0,$$

(see, e.g., Bickel and Freedman, 1981). This of course implies

$$\int_{-\infty}^{t} (F_n(x) - F(x))_+ \, \mathrm{d}x \to 0 \quad \text{and} \quad \int_{-\infty}^{t} (F_n(x) - F(x))_- \, \mathrm{d}x \to 0.$$

The if-part thus follows from (2.3).

For the only-if-part first assume that X, Y are bounded. Then define for any $k \in \mathbb{N}$

$$X_k := \frac{i}{k}, \text{ if } \frac{i}{k} \leq X < \frac{i+1}{k}, \quad Y_k := \frac{i+1}{k}, \text{ if } \frac{i}{k} \leq Y < \frac{i+1}{k}, \ i \in \mathbb{Z}. \tag{A.1}$$

Then X_k and Y_k have finite support, $X_k \leq X$ and $Y \leq Y_k$ almost surely, and therefore $X_k \leq_{(1+\gamma)\text{-SD}} X \leq_{(1+\gamma)\text{-SD}} Y \leq_{(1+\gamma)\text{-SD}} Y_k$ and $X_n \Rightarrow X, \quad Y_n \Rightarrow Y$.

If X, Y are unbounded, then approximate it by bounded X_n, Y_n as follows: define

$$X_n := \begin{cases} x_n^*, & \text{if } X < -n, \\ X, & \text{if } -n \leq X \leq n, \\ n, & \text{if } n < X, \end{cases}$$

and

$$Y_n := \begin{cases} -n, & \text{if } Y < -n, \\ Y, & \text{if } -n \leq Y \leq n, \\ n, & \text{if } n < Y, \end{cases}$$

where

$$x_n^* := -n - \frac{\int_{-\infty}^{-n} (G(x) - F(x))_- \, \mathrm{d}x}{P(X < -n)}.$$

Now define for the corresponding distribution functions F_n, G_n the functions

$$\psi_n(t) := \int_{-\infty}^{t} (G_n(x) - F_n(x))_+ \, \mathrm{d}x \quad \text{and } \xi_n(t) := \gamma \int_{-\infty}^{t} (G_n(x) - F_n(x))_- \, \mathrm{d}x$$

and similarly ψ and ξ with the distribution functions F, G. An easy calculations shows that these functions fulfill

$$\psi_n(t) = \begin{cases} 0, & \text{if } t < -n, \\ \psi(t) - \psi(-n), & \text{if } -n \leq t \leq n, \\ \psi(n) - \psi(-n), & \text{if } n < t, \end{cases}$$

and

$$\xi_n(t) = \begin{cases} \xi(t), & \text{if } -n \leq t \leq n, \\ \xi(n), & \text{if } n < t. \end{cases}$$

Thus $\psi(t) \leq \xi(t)$ for all $t \in \mathbb{R}$ implies $\psi_n(t) \leq \xi_n(t)$ for all $t \in \mathbb{R}$, and from Theorem 2.4 it then follows that $X \leq_{(1+\gamma)\text{-SD}} Y$ implies $X_n \leq_{(1+\gamma)\text{-SD}} Y_n$, and obviously X_n, Y_n are bounded and $X_n \Rightarrow X, \quad Y_n \Rightarrow Y$.

For each fixed n we can approximate X_n and Y_n by sequences (X_{nk}) and (Y_{nk}) as in (A.1). The sequences (X_{nn}) and (Y_{nn}) then fulfill the conditions of the theorem. □

Proofs of Section 4

Proof of Theorem 4.3. We consider the case $\gamma_{\text{cv}} > \gamma_{\text{cx}}$. Case $\gamma_{\text{cx}} > \gamma_{\text{cv}}$ is similar, and case $\gamma_{\text{cv}} = \gamma_{\text{cx}}$ is proved in Leshno and Levy (2002). Then the inequality (4.2) becomes

$$\int_{-\infty}^{t} (G(x) - F(x))_+ \, \mathrm{d}x + \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}} \int_{t}^{\infty} (G(x) - F(x))_+ \, \mathrm{d}x \\ \leq \gamma_{\text{cv}} \int_{-\infty}^{t} (F(x) - G(x))_+ \, \mathrm{d}x + \gamma_{\text{cx}} \int_{t}^{\infty} (F(x) - G(x))_+ \, \mathrm{d}x. \tag{A.2}$$

We will now show that (A.2) implies $F \leq_{(1+\gamma_{\text{cv}}, 1+\gamma_{\text{cx}})\text{-SD}} G$. Let $u \in \mathscr{U}_{\gamma_{\text{cv}}, \gamma_{\text{cx}}}$. Without loss of generality we can assume that $\sup_{x \in \mathbb{R}} u'(x) = 1$.

For a fixed $\varepsilon \in (0,1)$ call K the largest k for which

$$(1-\varepsilon)^k \left(1 - \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right) + \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}} \geq \inf_{x\in\mathbb{R}} u'(x).$$

Notice that K is finite only if

$$\inf_{x\in\mathbb{R}} u'(x) > \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}.$$

If K is finite then define a finite partition of the real line into intervals $[z_k, z_{k+1}]$ as follows: let $z_0 = -\infty$, $z_{K+1} = \infty$, and

$$z_k := \sup\left\{x : u'(x) \geq (1-\varepsilon)^k \left(1 - \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right) + \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right\}, \quad k = 1, \dots, K.$$

Let

$$m_k := \sup\{u'(x) : z_{k-1} \leq x \leq z_k\} = (1-\varepsilon)^{k-1} \left(1 - \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right) + \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}.$$

From (4.1) we get

$$\gamma_{\mathrm{cv}} m_{k+1} \leq u'(x) \leq m_k,$$

which is equivalent to

$$\gamma_{\mathrm{cv}} \left(\left(m_k - \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right)(1-\varepsilon) + \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}\right) \leq u'(x) \leq m_k,$$

hence

$$\gamma_{\mathrm{cv}}(1-\varepsilon) m_k + \varepsilon \gamma_{\mathrm{cx}} \leq u'(x) \leq m_k$$

for all $x \in [z_{k-1}, z_k]$ and all $k = 1, \dots, K+1$.

This implies

$$\begin{aligned}
&\int_{z_{k-1}}^{z_k} u'(x) \left(F(x) - G(x)\right) \,\mathrm{d}x \\
&\quad = \int_{z_{k-1}}^{z_k} u'(x)(F(x) - G(x))_+ \,\mathrm{d}x - \int_{z_{k-1}}^{z_k} u'(x)(G(x) - F(x))_+ \,\mathrm{d}x \\
&\quad \geq (\gamma_{\mathrm{cv}}(1-\varepsilon) m_k + \varepsilon\gamma_{\mathrm{cx}}) \int_{z_{k-1}}^{z_k} (F(x) - G(x))_+ \,\mathrm{d}x - m_k \int_{z_{k-1}}^{z_k} (G(x) - F(x))_+ \,\mathrm{d}x \\
&\quad = m_k T_k - \varepsilon c_k,
\end{aligned}$$

where

$$T_k := \gamma_{\mathrm{cv}} \int_{z_{k-1}}^{z_k} (F(x) - G(x))_+ \, \mathrm{d}x - \int_{z_{k-1}}^{z_k} (G(x) - F(x))_+ \, \mathrm{d}x$$

and

$$c_k := (\gamma_{\mathrm{cv}} m_k - \gamma_{\mathrm{cx}}) \int_{z_{k-1}}^{z_k} (G(x) - F(x))_- \, \mathrm{d}x.$$

If K is not finite, then the sequence (z_k) as defined above has a finite limit $z_\infty := \lim_{k\to\infty} z_k$, and we get an infinite partition of the real line in intervals $[z_k, z_{k+1}]$ with $k \in \mathbb{N}_0$ and an additional interval $[z_\infty, \infty)$. However, as we have assumed $\sup_{x\in\mathbb{R}} u'(x) = 1$, it follows from (4.1) that for $z_\infty \leq z < \infty$ we have

$$\gamma_{\mathrm{cx}} \leq u'(z) \leq m_\infty := \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}},$$

hence

$$\gamma_{\mathrm{cv}} m_\infty \leq u'(z) \leq m_\infty.$$

Similarly we get

$$\int_{z_\infty}^{\infty} u'(x) \left(F(x) - G(x)\right) \, \mathrm{d}x \geq m_\infty T_\infty,$$

where

$$T_\infty := \gamma_{\mathrm{cv}} \int_{z_\infty}^{\infty} (F(x) - G(x))_+ \, \mathrm{d}x - \int_{z_\infty}^{\infty} (G(x) - F(x))_+ \, \mathrm{d}x.$$

With a little abuse of notation in the case $K = \infty$ the integral over the whole real line can now be written as a sum over the partition as follows:

$$\int_{-\infty}^{\infty} u'(x) \left(F(x) - G(x)\right) \, \mathrm{d}x = \sum_{k=1}^{K+1} \int_{z_{k-1}}^{z_k} u'(x) \left(F(x) - G(x)\right) \, \mathrm{d}x.$$

The inequality (A.2) implies

$$\sum_{i=1}^{k} T_i + \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}} \sum_{i=k+1}^{K+1} T_i \geq 0$$

for all $k \leq K+1$, which in turn implies $\sum_{k=1}^{K+1} m_k T_k \geq 0$ for all decreasing sequences m_k such that

$$1 \geq m_k \geq \frac{\gamma_{\mathrm{cx}}}{\gamma_{\mathrm{cv}}}.$$

To see this, denote

$$B(0) = \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}} \sum_{k=1}^{K+1} T_k,$$

$$B(k) = \sum_{i=1}^{k} T_i + \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}} \sum_{i=k+1}^{K+1} T_i = \left(1 - \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}}\right) T_k + B(k-1),$$

so that

$$T_k = \frac{B(k) - B(k-1)}{1 - \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}}}$$

and denote

$$a_k = \frac{m_k - \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}}}{1 - \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}}}.$$

Then

$$\begin{aligned} \sum_{k=1}^{K+1} m_k T_k &= \sum_{k=1}^{K+1} a_k \left(1 - \frac{\gamma_{\text{cx}}}{\gamma_{\text{cv}}}\right) T_k + B(0) \\ &= \sum_{k=1}^{K+1} a_k \left(B(k) - B(k-1)\right) + B(0) \\ &= \sum_{k=1}^{K+1} \left(a_k - a_{k+1}\right) B(k) + (1 - a_1) B(0). \end{aligned}$$

Since $B(k) \geq 0$ for $k \geq 0$ and the sequence a_k is decreasing with $a_1 \leq 1$, $\sum_{k=1}^{K+1} m_k T_k$ is non-negative. Thus

$$\int_{-\infty}^{\infty} u'(x)(F(x) - G(x))\, \mathrm{d}x \geq \sum_{k=1}^{K+1} (m_k T_k - \varepsilon c_k) \geq -\varepsilon \sum_{k=1}^{K+1} c_k = -\varepsilon(\gamma_{\text{cv}} m_k - \gamma_{\text{cx}}) \int (G(x) - F(x))_{-}\, \mathrm{d}x.$$

As this holds for arbitrary $\varepsilon > 0$, we get

$$\int_{-\infty}^{\infty} u \,\mathrm{d}G - \int_{-\infty}^{\infty} u \,\mathrm{d}F = \int_{-\infty}^{\infty} u'(x) \left(F(x) - G(x)\right)\, \mathrm{d}x \geq 0. \qquad \square$$

The following theorems follow easily from Müller (1997, Theorem 4.2). We report the proofs for the sake of completeness.

Proof of Theorem 4.8. Let $u_1 \in \mathscr{U}_{\gamma_{\mathrm{cv}}^{(1)},\gamma_{\mathrm{cx}}^{(1)}}$, $u_2 \in \mathscr{U}_{\gamma_{\mathrm{cv}}^{(2)},\gamma_{\mathrm{cx}}^{(2)}}$ and let $u(x) := u_1(u_2(x))$. For all $x \le y$ we have

$$u'(x) = u_1'(u_2(x))u_2'(x) \ge \gamma_{\mathrm{cv}}^{(1)} u_1'(u_2(y))\gamma_{\mathrm{cv}}^{(2)} u_2'(y) = \gamma_{\mathrm{cv}}^{(1)} \times \gamma_{\mathrm{cv}}^{(2)} u'(y)$$

and, by a similar argument,

$$u'(x) \le \frac{1}{\gamma_{\mathrm{cx}}^{(1)} \times \gamma_{\mathrm{cx}}^{(2)}} u'(y).$$

Therefore $u \in \mathscr{U}_{\gamma_{\mathrm{cv}}^{(1)}\times\gamma_{\mathrm{cv}}^{(2)},\gamma_{\mathrm{cx}}^{(1)}\times\gamma_{\mathrm{cx}}^{(2)}}$. This implies the desired result. □

Proof of Theorem 4.9. For any $u \in \mathscr{U}_{\gamma_{\mathrm{cv}},\gamma_{\mathrm{cx}}}$ we have

$$\mathbb{E}[u(X)] = \mathbb{E}[\mathbb{E}[u(X) \mid \Theta] \le \mathbb{E}[\mathbb{E}[u(Y) \mid \Theta] = \mathbb{E}[u(Y)].$$ □

Proof of Theorem 4.10. Let $u \in \mathscr{U}_{\gamma_{\mathrm{cv}},\gamma_{\mathrm{cx}}}$. Define $\phi(z) = \mathbb{E}[u(X+z)]$ and $\psi(z) = \mathbb{E}[u(Y+z)]$. The function $x \mapsto u(x+z)$ is in $\mathscr{U}_{\gamma_{\mathrm{cv}},\gamma_{\mathrm{cx}}}$ for all $z \in \mathbb{R}$, therefore

$$\mathbb{E}[u(X+Z)] = \mathbb{E}[\phi(Z)] \le \mathbb{E}[\psi(Z)] = \mathbb{E}[u(Y+Z)].$$ □

B Further results

From Corollary 2.5, it is easy to check for $(1+\gamma)$-SD when F and G cross exactly once, which is often the case. To cover situations where F and G cross more than once, Corollary B.1, which generalizes the single-crossing result to the case where F and G cross multiple times, can be applied. The easy proof is omitted.

Corollary B.1. *Suppose there exist $M \in \mathbb{N}$ and $x_1 \le x_2 \le \ldots \le x_M$, with $x_0 = -\infty$ and $x_{M+1} = \infty$, such that $F(x) \ge G(x)$ for $x_{i-1} \le x < x_i$ if i is odd and $F(x) \le G(x)$ for $x_{i-1} \le x < x_i$ if i is even, $i = 1, \ldots, M+1$. Denote*

$$A_i = \int_{x_{i-1}}^{x_i} (F(x) - G(x))_+ \,\mathrm{d}x \quad \text{and} \quad B_i = \int_{x_{i-1}}^{x_i} (G(x) - F(x))_+ \,\mathrm{d}x \quad \text{for } i = 1, \ldots, M+1.$$

Then $F \le_{(1+\gamma)\text{-SD}} G$ if and only if

$$\gamma \ge \frac{\sum_{i=1}^{j+1} B_i}{\sum_{i=1}^{j+1} A_i} \quad \text{for all } j = 1, \ldots, M. \tag{B.1}$$

Corollary B.1 just necessitates the comparison of areas at a finite number of points, and should suffice to check for $(1+\gamma)$-SD for practical purposes, because infinite crossings are quite exotic. If F and G cross only once, then $M = 1$ and Corollary B.1 reduces to Corollary 2.5. Notice also that in Corollary B.1 we have $B_k = 0$ if k is odd, and $A_k = 0$ if k is even. So the condition in (B.1) could also be written in the form $\gamma \geq \sum B_{2k} / \sum A_{2k-1}$. In particular, it is a sufficient (but not necessary) condition that $\gamma \geq B_{2k}/A_{2k-1}$ for all k.

Example B.2 (Multiple crossing)**.** Suppose that F is continuous and G is a two-point distribution with equally likely outcomes $\mu \pm \sigma$. Let $\mathsf{M}(X)$ be the median of F, and assume that $\mu - \sigma < \mathsf{M}(X) < \mu + \sigma$, in which case F and G cross more than once as long as $\mu - \sigma$ and/or $\mu + \sigma$ are inside the support of F.

By Corollary B.1, $G \leq_{(1+\gamma)\text{-SD}} F$ if and only if $F(\mu - \sigma) = 0$ and

$$\int_{\mathsf{M}(X)}^{\mu+\sigma} (F(x) - 0.5)\, \mathrm{d}x \leq \gamma \int_{\mu-\sigma}^{\mathsf{M}(X)} (0.5 - F(x))\, \mathrm{d}x,$$

and $F \leq_{(1+\gamma)\text{-SD}} G$ if and only if $F(\mu - \sigma) > 0$ and the following two inequalities are satisfied:

$$\int_{\mu-\sigma}^{\mathsf{M}(X)} (0.5 - F(x))\, \mathrm{d}x \leq \gamma \int_{-\infty}^{\mu-\sigma} F(x)\, \mathrm{d}x,$$

$$\int_{\mu-\sigma}^{\mathsf{M}(X)} (0.5 - F(x))\, \mathrm{d}x + \int_{\mu+\sigma}^{\infty} (1 - F(x))\, \mathrm{d}x \leq \gamma \left(\int_{-\infty}^{\mu-\sigma} F(x)\, \mathrm{d}x + \int_{\mathsf{M}(X)}^{\mu+\sigma} (F(x) - 0.5)\, \mathrm{d}x \right).$$

The first inequality is similar to Example 2.10, with a sure payoff of $\mu - \sigma$ compared to the distribution with cdf $\min\{2F(x), 1\}$ (i.e., F truncated to the right of $\mathsf{M}(X)$). If the first inequality holds, then

$$\int_{\mu+\sigma}^{\infty} (1 - F(x))\, \mathrm{d}x \leq \gamma \int_{\mathsf{M}(X)}^{\mu+\sigma} (F(x) - 0.5)\, \mathrm{d}x,$$

which is similar to the case where a sure payoff of $\mu + \sigma$ is compared to the distribution with cdf $\max\{2F(x) - 1, 0\}$, i.e., F truncated to the left of $\mathsf{M}(X)$, is sufficient but not necessary for the second inequality to hold.

Alternatively, F can be presented as a 50–50 mixture of distributions with cdfs $\min\{2F(x), 1\}$ and $\max\{2F(x) - 1, 0\}$, and G as a 50–50 mixture of sure payoffs $\mu \pm \sigma$. If each component of a 50–50 lottery corresponding to G dominates each component of a 50–50 lottery corresponding to F, then $F \leq_{(1+\gamma)\text{-SD}} G$. Dominance of $\mu - \sigma$ over $\min\{2F(x), 1\}$ (i.e., dominance involving the “bad” part of each lottery) is a necessary condition for $F \leq_{(1+\gamma)\text{-SD}} G$. If this dominance is stronger (i.e.,

$\int_{\mu-\sigma}^{\mathsf{M}(X)}(0.5 - F(x))\ \mathrm{d}x \leq \gamma \int_{-\infty}^{\mu-\sigma} F(x)\ \mathrm{d}x$ is strict), then dominance of $\mu+\sigma$ over $\max\{2F(x)-1, 0\}$ (i.e., dominance involving the "good" part of each lottery) does not have to hold (i.e., it might be that $\int_{\mu+\sigma}^{\infty}(1 - F(x))\ \mathrm{d}x > \gamma \int_{\mathsf{M}(X)}^{\mu+\sigma}(F(x) - 0.5)\ \mathrm{d}x$).

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Figures

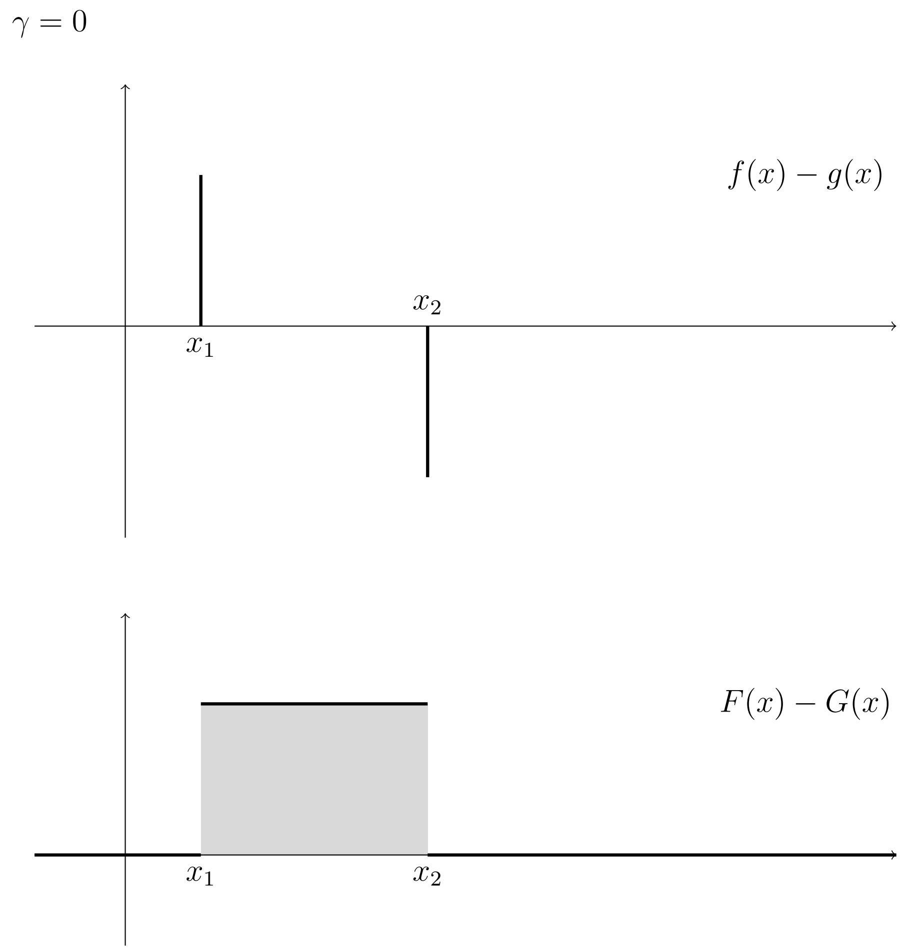

Figure 1: The differences $f - g$ and $F - G$ corresponding to an increasing transfer

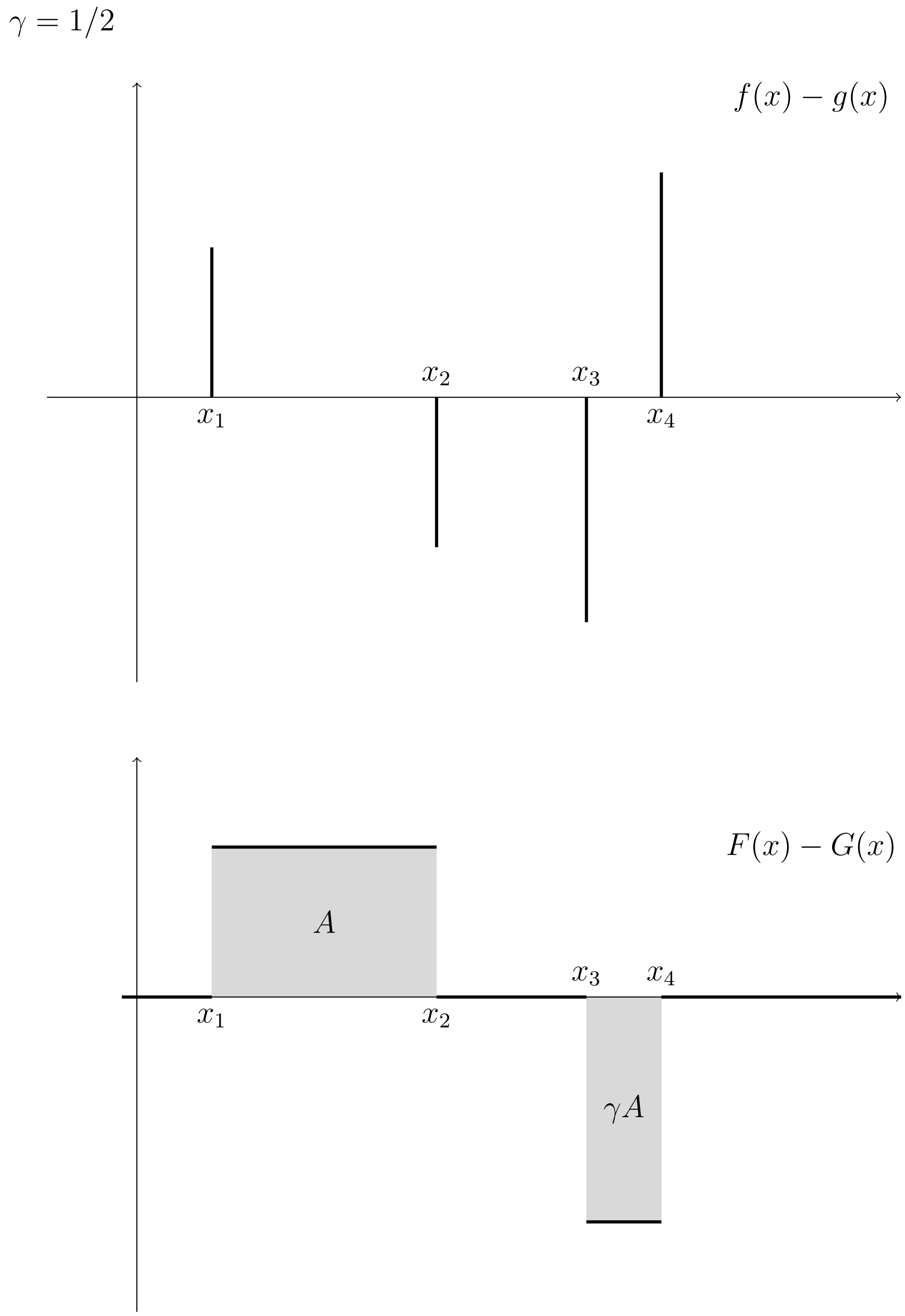

Figure 2: The differences $f - g$ and $F - G$ corresponding to a γ-transfer

$\gamma = 1$

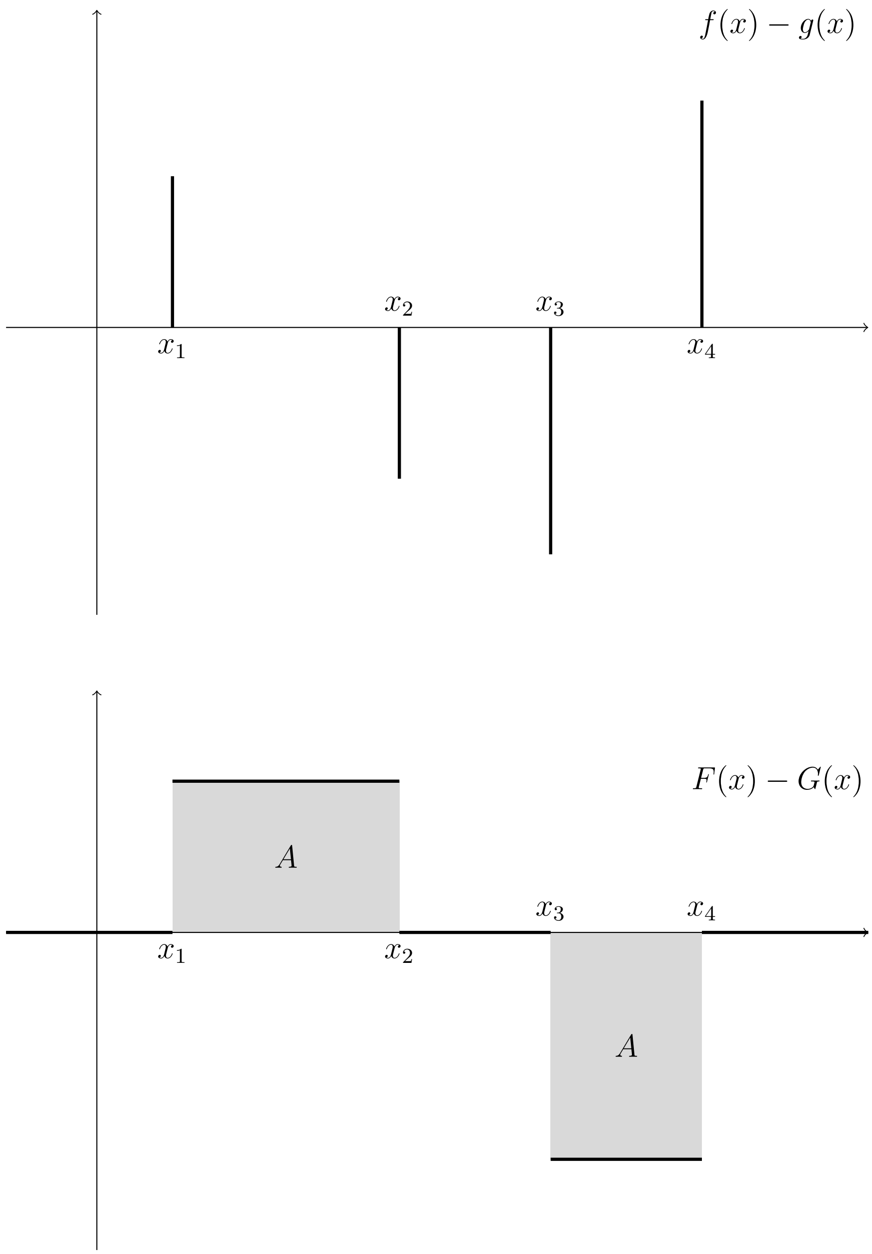

Figure 3: The differences $f - g$ and $F - G$ corresponding to a concave transfer

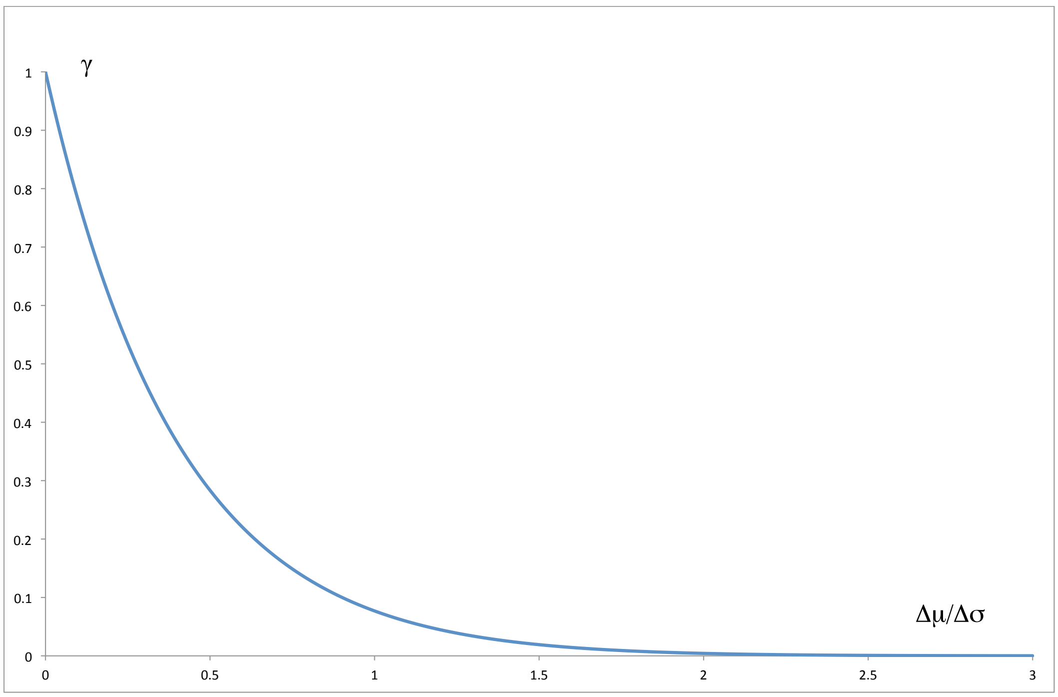

Figure 4: Comparing normal distributions: γ as a function of $\Delta\mu/\Delta\sigma$

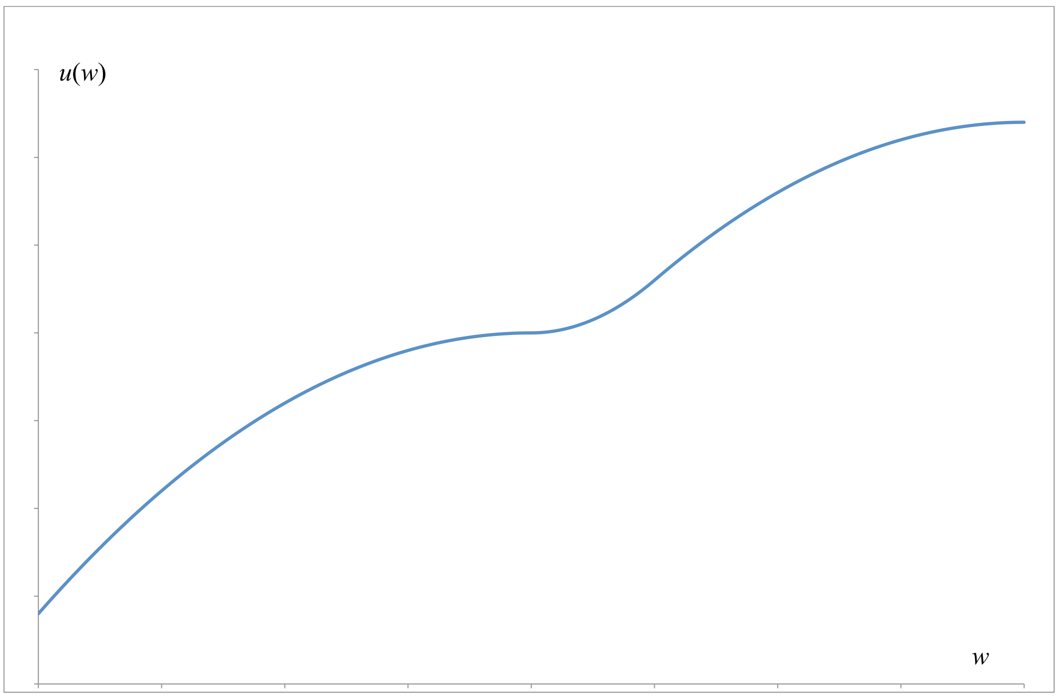

Figure 5: A concave-convex-concave utility function

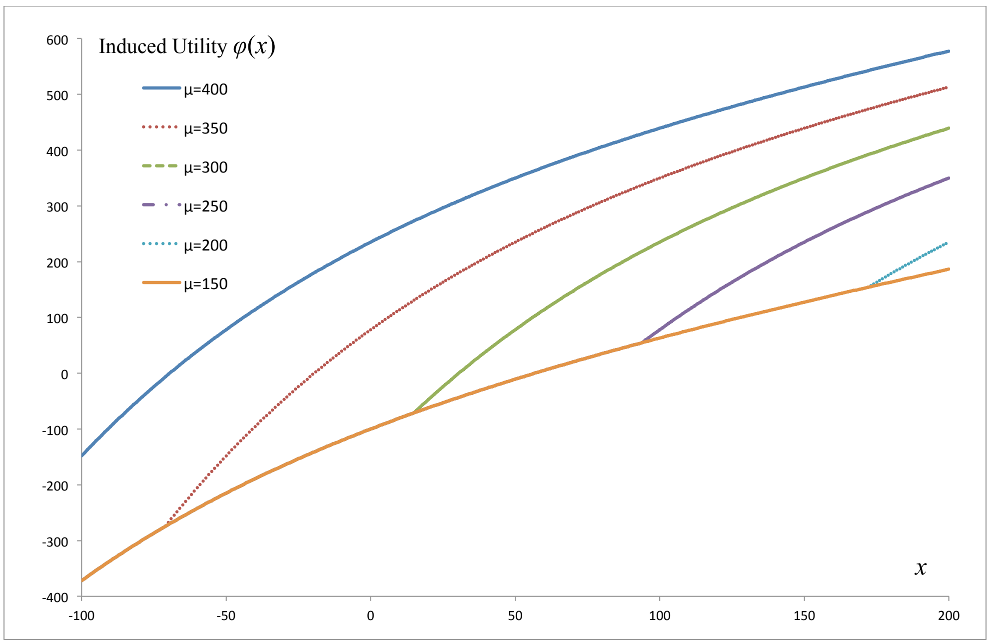

Figure 6: Induced utility with local convexities

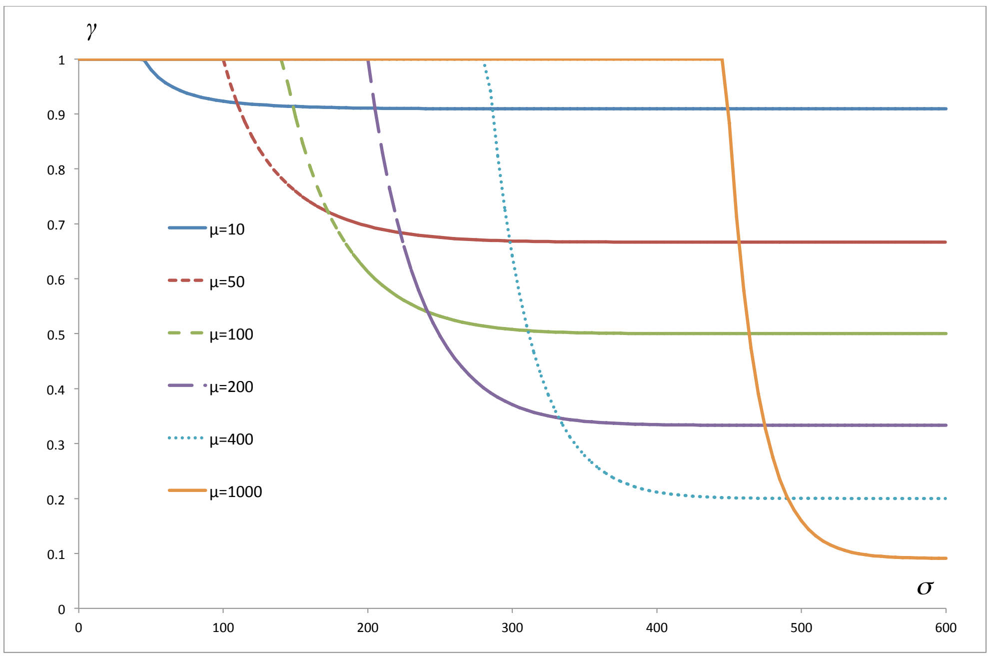

Figure 7: Whether to undertake a project: γ as a function of the standard deviation of the payoff