SIMPLIFYING AND SOLVING DECISION PROBLEMS BY STOCHASTIC DOMINANCE RELATIONS

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INTRODUCTION

The concept of expected utility provides a powerful framework for modeling decision making under risk. To compare two alternatives, we just need to compare their expected utilities. A potential obstacle here is that eliciting utility functions might be difficult. In a consulting context, a consultant might have to suggest a course of action without having the opportunity to elicit a corporate utility function. In a group decision making context, different parties might have different utilities, but if every group member prefers the same alternative, the difference in individual utilities is irrelevant.

Therefore, if two alternatives can be compared in a way that sets some of the particulars of the utility function aside, the situation is simplified, the resulting decision is more reliable, and might be acceptable to a wider group of involved parties. The concept of stochastic dominance offers that type of comparison. It is a formal statistical property, allowing partial ranking of probability distributions. If one alternative is preferred to another in the sense of the first-order stochastic dominance, then any decision maker with an increasing utility function would have the same preference. If one alternative is preferred to another in the sense of the second-order stochastic dominance, then any decision maker with an increasing and concave utility would have the same preference. Similar analogies hold for higher orders. In the case of multiple choices, even if stochastic dominance criterion does not provide complete ordering of all possible decisions, it might help to eliminate the dominated ones, so that more thorough analysis will focus on the subset of the most promising alternatives. The inferior alternative(s) can be formally identified via the concept of convex stochastic dominance [1,2].

Fishburn and Vickson [3], Whitmore and Findlay [4], and Levy [5] provide comprehensive surveys of stochastic dominance literature. Empirical tests for stochastic dominance efficiency using the notion of convex stochastic dominance are described in, e.g., Refs 6 and 7. Our article focuses on more recent results that highlight the link between stochastic dominance of different orders.

The next section presents several equivalent definitions of stochastic dominance, and explains the connection with the expected utility framework. After stating formal general results, we discuss and illustrate them at the intuitive level. The section titled "Stochastic Dominance as the Preference for Combining Good with Bad" shows that stochastic dominance of different orders can be connected via the preference for combining good with bad, while the next section presents the applications.

STOCHASTIC DOMINANCE AND Nth-DEGREE RISK

We start with a definition of stochastic dominance [8–10]: Assume that all random

variables have bounded supports contained within the interval $[a, b]$. Let F denote the cumulative distribution function (cdf) for such a random variable. Define $F^{(0)}(x) \equiv F(x)$ and define $F^{(i)}(x) \equiv \int_a^x F^{(i-1)}(t)\,\mathrm{d}t$ for $i \geq 1$.

Definition 1. Distribution F weakly dominates Distribution G in the sense of Nth-order stochastic dominance if

1. $F^{(N-1)}(x) \leq G^{(N-1)}(x)$ for all $a \leq x \leq b$,
2. $F^{(i)}(b) \leq G^{(i)}(b)$ for $i = 1, \ldots, N-2$.

We write F NSD G to denote that F dominates G via Nth-order stochastic dominance. If the random variables $\widetilde{X}$ and $\widetilde{Y}$ have cdfs F and G respectively, we will also say that $\widetilde{X}$ NSD $\widetilde{Y}$.

Integrals of cdfs, appearing in Definition 1, can be expressed via lower partial moments:

$$\int_a^x (x-t)^k\,\mathrm{d}F(t) = (k!)F^{(k)}(x), \ k = 1, 2, \ldots. \tag{1}$$

Equation (1) follows from integration by parts:

$$\int_a^x (x-t)^k\,\mathrm{d}F(t) = \left. (x-t)^k F(t)\right|_{t=a}^{t=x} + k\int_a^x (x-t)^{k-1}\,\mathrm{d}F^{(1)}(t),$$

where the first term is zero since $F^{(k)}(a) = 0$ and $\left.(x-t)^k\right|^{t=x} = 0$. Repeating the integration by parts $(k-1)$ times yields Equation (1).

The following theorem expresses the well-known links between stochastic dominance and expected utility. [Hadar and Russell [11] and Hanoch and Levy [12], introduced this notion into the economics literature for second-order stochastic dominance. See Refs 9 and 13, as well as [8], for extensions to higher orders of stochastic dominance.] Here $u(x)$ denotes the individual's utility function. For notational convenience, we use $u^{(n)}(x)$ to denote $\frac{\mathrm{d}^n u(x)}{\mathrm{d}x^n}$.

Theorem 1. *The following statements are equivalent:*

1. *F dominates G in the sense of Nth-order stochastic dominance,*
2. *$\int_a^b u(x)\,dF(x) \geq \int_a^b u(x)\,dG(x)$, for all functions u such that $\operatorname{sgn}\ u^{(n)}(x) = (-1)^{n+1}$ for $n = 1, \ldots, N$.*

Proof. Integrating by parts, the difference in expected utility for the distributions F and G can be expressed as

$$\int_a^b u(x)\,\mathrm{d}F(x) - \int_a^b u(x)\,\mathrm{d}G(x)$$

$$= u(x)\big[F(x) - G(x)\big]\Big|_{x=a}^{x=b}$$

$$+ (-1)\int_a^b u^{(1)}(x)\,\mathrm{d}\big[F^{(1)}(x) - G^{(1)}(x)\big]$$

$$= (-1)\int_a^b u^{(1)}(x)\big[F^{(0)}(x) - G^{(0)}(x)\big]\,\mathrm{d}x.$$

Repeating the integration by parts $(N-1)$ times yields

$$\int_a^b u(x)\,\mathrm{d}F(x) - \int_a^b u(x)\,\mathrm{d}G(x)$$

$$= \sum_{k=1}^{N-1} (-1)^k u^{(k)}(b)\big[F^{(k)}(b) - G^{(k)}(b)\big]$$

$$+ (-1)^N \int_a^b u^{(N)}(x)\big[F^{(N-1)}(x) - G^{(N-1)}(x)\big]\mathrm{d}x.$$

Therefore, the first claim of the theorem implies the second.

Equation (1) is useful to prove that the second claim of the theorem implies the first. Consider $u(t) = -(b-t)^k$, $k = 1, \ldots, N-2$. Then

$$\int_a^b u(t)\,\mathrm{d}F(t) \geq \int_a^b u(t)\,\mathrm{d}G(t)$$

$$\Leftrightarrow \int_a^b (b-t)^k\,\mathrm{d}G(t) \geq \int_a^b (b-t)^k\,\mathrm{d}F(t)$$

$$\Leftrightarrow F^{(k)}(b) \leq G^{(k)}(b).$$

Similarly, consider

$$u(t) = \begin{cases} -(x-t)^{N-1} & \text{if } t \leq x \\ 0 & \text{if } t > x \end{cases}.$$

Then

$$\int_a^b u(t)\mathrm{d}F(t) \geq \int_a^b u(t)\mathrm{d}G(t)$$

$$\Leftrightarrow \int_a^x (x-t)^{N-1}\mathrm{d}G(t) \geq \int_a^x (x-t)^{N-1}\mathrm{d}F(t)$$

$$\Leftrightarrow F^{(N-1)}(x) \leq G^{(N-1)}(x).$$

Stochastic dominance of order N implies stochastic dominance of any higher order. To isolate the Nth-order effect from the ones of lower orders, Ekern [14] introduced the concept of Nth-degree risk. The special case where $N = 2$ was introduced much earlier by Rothschild and Stiglitz [15], who call this case a "mean-preserving increase in risk."

Definition 2. Distribution F has more Nth-degree risk than Distribution G if

1. $F^{(N-1)}(x) \leq G^{(N-1)}(x)$ for all $a \leq x \leq b$,
2. $F^{(i)}(b) = G^{(i)}(b)$ for $i = 1, \dots, N-1$.

The following corollary to Theorem 1 relates the concept of Nth-degree risk to the expected utility framework.

Corollary 1. *The following statements are equivalent:*

1. *G has more N^{th}-degree risk than F,*
2. *$\int_a^b u(x)dF(x) \geq \int_a^b u(x)dG(x)$, for all functions u such that sgn $u^{(N)}(x) = (-1)^{N+1}$.*

As follows from Equation (1), G has more Nth-degree risk than F if and only if F NSD G and the first $N-1$ moments of F and G are identical.

How could we visualize those stochastic dominance relations at the intuitive level? To construct a pair of random variables ordered by the first-order stochastic dominance, we can take any random variable $\widetilde{X}$ and add to it a nonpositive random variable $\widetilde{Z}$. Then $\widetilde{X} + \widetilde{Z}$ is first-order dominated by $\widetilde{X}$. For the second-order dominance, $\widetilde{Z}$ can be a random variable with nonpositive conditional mean (i.e., $\mathbb{E}(\widetilde{Z}|\widetilde{X} = x) \leq 0$ for all x). Then $\widetilde{X} + \widetilde{Z}$ is second-order dominated by $\widetilde{X}$.

In the examples above, $\widetilde{X}$ can also be a degenerate random variable, that is, $\widetilde{X} \equiv c$. Then, for any random variable $\widetilde{Y}$, $\widetilde{Y} \leq c$, we can say that $\widetilde{Y}$ is first-order dominated by $\widetilde{X} \equiv c$, and $\widetilde{Y}$ is second-order dominated by $\widetilde{X} \equiv \mathbb{E}(\widetilde{Y})$. However, higher orders of stochastic dominance (or ordering by Nth degree risk with $N > 2$) cannot be constructed with one random variable being degenerate. Indeed, suppose that $\widetilde{X}$ and $\widetilde{Y}$ can be ordered by Nth degree risk with $N > 2$, while $\widetilde{X} \equiv c$ is degenerate. By Definition 2, the first $N-1$ moments of $\widetilde{X}$ and $\widetilde{Y}$ are identical. Thus, $Var(\widetilde{Y}) = Var(\widetilde{X}) = 0$, and $\widetilde{Y} = c$ almost surely. As an extension, if $\widetilde{X}$ and $\widetilde{Z}$ are two independent random variables, then a pair $(\widetilde{X} + \widetilde{Z}, \widetilde{X})$ is either ranked by the first-order or second-order stochastic dominance, or cannot be ranked by stochastic dominance of any order.

The above discussion suggests that for stochastic dominance of order three and higher, we cannot have a degenerate probability distribution; both lotteries must be risky. A good illustration is the result of an experiment by Mao [16], which was used by Menezes *et al.* [17] to motivate the concept of aversion to downside risk (i.e., prudence that requires positive third-order derivative of the utility function). Consider the following two lotteries. Lottery A pays 1000 with a probability of $\frac{3}{4}$ and pays 3000 with a probability of $\frac{1}{4}$. Lottery B pays zero with a probability of $\frac{1}{4}$ and pays 2000 with a probability of $\frac{3}{4}$. Note that both lotteries exhibit the same first two moments in their distributions, and lottery A third-order stochastically dominates B. Individuals who prefer lottery A to lottery B exhibit "downside risk aversion."

The concept of downside risk is related to the following decomposition: Define $\widetilde{X}_1 \equiv 2000$, $\widetilde{Y}_1 \equiv 1000$, $\widetilde{X}_2 \equiv 0$, and $\widetilde{Y}_2 \equiv [-1000, +1000]$, a 50–50 lottery. Note that $\widetilde{Y}_1$ is a first-order increase in risk over $\widetilde{X}_1$, $\widetilde{Y}_2$ is a second-order increase in risk over $\widetilde{X}_2$, A is the 50–50 lottery $[\widetilde{X}_1 + \widetilde{Y}_2, \widetilde{Y}_1 + \widetilde{X}_2]$, and B is the 50–50 lottery $[\widetilde{X}_1 + \widetilde{X}_2, \widetilde{Y}_1 + \widetilde{Y}_2]$. Thus, we can think of the following story for lotteries A and B: there are two equally likely states, and we receive 2000 in one state and 1000 in the other. Then, we are required to add a zero-mean risk $[-1000, 1000]$ to one of

the states. Lottery A (B) adds this risk to the state where we receive 2000 (1000). Third-order stochastic dominance (or downside risk aversion) is consistent with the preference to add risk to the higher-wealth state. An alternative view is related to the concept of "preceding changes in risk" [18]. We can start from lottery B, $[\widetilde{X}_1 + \widetilde{X}_2, \widetilde{Y}_1 + \widetilde{Y}_2]$, and make second-order improvement (replace $\widetilde{Y}_2$ with $\widetilde{X}_2$) at lower wealth level ($\widetilde{Y}_1 \equiv 1000$), followed by second-order deterioration of the same size (replace $\widetilde{X}_2$ with $\widetilde{Y}_2$) at a higher-wealth level ($\widetilde{X}_1 \equiv 2000$). That would yield lottery A. Here the second-order improvement precedes (in terms of wealth) second-order deterioration, and thus leads to the third-order improvement.

To conclude, the example above is consistent with the preference for combining good with bad, resulting in the lottery $[\widetilde{X}_1 + \widetilde{Y}_2, \widetilde{Y}_1 + \widetilde{X}_2]$, as opposed to combining good with good and bad with bad, leading to the lottery $[\widetilde{X}_1 + \widetilde{X}_2, \widetilde{Y}_1 + \widetilde{Y}_2]$, where $\widetilde{X}_i$ dominates $\widetilde{Y}_i$ in the sense of ith-order stochastic dominance, $i = 1, 2$. The next section states and proves how stochastic dominance of different orders (and different orders of risk) can be characterized by a particular preference for combining good with bad.

STOCHASTIC DOMINANCE AS THE PREFERENCE FOR COMBINING GOOD WITH BAD

Let $[A, B]$ denote a lottery that pays either A or B, each with probability one-half. Consider the mutually independent random variables $\widetilde{X}_N, \widetilde{Y}_N, \widetilde{X}_M,$ and $\widetilde{Y}_M$, and assume that $\widetilde{X}_i$ dominates $\widetilde{Y}_i$ via ith-order stochastic dominance for $i = M, N$. We wish to compare the 50–50 lotteries $[\widetilde{X}_N + \widetilde{Y}_M, \widetilde{Y}_N + \widetilde{X}_M]$ and $[\widetilde{X}_N + \widetilde{X}_M, \widetilde{Y}_N + \widetilde{Y}_M]$.

Theorem 2. *[19]. Suppose that $\widetilde{X}_i$ dominates $\widetilde{Y}_i$ via ith-order stochastic dominance for $i = M, N$. The lottery $[\widetilde{X}_N + \widetilde{Y}_M, \widetilde{Y}_N + \widetilde{X}_M]$ dominates the lottery $[\widetilde{X}_N + \widetilde{X}_M, \widetilde{Y}_N + \widetilde{Y}_M]$ via $(N + M)$th-order stochastic dominance.*

Proof. Let T be a positive integer and define $\mathbb{U}_T \equiv \{u \mid sgn \ u^{(n)}(w) = (-1)^{n+1}$ for $n = 1, \ldots, T\}$. For an arbitrary function $u \in \mathbb{U}_{N+M}$, define $v(w) \equiv \mathbb{E}u(\widetilde{Y}_M + w) - \mathbb{E}u(\widetilde{X}_M + w)$, where $\mathbb{E}$ denotes the expectation operator. We first show that $v \in \mathbb{U}_N$. To see this, consider any integer k, $1 \leq k \leq N$. Observe that $(-1)^k u^{(k)} \in \mathbb{U}_{N+M-k} \subset \mathbb{U}_M$, and therefore $sgn \ v^{(k)}(w) = sgn \ [\mathbb{E}u^{(k)}(\widetilde{Y}_M + w) - \mathbb{E}u^{(k)}(\widetilde{X}_M + w)] = (-1)^{k+1}$. The second equality above follows from $(-1)^k u^{(k)} \in \mathbb{U}_M$ and $\widetilde{X}_M$ MSD $\widetilde{Y}_M$. Thus, $v \in \mathbb{U}_N$.

The condition that $\widetilde{X}_N$ dominates $\widetilde{Y}_N$ via Nth-order stochastic dominance, together with $v \in \mathbb{U}_N$, implies that $\mathbb{E}v(\widetilde{X}_N) \geq \mathbb{E}v(\widetilde{Y}_N)$, which by the definition of v is equivalent to

$$\mathbb{E}u(\widetilde{X}_N + \widetilde{Y}_M) - \mathbb{E}u(\widetilde{X}_N + \widetilde{X}_M)$$
$$\geq \mathbb{E}u(\widetilde{Y}_N + \widetilde{Y}_M) - \mathbb{E}u(\widetilde{Y}_N + \widetilde{X}_M).$$

Rearranging terms above, this inequality is equivalent to

$$\frac{1}{2}\{\mathbb{E}u(\widetilde{X}_N + \widetilde{Y}_M) + \mathbb{E}u(\widetilde{Y}_N + \widetilde{X}_M)\}$$
$$\geq \frac{1}{2}\{\mathbb{E}u(\widetilde{X}_N + \widetilde{X}_M) + \mathbb{E}u(\widetilde{Y}_N + \widetilde{Y}_M)\},$$

which is precisely the lottery preference claimed in the theorem.

Theorem 2 expresses a preference for lotteries that combine "relatively good" assets with "relatively bad" ones, where relatively good and bad are defined via stochastic dominance. It is analogous to the notion of "disaggregating the harms" as discussed by Eeckhoudt and Schlesinger [20], if we interpret the "harms" as sequentially replacing each of the $\widetilde{X}$ random variables with a $\widetilde{Y}$ random variable in the sum $\widetilde{X}_N + \widetilde{X}_M$.

These preferences lead to a partial ordering of the four alternative sums of random variables, based upon stochastic dominance criteria:

$$\widetilde{X}_N + \widetilde{X}_M \succ \widetilde{X}_i + \widetilde{Y}_j \succ \widetilde{Y}_N + \widetilde{Y}_M$$
$$\text{for } (i, j) \in \{(M, N), (N, M)\}. \quad (2)$$

Note that the sums $\widetilde{X}_M + \widetilde{Y}_N$ and $\widetilde{X}_N + \widetilde{Y}_M$ cannot be ordered via stochastic dominance.

In the spirit of Menezes and Wang [21], we can refer to these two sums as the "inner risks" and the sums $\widetilde{X}_N + \widetilde{X}_M$ and $\widetilde{Y}_N + \widetilde{Y}_M$ as the "outer risks." Theorem 2 thus expresses a preference for a 50–50 lottery over the two inner risks as opposed to a 50–50 lottery over the two outer risks. If all lotteries in Equation (2) are degenerate (in which case $M = N = 1$), preference for a lottery over the two inner risks as opposed to a lottery over the two outer risks reduces to the standard notion of risk aversion.

The following Corollary extends Theorem 2 to Ref. 14's ordering by Nth-degree risk.

Corollary 2. *Suppose that $\widetilde{Y}_i$ has more ith-degree risk than $\widetilde{X}_i$ for $i = M, N$. Then the lottery $[\widetilde{X}_N + \widetilde{X}_M, \widetilde{Y}_N + \widetilde{Y}_M]$ has more $(N + M)$th-degree risk than the lottery $[\widetilde{X}_N + \widetilde{Y}_M, \widetilde{Y}_N + \widetilde{X}_M]$.*

Stochastic Dominance and Correlation Aversion

Theorem 2 is also consistent with the notion of correlation aversion [22], as discussed in Ref. 23. Denote by I and I' two binary random variables, with marginal distributions taking either 0 or 1 with probability 1/2. Lottery $[\widetilde{X}_N, \widetilde{Y}_N]$ can be thought of as the random variable $I\widetilde{X}_N + (1 - I)\widetilde{Y}_N$, and lottery $[\widetilde{X}_M, \widetilde{Y}_M]$ can be thought of as the random variable $I'\widetilde{X}_M + (1 - I')\widetilde{Y}_M$. Summing these two random variables gives

$$I\widetilde{X}_N + (1 - I)\widetilde{Y}_N + I'\widetilde{X}_M + (1 - I')\widetilde{Y}_M. \quad (3)$$

Theorem 2 compares the 50–50 lotteries $A = [\widetilde{X}_N + \widetilde{Y}_M, \widetilde{Y}_N + \widetilde{X}_M]$ and $B = [\widetilde{X}_N + \widetilde{X}_M, \widetilde{Y}_N + \widetilde{Y}_M]$, where A combines good with bad, while B combines good with good and bad with bad. Note that Lottery A corresponds to Equation (3) with $I' = 1 - I$ and Lottery B corresponds to Equation (3) with $I' = I$. In that sense, A (B) corresponds to the perfect negative (positive) correlation between events I and I', that specify lotteries $[\widetilde{X}_N, \widetilde{Y}_N]$ and $[\widetilde{X}_M, \widetilde{Y}_M]$. More generally, denote correlation between I and I' by ρ. Then Equation (3) becomes

$$I\widetilde{X}_N + (1 - I)\widetilde{Y}_N + I'\widetilde{X}_M + (1 - I')\widetilde{Y}_M$$

$$= \begin{cases} \widetilde{X}_N + \widetilde{X}_M & \text{with probability } \frac{1}{4}(1 + \rho), \\ \widetilde{X}_N + \widetilde{Y}_M & \text{with probability } \frac{1}{4}(1 - \rho), \\ \widetilde{Y}_N + \widetilde{X}_M & \text{with probability } \frac{1}{4}(1 - \rho), \\ \widetilde{Y}_N + \widetilde{Y}_M & \text{with probability } \frac{1}{4}(1 + \rho); \end{cases}$$

$$= \begin{cases} [\widetilde{X}_N + \widetilde{X}_M, \widetilde{Y}_N + \widetilde{Y}_M] \\ \quad \text{with probability } \frac{1}{2}(1 + \rho), \\ [\widetilde{X}_N + \widetilde{Y}_M, \widetilde{Y}_N + \widetilde{X}_M] \\ \quad \text{with probability } \frac{1}{2}(1 - \rho). \end{cases}$$

As this expression shows, increasing ρ increases the probability of the lottery that combines good with good and bad with bad (i.e., Lottery B), and decreases the probability of the lottery that combines good with bad (Lottery A). Therefore, by Theorem 2, increasing the correlation ρ is a deterioration in the sense of $(N + M)$th-order stochastic dominance, consistent with the notion of correlation aversion.

The next section discusses how the concept of stochastic dominance, and Theorem 2 in particular, can be applied to decision making under uncertainty.

APPLICATIONS TO DECISION MAKING

As shown in the previous section, preferring more of an attribute to less and preferring to combine good lotteries with bad ones implies preferences consistent with stochastic dominance. Therefore, if the decision is about the alternatives that can be ordered by stochastic dominance, the choice can be made without computing the expected utility, and thus without knowing the particulars of the utility function. First, we discuss several contexts where the decision is about allocating good or bad random variables to a particular state. In such situations, we show how Theorem 2 can be used to determine the optimal allocation. Secondly, we discuss the connection between stochastic dominance (of arbitrary high order) and exponential utility.

Allocating Risks

The main building block in Theorem 2 is a lottery $[\widetilde{X}, \widetilde{Y}]$, resulting in either random

variable $\widetilde{X}$ or random variable $\widetilde{Y}$, with equal chances. There are several contexts where such a lottery might occur. One is a two-period consumption/saving model, where labor income in the first period is represented by the random variable $\widetilde{X}$, and in the second period is represented by the random variable $\widetilde{Y}$, as illustrated in the section titled "Precautionary Effects." A second is a corporation with two headquarters and taxable profits, as illustrated in the section titled "Corporation with Two Headquarters." A third is more "direct," where the decision maker receives either $\widetilde{X}$ or $\widetilde{Y}$ with equal chances. Such a context is very useful to construct examples of stochastic dominance of higher orders, and to consider the effect of an independent additive background risk, as illustrated in the section titled "Tempering Effects of Risk."

Precautionary Effects. Consider a simple two-period model of consumption and saving. An individual with time-separable preferences has a random labor income of $\widetilde{X}$ at date $t = 0$ and income $\widetilde{Y}$ at date $t = 1$. The individual decides to save some of her income at date $t = 0$ and to consume the rest. She must decide on how much to save before learning the realized value of $\widetilde{X}$. Thus, her consumption at date $t = 0$ is $\widetilde{X} - s$, where s is the amount saved. If $s < 0$, the consumer is borrowing money (i.e., negative savings) and consuming more than the realized value of $\widetilde{X}$ at date $t = 0$. We assume that the interest rate for borrowing or lending is zero and that there is no time-discounting for valuing consumption at date $t = 1$. At this date, the individual consumes her income plus any savings, $\widetilde{Y} + s$. Let s^* denote the individual's optimal choice for savings.

Suppose that $\widetilde{X}$ dominates $\widetilde{Y}$ via Nth-order stochastic dominance. For any nonnegative scalar $\varphi \geq 0$, since φ first-order dominates $-\varphi$, it follows from Theorem 2 that the 50–50 lottery $[(\widetilde{X} - \varphi), (\widetilde{Y} + \varphi)]$ dominates $[(\widetilde{X} + \varphi), (\widetilde{Y} - \varphi)]$ by $(N + 1)$th-order stochastic dominance. Reinterpreting the "50–50 lottery" $[A, B]$ as sequential consumption of A at $t = 0$ and B at $t = 1$, Theorem 2 implies that saving an arbitrary amount $\varphi \geq 0$ always dominates saving the amount

$-\varphi$, whenever preferences satisfy $(N + 1)$th-order stochastic dominance preference. It follows that we must have $s^* \geq 0$ for this individual.

For example, suppose that $\widetilde{X} \equiv \mathbb{E}(\widetilde{Y})$ and that preferences are given by expected utility. Then this result coincides with the classical case examined by Leland [24] and Sandmo [25], for the case where preferences display prudence, $u''' \geq 0$. This basic result extends in several directions:

(i) Suppose that one's labor income is risky in both periods, but it is riskier in the sense of a second-degree increase in risk at date $t = 1$. Anyone with preferences exhibiting an aversion to downside risk (i.e., prudence) would prefer to save money rather than to borrow money. If the second-period wealth is riskier in the sense of second-order stochastic dominance, a precautionary saving demand then requires both aversion to downside risk and aversion to mean-preserving spreads, as defined in Ref. 15.

(ii) As opposed to (i), suppose that income in the second period is stochastically lower in the sense of first-order stochastic dominance. In that case, aversion to downside risk is no longer required to generate a demand for precautionary saving. We only need someone to be risk averse, that is, averse to risk increases of degree two.

(iii) We obtain the general case stated above if first-period wealth dominates second-period wealth via Nth-order stochastic dominance: A precautionary demand is generated whenever preferences satisfy $(N + 1)$th-degree stochastic dominance preference.

Finally, suppose that there are two sources of future income risk (e.g., labor income and capital income) which might be correlated. How does an increase in correlation affect savings? In the setting described in section titled "Stochastic Dominance and Correlation Aversion," increasing correlation leads to $(N + M)$th-order deterioration. In turn, that will increase savings if preferences are

consistent with $(N + M + 1)$th-order stochastic dominance.

Corporation with Two Headquarters. We consider here an example applying Theorem 2 with $N = 1$ and $M = 2$, but one that is not an example of a precautionary effect. Consider a risk-neutral corporation with taxable profit $\widetilde{X}$ in country A and taxable profit $\widetilde{Y}$ in country B. We assume that the tax schedule is identical in both countries and that $\widetilde{X}$ second-order dominates $\widetilde{Y}$. The tax owed on realized profit π is denoted by $t(\pi)$. The tax schedule is assumed to be increasing with a marginal tax rate that is also increasing, but at a decreasing rate. This assumption is realistic since the marginal rate is often bounded by some maximum, such as 50% of additional profit, and certainly is strictly bounded by 100%. After-tax profits can thus be written as $u(\pi) = \pi - t(\pi)$. If $t(\cdot)$ is differentiable, our assumptions about t imply that $u''(\pi) < 0$ and $u'''(\pi) > 0$. Moreover, since we should also have $t'(\pi) < 1$ for any profit level π, it follows that $u'(\pi) > 0$ as well.

Suppose now that the corporation has a new project with a pre-tax distribution of profit $\widetilde{Z}$, where $\widetilde{Z} > 0$ almost surely. The corporation must decide whether to locate the project in country A or in country B. The after-tax total profit of the corporation is given by $u(\pi_A) + u(\pi_B)$, where π_i denotes the realized pre-tax profit in country i, for $i = A, B$. Since $\widetilde{Z}$ first-order dominates zero, it follows from Theorem 2 that $E[u(\widetilde{X}) + u(\widetilde{Y} + \widetilde{Z})] > E[u(\widetilde{X} + \widetilde{Z}) + u(\widetilde{Y})]$ because the valuation function u (i.e., after-tax profits) satisfies third-order stochastic dominance preference. Thus, the firm should locate the new project in country B in order to maximize its global after-tax profit.

Tempering Effects of Risk. Kimball [26] defines a key behavior consequence of temperance; namely, that an unavoidable risk will "lead an agent to reduce exposure to another (independent) risk." From Theorem 2, we obtain a variant of Kimball's tempering effect: an unavoidably higher level of risk in one state will lead one to reduce exposure to a second risk in that state. Here,

higher or lower "risk" is characterized via second-order stochastic dominance.

The notion of tempering effects extends to higher orders of risk. For example, Lajeri-Chaherli [27] studies the effects of background risks on precautionary savings and in doing so, examines the condition of decreasing absolute temperance. A necessary condition for this property within her expected utility framework is $u(5) > 0$, which she labels as "edginess." By choosing $M = 2$ and $N = 3$, we can interpret edginess as implying that a decrease in one risk (via second-order stochastic dominance) helps to temper the effects of an increase in downside risk of another additive risk.

Interestingly, we can obtain a second equivalence for temperance that has nothing to do with tempering effects, by letting $N = 3$ and $M = 1$. Let $\widetilde{Y}_3$ be an increase in downside risk over $\widetilde{X}_3$, and let $\widetilde{X}_1$ first-order dominate $\widetilde{Y}_1$. The (stochastically) higher wealth in $\widetilde{X}_1$ helps to mitigate the effects of the increased downside risk in $\widetilde{Y}_3$. Thus, we prefer to pair $\widetilde{X}_1$ together with $\widetilde{Y}_3$ in our lottery preference. In the special case, where $\widetilde{X}_1$ and $\widetilde{Y}_1$ are both constants, we get a precautionary effect as in the section titled "Precautionary Effects." But note how this equivalence for "temperance" does not describe any type of tempering effect. Instead, it implies that the property of temperance yields a precautionary effect in protecting against increase in downside risk of future labor income. Similarly, by choosing $M = 1$ and $N = 4$, edginess can be interpreted as implying a precautionary effect against increase in fourth-degree risk for future labor income.

Stochastic Dominance and Exponential Utility

A useful implication is that preference for the dominant lottery in Theorem 2 restricts potential forms of the utility function. Tsetlin and Winkler [28] extend this approach to a multiattribute setting. Consider a decision maker with a nondecreasing utility for wealth, $u'(x) \geq 0$. That means, for example, that $200 is better than $100. Consider a lottery [$100, $200], and suppose that we combine two such lotteries by adding outcomes.

This can be done by combining good with good and bad with bad, yielding [\$200, \$400] or by combining good with bad, yielding [\$300, \$300] which is \$300 for sure. For the choice between \$300 for sure and an uncertain outcome with a mean of \$300, a preference for combining good with bad is consistent with risk aversion: $u''(x) \leq 0$.

Now, for someone who is risk averse, we can say that \$300 is good and [\$200, \$400] is bad. Combining these two lotteries with an independent [\$100, \$200] lottery gives [[\$300, \$500], \$500] if we combine good with good and bad with bad, and [[\$400, \$600], \$400] if we combine good with bad. Up to a linear transformation, these are the same lotteries as A and B at the end of the first section (there we had [[0, 2000], 2000] and [[1000, 3000], 1000]), and combining good with bad is preferred (i.e., consistent with prudence) if $u'''(x) \geq 0$. As discussed in the section titled "Tempering Effects of Risk," the next steps would yield temperance ($u(4)(x) \leq 0$), and then edginess ($u(5)(x) \geq 0$). Overall, preference for combining good with bad implies that $u \in \mathbb{U}_\infty$, where $\mathbb{U}_\infty = \{u | (-1)^{k-1} u^{(k)} \geq 0, k = 1, 2, \ldots\}$.

Utility functions from $\mathbb{U}_\infty$ have been studied quite extensively [13,29–32], as they are consistent with stochastic dominance of any order. Whitmore (13, p. 78) mentions that "No compelling economic rationale for this extension is apparent although it can be shown that all functions in the class exhibit nonincreasing absolute risk aversion. Moreover, $\mathbb{U}_\infty$ can be shown to contain many of the families of utility functions which are used routinely by investigators, such as the linear, exponential, logarithmic, and power families. Furthermore, one might argue that a perfectly rational economic decision maker will have preferences which undergo the smooth and systematic change with wealth level which is implied by the conditions defining $\mathbb{U}_\infty$." The preference for combining good with bad, as opposed to combining good with good and bad with bad (Theorem 2), offers such economic rationale for $\mathbb{U}_\infty$.

Utility functions from $\mathbb{U}_\infty$ are also very tractable, since they can be characterized via Laplace transforms. Consider $u(x)$ defined on $(0, \infty)$. By Bernstein's lemma (33, p. 439,

Theorem 1a), $u \in \mathbb{U}_\infty$ if and only if its first derivative (i.e., the marginal utility) is a Laplace transform of a (not necessarily finite) measure ψ on $[0, \infty)$: $u'(x) = \int_0^\infty e^{-rx} d\psi(r)$. Then (29, Theorem 1)

$u \in \mathbb{U}_\infty$ if and only if $u(x)$

$$= u(x_0) + \int_0^\infty [(1 - e^{-r(x-x_0)})/r] \, d\psi(r). \quad (4)$$

By Equation (4), any utility function from $\mathbb{U}_\infty$ is a mixture (weighted average) of exponential utilities. This provides a quick way to check whether one alternative dominates another with respect to $\mathbb{U}_\infty$. These results are summarized in the theorem below, which is based on Ref. 31 (Proposition 1 and Proposition 4).

Theorem 3. *The following statements are equivalent:*

1. $\int_a^b u(x) \, dF(x) \geq \int_a^b u(x) \, dG(x)$ *for all* $u \in \mathbb{U}_\infty$,
2. $\int_a^b e^{-rx} \, dF(x) \leq \int_a^b e^{-rx} \, dG(x)$ *for all* $r \in [0, \infty)$,
3. *There exists N such that F weakly dominates G in the sense of Nth-order stochastic dominance.*

By Theorem 3, exponential utility [constant absolute risk aversion (CARA)] is a main building block for ranking risky alternatives. (Note that its second condition is equivalent to ranking moment-generating functions for distributions F and G.) Even if the second condition does not hold for all r, but for a wide enough range of risk aversion levels considered to be "reasonable," choosing F over G would be justified. This is consistent both with Kirkwood's observation [34] that most decision analysis applications use exponential utility function and with his suggested framework of using exponential utility for sensitivity analysis with respect to risk tolerance levels. This is good news for decision analysts, since exponential utility is, arguably, the easiest to use, in particular because the certainty equivalent of a project does not depend on the presence of independent additive background risk.

SUMMARY

Probability distributions can be partially ranked via stochastic dominance (Definition 1). Within an expected utility framework, preferences are consistent with stochastic dominance of order N if the first N derivatives of the utility function alternate in sign (Theorem 1). If preferences are consistent with stochastic dominance of arbitrary high order, the utility function is a weighted average (mixture) of exponential utilities (Eq. 4).

If preferences are consistent with stochastic dominance, the choice between alternatives might be easier to determine and be somewhat independent of the particulars of the decision maker's utility function, as discussed in the section titled "Stochastic Dominance and Exponential Utility." But why should anybody's preferences satisfy stochastic dominance? Alternatively, how could we check for that?

Theorem 2 shows that the decision maker's preferences are consistent with stochastic dominance of any order if the decision maker prefers more of the attribute to less, and prefers to combine good random variables with bad ones, as opposed to combining good with good and bad with bad. This condition is similar in spirit to risk aversion and correlation aversion in the sense that it reflects an aversion to the combination of bad with bad. Given the partial ordering in Equation (2), a 50–50 gamble between two intermediate outcomes is preferred to a gamble between two extreme outcomes. Preference for combining good with bad seems intuitive in many instances, especially in the wealth or consumption context. In the assessment process, the decision maker should be presented with choices that highlight the issue of preferences for combining good lotteries with bad versus combining good with good and bad with bad.

Theorem 2 also can be used to add intuition to many other extant concepts in the literature, such as skewness preference [18] and transfer principles in income redistribution [35,36]. Furthermore, in some instances, decisions are indeed about how to allocate risks across two states (or two time periods, or two firms, or two groups of people). The section titled "Allocating Risks" shows how such decisions can often be made independent of a particular utility function or even a particular preference function, as long as we believe that valuations are consistent with stochastic dominance criteria.

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