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When to Abandon a Research Project and Search for a New One

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We investigate the cost of the opportunity delayed by working on one project with uncertain success rather than searching for a new project. We answer the question: how long should a firm work on a research project with uncertain success before abandoning it if the only alternative is to search for a new project to work on? Rather than treating the opportunity as an exogenous alternative, this approach endogenizes the opportunity value and the attendant cost of its delay. We consider cases with both a finite and an infinite number of potential projects. We derive the optimal stopping time and show that it increases with the discount rate and with the prior probability that the project can be successfully completed, and it decreases with the rates of new project arrival and project completion.

Keywords: Project management; Dynamic programming; Uncertainty; Bayesian updating; Optimal stopping.

1. Introduction

Item: After a three year effort, Merck announced in March 2012 that based on the "projected development timeline" it was abandoning an oral formulation of Vernakalant, a drug for the prevention of atrial-fibrillation (Medscape, March 19 2012).

Item: In late July of 2015, the European Medicines Agency approved the sale of Mosquirio, an anti-malarial vaccine developed by GlaxoSmithKline (Fortune, July 24 2015). The vaccine had been in development at Glaxo for over 30 years: Moncef Sloui, head of Glaxo's vaccine division, had worked on it for 27 years.

A researcher can spend her time in one of two activities: working on an existing research project or generating an idea for a new research project. There is some prior probability that the existing research project can be successfully completed if it is being worked on; the updated probability of successful completion declines over time as the project is worked on and success does not arrive. If the researcher is not working on a project, she can search for new ideas, i.e., potential new projects. Questions addressed in this paper include: what is the opportunity cost to not generating new ideas while the researcher is working on an existing project?; and how long should a researcher work on a project before giving it up and returning to generating new ideas: three years as in the case of Vernakalant, or more than thirty years as in the case of Mosquirio?

The decision to abandon a project is very important and difficult. Some attention has been given to exit decisions for ailing business ventures (e.g., Staw and Ross 1987, Horn, Lovallo, and Viguerie 2006, Elfenbein and Knott 2015), and the key finding is that companies tend to stick with a failing venture too long. Several explanations are posited for this inertia, including status quo bias, the sunk-cost fallacy, escalation of commitment, confirmation bias, and unwillingness to admit failure. Elfenbein and Knott (2015, p. 974) conclude that since their evidence "suggests there is considerable exit beyond the optimum, firms could likely benefit from automated decision rules ... in making exit decisions. ... While firms could choose to override the exit rule, having the rule would implicitly make exit the default, rather than the current practice of continuation as the default." Ryan and Lippman (2005) develop a model to determine the optimal exit time from a deteriorating project.

The decision to abandon a research project shares the conditions making it hard to exit an ailing business, in addition to other considerations that can make the abandonment decision even more difficult. With ailing business ventures, very concrete information that is easily understandable to decision makers is regularly available through statistics signaling, e.g., declining market share and decreasing profits or increasing losses. Information about progress on a research project, to the extent it exists, is often much more vague and opaque to the decision maker, often consisting primarily of progress reports. Those reports come from the researchers themselves, who have a vested interest in having the project continue and may tend to paint an overly rosy picture. Typically "the project manager knows more about the quality of the idea than the senior executives that do the screening" and "the project manager's development effort is not entirely observable by senior executives." (Chao, Lichtendahl, and Grushka-Cockayne 2014, p. 1286.) The decison maker often does not have sufficient expertise to be able to verify the information in the progress reports.

There is a large literature on whether to start and when to stop investing time and effort on a research project, and what is the optimal path of implementing/investing in it (see, for example, Childs and Triantis (1999) which takes a contingent claims approach, and the references therein). And there are consulting firms and managerial processes such as Stage-gate (Cooper and Edgett 2015) that assist firms in the task of dropping unsuccessful projects. These decisions depend on the project characteristics (such as the probability that the project can be successfully completed, the rate at which success arrives if it is possible, the fixed amount of time that must be spent before success is possible, and the payoff if the project is successfully completed) and on the opportunity cost (what the decision maker could earn instead of the project at hand). In that literature the opportunity cost is mostly exogenous (often 0), and it seems that in many instances determining this opportunity cost would be quite challenging. The idea of this paper is to make opportunity cost somewhat endogenous, in the sense that if the current project is abandoned, the only option is to search for similar projects. We study this problem through a variety of continuous-time models: single projects versus multiple projects versus an infinity of projects, single rewards versus multiple rewards, and risk neutral versus risk averse. One purpose of studying the various approaches is to explore the robustness of the results.

Our focus is on processes where time and effort are the major inputs, or where the flow rate of effort and expenditures while carrying out the research is nearly constant; and the only signal the firm receives is that success has not arrived – until it does. Processes that fit this description include the early stage of pharmaceutical discovery or oil and gas exploration, academic research, and new product development. More broadly, our model applies to any search activity that involves identifying potential opportunities (e.g., prospective suppliers or business partners) and then closing the deal. We build a simple model of such processes and derive expressions for the optimal abandonment time that could be used as automated decision rules. We provide comparative statics on the abandonment time: for example, we find that the optimal stopping time is relatively insensitive to changes in the prior probability of success, except when that probability is quite large. On the other hand, the optimal stopping time is relatively sensitive to changes in the arrival rate of new projects and in the arrival rate of successful completion.

The paper is organized as follows. We begin in the next section with a brief literature review. In §3 we develop a parsimonious model with an exogenously specified opportunity value. We start by assuming there is only one available project and then extend to a finite number of projects, all of which could prove successful and provide benefit. As long as there are projects remaining, the choice in each instant is between working on the project in hand, or abandoning it and searching for a new project. If the latter action is chosen, we assume that new projects are generated following a Poisson process. We characterize the value function and the optimal policy. For instance, we show that the optimal abandonment time is decreasing in the number of projects remaining. In §4, we endogenize the opportunity value by considering the limiting case of an infinite number of available projects. We characterize the value function and determine the optimal stationary policy. The special case in which only one reward can be earned (though there are an infinite number of available projects) is considered in §4.1. We show that the optimal abandonment time is identical to the case in which an unlimited number of rewards can be earned. Other variations of the problem and comparative statics are provided in the remainder of §4. We conclude in §5.

2. Literature review

Our paper is related to the vast literature on research and development and technology adoption. Here we restrict attention to only a handful of the papers most closely related to the current paper.

McCardle (1985) develops a dynamic-programming model of technology adoption: in each period a firm can either experiment with (gather information about) a technology of uncertain value, adopt the technology, or reject the technology. As the firm gathers information about the possible success of the technology, it updates its prior beliefs using Bayes rule. Eventually the firm has enough information (its posterior distribution has low enough variance) that it can make an adopt or reject decision. If the firm rejects the technology, it earns 0 and the problem comes to an end. This work was extended to general distributions by Ulu and Smith (2009) and Smith and Ulu (2012), and to allow risk aversion by Smith and Ulu (2015). Rather than having a two-dimensional state space (mean and precision) as in McCardle, more general probability distributions are allowed.

In a set of clever papers, Alizamir, de Vericourt, and Sun (2013, 2014) model a related problem (diagnostic testing of patients in a queue) in which additional information is gathered, i.e., another test performed, or the patient is sent home or remanded for treatment. Their model has the added complication of congestion. Rather than the cone-shaped optimal policy developed in McCardle (1985), they establish the optimality of an M-shaped policy. The inverted cone of the "M" is a result of the congestion.

Kornish and Keeney (2008) consider information acquisition in the context of designing a flu vaccine – a decision making situation with a deadline and with a finite number of alternatives to choose among.

The closest antecedent to the current paper is Lippman and McCardle (1991), which extends the model in McCardle (1985) to allow for repeated search. In Lippman and McCardle (1991), if the firm rejects one technology, it returns to the pool of available technologies and draws again. They establish results regarding the monotonicity and convexity of the value function and characterize the optimal policy for general probability distributions satisfying certain Bayesian updating conditions. In the current paper, we are able to establish tighter results by imposing slightly stronger conditions in a continuous-time model (the model in Lippman and McCardle (1991) is discrete time). For example, we are able to solve for and characterize the optimal abandonment time (see Figure 3) knowing only the prior probability of successful project completion, the rate at which success conditionally arrives, and the rate at which a new project can be found.

3. Setting the stage: exogenous opportunity cost

We begin with a simple, infinite-horizon, continuous-time model with only one available project and an assumption that the opportunity cost is exogenously specified. The current time period corresponds to $t = 0$. The instantaneous discount rate is $e^{-\delta t}$ for some fixed $\delta > 0$.

A risk-neutral decision maker has before her a risky research project that she could devote her time and effort to. There is a chance that any effort devoted to the project will be fruitless – the project could be impossible to complete or the researcher might not have the necessary expertise. Denote by p the researcher's prior probability that she can successfully complete the project. The researcher will Bayesianly update the prior p as she works on the project and success is not achieved. Successful completion, if possible, arrives exponentially at rate λ. The probability that the project is completed by time t, given that it can be successfully completed, is $1 - e^{-\lambda t}$. There is a minimum time Δ that must be spent working on the project before completion is even possible. This is meant to capture the fixed costs of the project, such as time spent out-fitting a laboratory or recruiting subjects for a study, or in the case of a PhD student, the time spent searching the literature. That is, if the project is undertaken, first one has to work on it for an amount of time Δ; thereafter if the project can be successfully completed, completion arrives at rate λ. The project pays reward $R > 0$ if and when it is successfully completed. (Our results do not change if the rewards have a distribution and R is interpreted as an expected value.)

There is an available non-project activity yielding a deterministic cash flow that the decision maker could earn if she works on that in lieu of working on the project. Denote by π the net present value at time 0 of the future cash flows from the non-project activity; that is, if she chooses not to begin work on the project, π is earned from the non-project activity. By stationarity, the net present value of future cash flows from the non-project activity if it is begun at some later time t is $\pi e^{-\delta t}$. Given the discount rate $e^{-\delta}$ and the non-project activity value π, the opportunity cost of a one-period delay of the non-project activity is $\pi(1 - e^{-\delta})$. For example, if time is measured discretely and the non-project activity yields an annuity that pays α per period, then π is given by $\alpha/(1 - e^{-\delta})$, and the opportunity cost of a one-period delay is $\pi(1 - e^{-\delta}) = \alpha$.

3.1. Abandoning the project once started

Suppose the decision maker has started work on the risky project. Because we assume that a project once abandoned cannot be restarted, the question before the decision maker is when, if ever, to abandon work on the project and divert all her effort to the outside activity. Let $\Delta + t$ be the abandonment time. Note that because there is no chance that the project can be completed prior to Δ, it cannot be optimal to abandon the project prior to Δ if it was beneficial to start. The expected value at time 0 of the current project to the researcher if the plan is to abandon at $\Delta + t$ is given by

$$\begin{aligned} v(t) &= p\int_0^t (R+\pi)e^{-\delta(\Delta+z)}\lambda e^{-\lambda z}dz + (1-p(1-e^{-\lambda t}))\pi e^{-\delta(\Delta+t)} \\ &= e^{-\delta\Delta}\left[p(R+\pi)\frac{\lambda}{\lambda+\delta}(1-e^{-(\lambda+\delta)t}) + (1-p+pe^{-\lambda t})\pi e^{-\delta t}\right]. \end{aligned} \tag{1}$$

The first term on the right-hand side of the first line of (1) represents the expected value if the project is successfully completed before time $\Delta + t$; the prior probability of successful completion is p; and the return on completion is the reward to the project R plus the value of the outside activity π, appropriately discounted for the arrival time before $\Delta + t$. Note that once the researcher successfully completes the project and earns R, she immediately turns her attention and efforts to the outside activity and earns the present value π.

The second term on the right-hand side of the first line of (1) represents the expected value if the project is not successfully completed by time t, and is then abandoned. This happens with prior probability $(1 - p(1 - e^{-\lambda t}))$. The researcher then turns to the non-project activity and earns π. For example, suppose a firm has a product it can produce and sell, earning π in net-present value; or the firm could attempt to develop an improved product that would earn $R + \pi$ in net-present value, if the improvement can be successfully completed. How long, if at all, should the firm pursue the product improvement? And what is the expected value of pursuing the improvement project?

Lemma 1 *The expected value of the project $v(t)$ is bounded above by $R + \pi$. It is increasing in the prior p, the reward R, the opportunity value π; and it is decreasing in the fixed cost Δ and the discount rate δ.*

Proof: The firm earns $R + \pi$ if there are no fixed costs, $\Delta = 0$, no time-discounting, $\delta = 1$, and if the research is successful. Differentiating $v(t)$ with respect to Δ and δ establishes that $v(t)$ is decreasing in both. Hence, $R + \pi$ is an upper bound. Differentiating $v(t)$ with respect to p and R completes the proof. ∎

Let $p(t)$ represent the posterior probability at time $\Delta + t$ that the project can be successfully completed given that it has been worked on for $\Delta + t$ without success. Then $p(t)$ is given by

$$p(t) = \frac{pe^{-\lambda t}}{1 - p(1 - e^{-\lambda t})}.$$

In order to find the optimal abandonment time t^*, differentiate $v(t)$ in (1) with respect to t yielding

$$\begin{aligned}
\frac{dv(t)}{dt} &\propto p(R + \pi)\lambda e^{-(\lambda+\delta)t} - \pi\left[(1 - p)\delta e^{-\delta t} + p(\lambda + \delta)e^{-(\lambda+\delta)t}\right] \\
&\propto p(R + \pi)\lambda e^{-\lambda t} - \pi\left[(1 - p)\delta + p(\lambda + \delta)e^{-\lambda t}\right] \\
&= pR\lambda e^{-\lambda t} - \pi\delta(1 - p + pe^{-\lambda t}) \\
&\propto p(t)R - \pi\frac{\delta}{\lambda}. \qquad (2)
\end{aligned}$$

Setting the derivative in (2) equal to zero and solving yields the optimal abandonment time when $pR\lambda > \pi\delta$ (which we assume henceforth):

$$t^* = -\frac{1}{\lambda}\ln\left[\frac{1 - p}{p}\frac{\pi\delta}{R\lambda - \pi\delta}\right]. \qquad (3)$$

Observe that this equates the posterior probability of successfully completing the project and receiving the associated reward R with the cost of the delay in receiving the sure π.

Lemma 2 *The optimal abandonment time, $\Delta + t^*$, decreases with the opportunity value π and the discount rate δ; and increases with the prior probability of successful completion p, and the reward from successful completion R. The optimal abandonment time is increasing (decreasing) for small (large) values of λ.*

Proof: The assertions regarding π, δ, p, and R follow on taking the appropriate derivatives of $t^* = -\frac{1}{\lambda} \ln \left[\frac{1-p}{p} \frac{\pi\delta}{R\lambda - \pi\delta} \right]$.

To establish the result for λ, denote $p(t, \lambda) = \frac{pe^{-\lambda t}}{1-p(1-e^{-\lambda t})}$. The optimal stopping time t^* is given by equation $\lambda p(t^*, \lambda) = \pi\delta/R$. Therefore, $\frac{dt^*}{d\lambda} = -\frac{\frac{\partial}{\partial \lambda}(\lambda p(t,\lambda))}{\frac{\partial}{\partial t}(\lambda p(t,\lambda))}$. As $\frac{\partial}{\partial t}(\lambda p(t, \lambda)) < 0$, $\frac{dt^*}{d\lambda}$ has the same sign as $\frac{\partial}{\partial \lambda}(\lambda p(t, \lambda))$. Observe that $\frac{\partial}{\partial \lambda}(\lambda p(t, \lambda)) = \frac{\partial}{\partial \lambda}\left(\lambda \frac{pe^{-\lambda t}}{1-p(1-e^{-\lambda t})}\right) = p\frac{e^{-t\lambda}}{\left(pe^{-t\lambda}-p+1\right)^2}\left(pe^{-t\lambda} - t\lambda - p + pt\lambda + 1\right)$.

Therefore, $\frac{\partial}{\partial \lambda}(\lambda p(t^*, \lambda))$ has the same sign as $pe^{-t^*\lambda} - t^*\lambda - p + pt^*\lambda + 1 = pe^{-t^*\lambda} + (1 - p)(1 - \lambda t^*)$, which is positive (negative) if $(\lambda t^* - 1)\, e^{\lambda t^*} < (>) \frac{p}{1-p}$. ∎

Alternatively, we can characterize the optimal decision by considering the posterior probability $p(t)$. Because $p(t)$ is monotonically decreasing, $p(t^*) = \pi\delta/R\lambda$ represents the threshold probability at which the firm should abandon the project. As the formula for $p(t)$ does not involve R, π, or δ, it is much easier to establish several of the assertions of Lemma 2: the firm should abandon the project when the posterior probability of success falls below $p(t^*)$, and $p(t^*)$ is increasing in π and δ and decreasing in R.

The only remaining question is whether or not the researcher should begin work on the project, and the answer is obvious: if $v(t^*) > \pi$, start work; otherwise, do not.

3.2. A finite number of potential projects

In the previous section, we set the stage by considering a model with only one project and an exogenous opportunity, characterized by the value π of the alternative. In this section we begin to make the opportunity endogenous by assuming that if the firm abandons the research project at hand, it will search for a new project to work on. We begin by assuming that there are a finite number n of projects available. (§3.1 corresponds to the case $n = 1$.) Imagine, for example, searching for a supplier among a finite number of potential suppliers; here successful completion would be closing a deal and having the supplier successfully deliver the needed part. While the firm engages in search, a new project arrives exponentially at rate λ_0. (If $\lambda_0 = \infty$, new projects arrive instantaneously; this is equivalent to the case in which the firm has the research projects already in hand.) As long as the exogenous opportunity value when there is only one project remaining is positive, a finite number of potential projects is tantamount to a finite horizon. (In the next section we let the number of projects, hence the horizon, recede to infinity.)

Each time the firm successfully completes a project, it earns R suitably discounted. Once a project is successfully completed or abandoned, the firm searches out a new project as long as there are projects remaining.

Let π_k for $k \leq n$ be the value of searching for a new project when there are k projects remaining that have not been investigated and either successfully completed or abandoned. Without loss of generality, fix π_0; as in the previous section, any small positive value will do. Let $v_k(t)$ be the expected value when the firm has begun investigation of the k^{th} remaining project, and it plans to abandon the project if success is not achieved by time $\Delta + t$. We now set up an interleaved set of recursions: one for $v_k(t)$, suitably optimized, and one for π_k.

Begin by letting $v_1(t) = v(t)$ as given in (1), substituting π_0 for π. Denote the optimal abandonment time by t_1^*. Let $v_1 = v_1(t_1^*)$ be the optimal value of working on the last remaining project. Now let π_1 be the value of searching for the last remaining project:

$$\pi_1 = \int_0^\infty \lambda_0 e^{-\lambda_0 x} e^{-\delta x} v_1 dx = \frac{\lambda_0}{\lambda_0 + \delta} v_1.$$

Iteratively define $v_2(t), v_2(t_2^*), \pi_2, v_3(t), v_3(t_3^*), \pi_3, \ldots$. Then $v_k(t)$ represents the value of starting work on the k^{th} project when the plan is to abandon it at time $\Delta + t$ if it has not been successfully completed prior to that time:

$$v_k(t) = e^{-\delta\Delta}\left[p(R + \pi_{k-1})\frac{\lambda}{\lambda+\delta}(1 - e^{-(\lambda+\delta)t}) + (1 - p + pe^{-\lambda t})\pi_{k-1}e^{-\delta t}\right]. \tag{4}$$

As in the previous case, optimize (4) by differentiating and solving for t_k^*:

$$t_k^* = -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda - \pi_{k-1}\delta}\right]. \tag{5}$$

The optimal payoff at the start of the k^{th} project is $v_k = v_k(t_k^*)$:

$$v_k = v_k(t_k^*) = e^{-\delta\Delta}\left[p(R + \pi_{k-1})\frac{\lambda}{\lambda+\delta}(1 - e^{-(\lambda+\delta)t_k^*}) + (1 - p + pe^{-\lambda t_k^*})\pi_{k-1}e^{-\delta t_k^*}\right].$$

As shown in the Appendix, substituting in the value of t_k^* from above and simplifying yields v_k as a function of π_{k-1}:

$$v_k = e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[R + \pi_{k-1} + \pi_{k-1}\left(\frac{1-p}{p}\right)\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda - \pi_{k-1}\delta}\right)^{\delta/\lambda}\right]. \tag{6}$$

Finally, let π_k be the value of searching for the k^{th}-to-last remaining project.

$$\begin{aligned}\pi_k &= \int_0^\infty \lambda_0 e^{-\lambda_0 x} e^{-\delta x} v_k dx = \frac{\lambda_0}{\lambda_0+\delta} v_k \qquad (7)\\ &= \frac{\lambda_0}{\lambda_0+\delta}e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[R + \pi_{k-1} + \pi_{k-1}\left(\frac{1-p}{p}\right)\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda - \pi_{k-1}\delta}\right)^{\delta/\lambda}\right],\end{aligned}$$

which gives π_k as a function of π_{k-1}. Figure 1 displays π_k as a function of k for various values of the prior probability of success p with $\lambda = 1, \lambda_0 = 2, \delta = 0.1, \Delta = 0, R = 1$ and $\pi_0 = 0$. From the figure we see that π_k appears to be bounded, concave, and increasing in k; and it appears to be increasing in p. The next Proposition establishes these observations and several others.

*************************** INSERT FIGURE 1 HERE ************************

Proposition 1 *At each project stage the optimization is well-defined. The value function π_k is bounded uniformly in k. The value function π_k is monotonically increasing in k; it is increasing in the prior p and the reward R, and it is decreasing in the discount rate δ and the fixed cost Δ.*

Proof: A bound on the value function is given by $pR(\lambda/\delta) + \pi_0$. To establish that π_k is increasing in k, note that the firm could immediately abandon the k^{th} project and earn π_{k-1}; behaving optimally must earn more than that. The comparative statics with regard to p, R, δ and Δ are established through a straightforward recursion following Lemma 1. ∎

Corollary 1 *The optimal abandonment time t_k^* is decreasing in k.*

Proof: The claim follows immediately from Proposition 1 and Lemma 2. ∎

Read another way, Corollary 1 says that the fewer the number of projects remaining to be investigated, the more time the firm should spend on each one prior to abandoning it. The fewer the number of projects remaining, the lower the opportunity value foregone by working on the project at hand; so work longer. Figure 2 displays t_k^* as a function of k for different values of p with other parameters as in Figure 1.

*****************************INSERT FIGURE 2 HERE ***********************

4. Endogenizing the opportunity value: An infinite number of potential projects

For the case with an infinite number of potential projects, each of which if successful earns R, formally let $k \to \infty$ in equations (4) through (7). We need to confirm the legitimacy of that formality. In this case $v_\infty(t)$ is the value to starting work on a project when there are an infinity of projects, and the plan is to abandon the current project at time $\Delta + t$ and then to behave optimally thereafter:

$$v_\infty(t) = e^{-\delta\Delta}\left[p(R+\pi_\infty)\frac{\lambda}{\lambda+\delta}(1-e^{-(\lambda+\delta)t}) + (1-p+pe^{-\lambda t})\pi_\infty e^{-\delta t}\right]. \quad (8)$$

Differentiating (8) with respect to the abandonment time t and setting it equal to zero yields the optimal abandonment time t^*_∞:

$$t^*_\infty = -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi_\infty\delta}{R\lambda - \pi_\infty\delta}\right]. \tag{9}$$

Finally, π_∞ is the value of searching for the next project:

$$\pi_\infty = \int_0^\infty \lambda_0 e^{-\lambda_0 x} e^{-\delta x} v_\infty(t^*_\infty)dx = \frac{\lambda_0}{\lambda_0+\delta} v_\infty(t^*_\infty). \tag{10}$$

If we let $v(t) = v_\infty(t)$, $t^* = t^*_\infty$, and $\pi = \pi_\infty$, then (8) and (9) with an infinite number of projects reduce to (1) and (3) with an exogenous opportunity value π, with the added condition that π satisfy a fixed point equation derived by substituting in (10) and simplifying:

$$\pi = \frac{\lambda_0}{\lambda_0+\delta} e^{-\delta\Delta} p \frac{\lambda}{\lambda+\delta}\left[(R+\pi) + \pi\frac{1-p}{p}\left(\frac{1-p}{p}\frac{\pi\delta}{R\lambda - \pi\delta}\right)^{\delta/\lambda}\right]. \tag{11}$$

Alternatively, rather than solving the fixed point equation for π (that is, π not as a function of t^*), one could solve a fixed point for t^*. A fair bit of algebra (see the Appendix) leads to the following:

$$\frac{1}{\lambda}e^{\lambda t^*} + \frac{1}{\delta}e^{-\delta t^*} = \frac{\lambda+\delta}{\lambda\delta(1-p)}\left(\frac{\lambda_0+\delta}{\lambda_0}e^{\delta\Delta} - p\right). \tag{12}$$

Proposition 2 *The value function π_∞ is well-defined; it is the unique fixed point of (11), it is increasing in the prior p and the reward R, and it is decreasing in the discount rate δ and the fixed cost Δ.*

Proof: Proposition 1 establishes that π_k is uniformly bounded in k and is monotonically increasing in k. Hence, by the monotone convergence theorem, $\pi_\infty = \lim_k \pi_k$ is also bounded. The comparative statics are proved as before. ▮

Corollary 2 *The optimal abandonment time t^* is increasing in p, δ and Δ; it is decreasing in λ_0 and λ; and it is unaffected by changes in R.*

Proof: The left-hand side of (12), which defines the optimal abandonment time, is increasing in t^*. Straightforward differentiation shows that the right-hand side of (12) is increasing in p and Δ, decreasing in λ_0, and unaffected by changes in R, establishing the corollary for those parameters.

To establish the claim regarding λ, define

$$F(\lambda,t) = \frac{\lambda e^{-t\delta} + \delta e^{t\lambda}}{\lambda+\delta} - \frac{1}{1-p}\left(\left(1+\frac{\delta}{\lambda_0}\right)e^{\delta\Delta} - p\right).$$

Then equation (12) is equivalent to $F(\lambda, t^*) = 0$, and

$$\frac{dt^*}{d\lambda} = -\frac{\partial F/\partial \lambda}{\partial F/\partial t^*}.$$

The denominator is positive: $\partial F/\partial t^* = \frac{\lambda\delta}{\lambda+\delta}\left(e^{t\lambda} - e^{-t\delta}\right) > 0$. It remains to show that the numerator is also positive:

$$\begin{aligned}\partial F/\partial \lambda &= \frac{e^{-t^*\delta} + \delta t^* e^{t^*\lambda}}{\lambda+\delta} - \frac{\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}}{(\lambda+\delta)^2} = \frac{\delta e^{-t^*\delta} + \delta t^*(\lambda+\delta)e^{t^*\lambda} - \delta e^{t^*\lambda}}{(\lambda+\delta)^2} \\ &= \frac{\delta}{(\lambda+\delta)^2} e^{t^*\lambda}\left(e^{-t^*(\lambda+\delta)} + t^*(\lambda+\delta) - 1\right) > 0.\end{aligned}$$

To establish the claim with respect to δ, as above, define

$$F(\delta, t) = \frac{\lambda e^{-t\delta} + \delta e^{t\lambda}}{\lambda+\delta} - \frac{1}{1-p}\left(\left(1 + \frac{\delta}{\lambda_0}\right)e^{\delta\Delta} - p\right).$$

Then equation (12) is equivalent to $F(\delta, t^*) = 0$, and

$$\frac{dt^*}{d\delta} = -\frac{\partial F/\partial \delta}{\partial F/\partial t^*}.$$

Again, the denominator is positive: $\partial F/\partial t^* = \frac{\lambda\delta}{\lambda+\delta}\left(e^{t^*\lambda} - e^{-t^*\delta}\right) > 0$. It remains to show that $\partial F/\partial \delta < 0$. Details are in the Appendix. ∎

4.1. Only a single reward can be earned

The model developed above and in §2 allows the firm to earn the reward R for as many projects as are successfully completed. Alternatively, one can consider the case wherein only one reward can be earned. For example, PhD students submit only one dissertation to earn their degree. Indeed, in many cases, only one solution to a particular technological or research problem is needed, only one partner (e.g, supplier, employer or employee) to be found. In this section, we consider the case of a single reward, and use superscript ^{1} to distinguish it from the case of multiple rewards considered earlier.

The recursive equation (4) becomes

$$v_k^1(t) = e^{-\delta\Delta}\left[p(R+\pi_0)\frac{\lambda}{\lambda+\delta}(1 - e^{-(\lambda+\delta)t}) + (1 - p + pe^{-\lambda t})\pi_{k-1}^1 e^{-\delta t}\right], \tag{13}$$

with π_{k-1} in the beginning of the right-hand side of equation (4) changed to π_0, reflecting the fact that once the reward is earned, future payoffs are only from the non-project activity π_0. Then

$$\pi_k^1 = \int_0^\infty \lambda_0 e^{-\lambda_0 x} e^{-\delta x} v_k^1 dx = \frac{\lambda_0}{\lambda_0+\delta} v_k^1,$$

and the optimal stopping time t_k^{1*} is given by

$$t_k^{1*} = -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi_{k-1}^1\delta}{(R+\pi_0)\lambda - \pi_{k-1}^1(\lambda+\delta)}\right].$$

If only one potential project is available (i.e., $k = 1$), then the problems with one reward and multiple rewards are equivalent; that is, for $k = 1$, Equations (4) and (13) are the same, hence, their solutions are the same. In particular, the optimal stopping times are the same, $t_1^{1*} = t_1^*$. However, for $k > 1$, k finite, equations (4) and (13) are different, and thus the equations for the optimal stopping times are different. For instance, for $k = 2$,

$$t_2^{1*} = -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi_1^1\delta}{(R+\pi_0)\lambda - \pi_1^1(\lambda+\delta)}\right] < -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi_1\delta}{R\lambda - \pi_1}\right] = t_2^*.$$

That is, if you have just begun the second-to-last possible project, you will continue working on that project longer if you are able to earn rewards from both of the remaining projects than if you can earn a reward only from one of the remaining two. It is worth spending a little longer on the project in hand because its reward can be banked and added to the reward from the last remaining project.

If there are an infinite number of potential projects but only a single reward can be earned, equation (8) becomes

$$v^1(t) = e^{-\delta\Delta}\left[p(R+\pi_0)\frac{\lambda}{\lambda+\delta}(1-e^{-(\lambda+\delta)t}) + (1-p+pe^{-\lambda t})\pi^1 e^{-\delta t}\right]. \tag{14}$$

Optimizing with respect to t yields

$$t^{1*} = -\frac{1}{\lambda}\ln\left[\frac{1-p}{p}\frac{\pi^1\delta}{(R+\pi_0)\lambda - \pi^1(\lambda+\delta)}\right], \tag{15}$$

and then completing the recursion for π^1:

$$\pi^1 = \frac{\lambda_0}{\lambda_0+\delta}v^1(t^{1*}).$$

Note the close resemblance of (14) and (15) to the value function and optimal abandonment equations (8 and 9) for the case in which many rewards can be earned. Indeed, the optimal stopping time is the same in both cases: $t^{1*} = t_\infty^*$. To see this, follow the same procedure as in §3 above to yield exactly the same equation (12) for the optimal stopping time (see the Appendix for details). This result is rather surprising and unlike the finite ($k \geq 2$) case: as long as there are an infinite number of possible projects it does not matter whether the firm can earn one or many rewards. In each case, the firm spends the same amount of time on a project before abandoning it and searching for a new project. The opportunity value is greatly diminished when there is only one possible reward, but that diminishment equally affects the value of the project at hand.

Corollary 3 *The optimal abandonment time t^{1*} is increasing in p, δ and Δ; it is decreasing in λ_0 and λ; and it is unaffected by changes in R.*

Corollary 3 follows immediately from Corollary 2 and the equivalence of the optimal abandonment time solution (12) for both the one and many rewards cases.

The model with one reward also allows us to get insights into the behavior of a risk-averse searcher. For simplicity, assume that the payoff from the non-project activity $\pi_0 = 0$; i.e., completing the project in infinite time and abandoning it immediately yield the same payoff. Then the only uncertainty is the completion time. If the project is completed at time τ, the payoff is $e^{-\delta\tau}R$. Utility for wealth $u_w(e^{-\delta\tau}R)$ can be thought of as utility for completion time $u_\tau(\tau) = u_w(e^{-\delta\tau}R)$. If the decision maker is risk averse with constant relative risk aversion, $u_w(w) = w^{1-\rho}$ for $0 < \rho < 1$, and therefore $u_\tau(\tau) = (e^{-\delta\tau}R)^{1-\rho} = R^{1-\rho}e^{-\delta(1-\rho)\tau}$. Constant relative risk aversion equates to decreasing the discount factor from δ to $\delta(1-\rho)$: a risk-averse decision maker with constant relative risk averse utility will behave like a risk-neutral decision maker with a smaller discount factor. Per Corollary 2, a smaller discount factor leads to an earlier optimal abandonment time.

Constant relative risk aversion with $\rho = 1$ corresponds to logarithmic utility, $u_w(w) = \ln(w)$, and in this case $u_\tau(\tau) = \ln(e^{-\delta\tau}R) = \ln(R) - \delta\tau$. That is, a decision maker with logarithmic utility will use the same stopping time irrespective of the discount factor δ. In this case, maximizing $E[u_\tau(\tau)] = \ln(R) - \delta E(\tau)$ is equivalent to minimizing the expected completion time $E(\tau)$.

Also note that our results go through for the case of $\rho < 0$, in which case $u_w(w) = w^{1-\rho}$ corresponds to risk-seeking preferences. In this case, a more risk-seeking decision maker would behave as if he were a risk neutral decision maker whose discount factor is greater than δ. An increase in δ leads to an increase in the optimal abandonment time.

4.2. Deriving the lowest bound on the stopping time

In this section we discuss the optimal stopping time for the case of an infinite number of potential projects (i.e., $k = \infty$) with no discounting (i.e., $\delta \to 0$). This case is interesting because 1) it provides a lower bound on the optimal stopping time, as t^* is increasing with δ and decreasing with k; 2) this optimal stopping time is the same for the cases with multiple rewards and single reward; 3) in the case of a single reward, this optimal stopping time corresponds to the behavior of a risk-averse decision maker with logarithmic utility function, regardless of the discount factor itself; 4) in the case of a single reward, this optimal stopping time corresponds to the optimal behavior under the objective of minimizing the expected time to completion.

To derive the equation for t^* with $\delta \to 0$, we start with (12), multiplying both sides by δ, and then apply a Taylor series expansion, keeping only the terms that are linear in δ. Details are in the Appendix. This yields

$$e^{\lambda t^*} - \lambda t^* - 1 = \frac{\lambda}{1-p}\left(\frac{1}{\lambda_0} + \Delta\right). \tag{16}$$

Figure 3 plots the optimal stopping time for $\lambda = 1$ as a function of p, for different values of λ_0 with $\Delta = 0$. Note that, as long as $p < 0.8$, the optimal stopping time does not change much with p. Consistent with Corollary 3, t^* is decreasing in λ_0. Also, from Corollary 3, t^* is decreasing in λ.

Equation (16) allows us to compute the optimal stopping time by knowing only λ and λ_0 as long as p is not too large. In many instances, it might be easier to estimate λ and λ_0 instead of directly deciding whether to stop the project at time t. As mentioned in §1, because of organizational inertia the decision to stop a project is often difficult, while the decision to continue is easy. Thus, it might be useful to see whether the decision to continue is consistent with the existing estimates of λ and λ_0. If these estimates were to increase, e.g., because of acquired proficiency, then by Corollary 3, one should stop earlier.

****************************INSERT FIGURE 3 HERE *************************

4.3. Multiple types of projects

A project is defined by the prior probability p that it can be successfully completed; the rate λ at which success arrives if possible; the fixed cost Δ before success can be achieved; and the reward R earned on successful completion. Heretofore all projects are of the same type in that they all share the same parameter values (p, λ, Δ, R). Suppose instead that there are multiple project types. For ease of exposition, we deal with the case of two project types: $(p_1, \lambda_1, \Delta_1, R_1)$ and $(p_2, \lambda_2, \Delta_2, R_2)$. Type one projects occur with probability q. Once a project has been generated, the firm knows its type. The recursions build as in §2.2; we deal only with the limiting ($k = \infty$) case here. If the firm has a project of type 1 in hand, the firm solves (1) with the appropriate values of $(p_1, \lambda_1, \Delta_1, R_1)$; and if the firm has a project of type 2 in hand, it solves (1) with the appropriate values of $(p_2, \lambda_2, \Delta_2, R_2)$. Denote those value functions as $v_1(t)$ and $v_2(t)$ respectively; and denote the optimal abandonment times as t_1^* and t_2^* respectively. Then the fixed point equation for the expected value of searching for a project is

$$\pi = \frac{\lambda_0}{\lambda_0 + \delta}\left(q v_1(t_1^*) + (1-q) v_2(t_2^*)\right), \tag{17}$$

where, for $i = 1, 2$,

$$t_i^* = -\frac{1}{\lambda_i} \ln \left[\frac{1 - p_i}{p_i} \frac{\pi\delta}{R_i\lambda_i - \pi\delta} \right]. \tag{18}$$

For example, university promotion committees often assign point values to journals, explicitly or implicitly: there are A journals and B journals. An academic should spend a longer amount of time before abandonment on a project for an A journal than on a project for a B journal, other things being equal.

In the case of a single project type, one needs to estimate λ_0 in order to derive the optimal stopping time. For multiple types of projects, the stopping times given by (18) must be consistent. For example, if the decision maker believes that projects of type 1 have to be stopped at t_1^*, then she can derive the corresponding π, which in turn will determine t_2^*, thus providing an extra consistency check. Figure 4 plots the optimal stopping time t_2^* for different values of R_2 as a function of t_1^*, the optimal stopping time for a project with $R_1 = 1$, giving a sense of the consistency that would be required (with the rest of the parameters at $\lambda_1 = \lambda_2 = 1, p_1 = p_2 = 0.5, \Delta_1 = \Delta_2 = 0$, and $\delta = 0.1$). For example, if it is optimal to stop the project with $R_1 = 1$ at $t_1^* = 1$, then one should stop the project with $R_2 = 5$ at $t_2^* = 2.87$, while the project with $R_2 = 0.5$ should not be started at all.

*******************************INSERT FIGURE 4 HERE***************************

5. Conclusion

We developed a stylized model of time and effort allocation to the creative process, a process rife with uncertainty regarding the chance of success of any individual project, the length of time one might wait for a new project to appear, and the amount of time required to bring a potentially successful project to completion. Whenever a researcher is working on a project, she must consider alternative uses of her time and effort – what is the opportunity cost of continuation? If the only alternative facing the researcher is to search out and begin devoting effort to some new project, ex-ante identical to the one she is currently working on, then the opportunity cost becomes endogenous. Our attention was focused on the timing decision: when to give up on the project at hand in order to search for a new project. Our focus on the timing of the quit decision yielded several insights. For a large range of the probability p of being able to successfully complete a project, the optimal solution is relatively insensitive to the probability. That is, for $0 < p < 0.5$, the optimal stopping time t^* is relatively flat in p. Most research projects of the nature modeled here (e.g, pharmaceutical discovery) do not have very high initial probabilities of success, in which case precise estimation of p is not important. Given that such probabilities are often difficult to assess, this result is important for prescriptive purposes. On the other hand, the optimal stopping time t^* is sensitive to the arrival rate of new projects (when the firm is searching) and the arrival rate of successful completion for a project that can be successfully completed.

Given the empirical evidence of organizational inertia and personal procrastination, which lead to continuing projects long after they should have been aborted, one practical application of our results would be for setting an alarm clock: if a project has not been successfully completed by the time its alarm goes off, the project should be terminated and efforts should be devoted to searching for something new.

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Appendix

Derivation of equation (6)

Begin by substituting (5) into (4):

$$\begin{aligned}
v_k &= v_k(t_k^*) = e^{-\delta\Delta}\left[p(R+\pi_{k-1})\frac{\lambda}{\lambda+\delta}(1-e^{-(\lambda+\delta)t_k^*}) + (1-p+pe^{-\lambda t_k^*})\pi_{k-1}e^{-\delta t_k^*}\right] \\
&= e^{-\delta\Delta}\left[p(R+\pi_{k-1})\frac{\lambda}{\lambda+\delta} - p(R+\pi_{k-1})\frac{\lambda}{\lambda+\delta}e^{-(\lambda+\delta)t_k^*} + (1-p)\pi_{k-1}e^{-\delta t_k^*} + p\pi_{k-1}e^{-(\lambda+\delta)t_k^*}\right] \\
&= pe^{-\delta\Delta}\frac{\lambda}{\lambda+\delta}\left[(R+\pi_{k-1}) - \frac{1}{\lambda}(R\lambda-\pi_{k-1}\delta)\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda-\pi_{k-1}\delta}\right)^{\frac{\lambda+\delta}{\lambda}}\right. \\
&\qquad \left. + \left(\frac{1-p}{p}\right)\left(\frac{\lambda+\delta}{\lambda}\right)\pi_{k-1}\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda-\pi_{k-1}\delta}\right)^{\delta/\lambda}\right]
\end{aligned}$$

$$
\begin{aligned}
&= pe^{-\delta\Delta}\frac{\lambda}{\lambda+\delta}\left[(R+\pi_{k-1})-\left(\frac{1-p}{p}\right)\pi_{k-1}\frac{\delta}{\lambda}\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda-\pi_{k-1}\delta}\right)^{\delta/\lambda}\right.\\
&\quad\left.+\left(\frac{1-p}{p}\right)\left(\frac{\lambda+\delta}{\lambda}\right)\pi_{k-1}\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda-\pi_{k-1}\delta}\right)^{\delta/\lambda}\right]\\
&= pe^{-\delta\Delta}\frac{\lambda}{\lambda+\delta}\left[R+\pi_{k-1}+\pi_{k-1}\left(\frac{1-p}{p}\right)\left(\frac{1-p}{p}\frac{\pi_{k-1}\delta}{R\lambda-\pi_{k-1}\delta}\right)^{\delta/\lambda}\right].
\end{aligned}
$$

∎

Derivation of (12)

To get equation (12) for t^*, solve (9) with respect to $R\lambda$:

$$
t^* = -\frac{1}{\lambda}\ln\left(\frac{1-p}{p}\frac{\pi\delta}{R\lambda-\pi\delta}\right) \Leftrightarrow e^{-\lambda t^*} = \frac{1-p}{p}\frac{\pi\delta}{R\lambda-\pi\delta} \Leftrightarrow R\lambda = \left(\frac{1-p}{p}e^{\lambda t^*}+1\right)\pi\delta.
$$

Substitute for $R\lambda$ in (11)

$$
\begin{aligned}
\pi &= \frac{\lambda_0}{\lambda_0+\delta}e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[R+\pi+\pi\frac{1-p}{p}\left(\frac{1-p}{p}\frac{\pi\delta}{R\lambda-\pi\delta}\right)^{\delta/\lambda}\right]\\
&= \frac{\lambda_0}{\lambda_0+\delta}e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[\left(\frac{1-p}{p}e^{\lambda t^*}+1\right)\pi\delta\frac{1}{\lambda}+\pi+\pi\frac{1-p}{p}e^{-\delta t^*}\right].
\end{aligned}
$$

Divide through by π:

$$
\begin{aligned}
1 &= \frac{\lambda_0}{\lambda_0+\delta}e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[\left(\frac{1-p}{p}e^{\lambda t^*}+1\right)\frac{\delta}{\lambda}+1+\frac{1-p}{p}e^{-\delta t^*}\right]\\
&= \frac{\lambda_0}{\lambda_0+\delta}e^{-\delta\Delta}p\frac{\lambda}{\lambda+\delta}\left[\frac{\delta}{\lambda}+1+\frac{1-p}{p}\left(\frac{\delta}{\lambda}e^{\lambda t^*}+e^{-\delta t^*}\right)\right].
\end{aligned}
$$

Rearrange terms:

$$
\frac{\delta}{\lambda}e^{\lambda t^*}+e^{-\delta t^*} = \left(\frac{\lambda_0+\delta}{\lambda_0}e^{\delta\Delta}\frac{\lambda+\delta}{\lambda}\frac{1}{p}-\frac{\lambda+\delta}{\lambda}\right)\frac{p}{1-p}.
$$

Finally, multiply through by $1/\delta$ to yield (12) ∎

Corollary 2: t^* is increasing in δ.

To finish the proof of Corollary 2, it remains to show that $\partial F/\partial\delta < 0$.

$$
\partial F/\partial\delta = \frac{-\lambda t^* e^{-t^*\delta}+e^{t^*\lambda}}{\lambda+\delta} - \frac{\lambda e^{-t^*\delta}+\delta e^{t^*\lambda}}{(\lambda+\delta)^2} - \frac{1}{1-p}\frac{1}{\lambda_0}e^{\delta\Delta} - \frac{1}{1-p}\left(1+\frac{\delta}{\lambda_0}\right)\Delta e^{\delta\Delta}.
$$

As we can see, $\partial F/\partial\delta$ decreases with Δ, so it will be enough to show that $\partial F/\partial\delta < 0$ when $\Delta = 0$. For $\Delta = 0$,

$$
F(\delta, t^*) = \frac{\lambda e^{-t^*\delta}+\delta e^{t^*\lambda}}{\lambda+\delta} - \frac{1}{1-p}\left(1+\frac{\delta}{\lambda_0}-p\right) = \frac{\lambda e^{-t^*\delta}+\delta e^{t^*\lambda}}{\lambda+\delta} - 1 - \beta\delta,
$$

where $\beta = \frac{1}{(1-p)\lambda_0}$.

Then $\partial F / \partial \delta = \frac{-\lambda t^* e^{-t^*\delta} + e^{t^*\lambda}}{\lambda+\delta} - \frac{\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}}{(\lambda+\delta)^2} - \beta$. Multiplying by $(\lambda+\delta)^2$, we obtain

$$(\lambda+\delta)^2 \partial F/\partial\delta = (\lambda+\delta)\left(-\lambda t^* e^{-t^*\delta} + e^{t^*\lambda}\right) - \left(\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}\right) - \beta(\lambda+\delta)^2 .$$

From equation (12) with $\Delta = 0$, $\beta = \frac{1}{\delta}\left(\frac{\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}}{\lambda+\delta} - 1\right)$. Then

$$\begin{aligned}(\lambda+\delta)^2 \partial F/\partial\delta &= (\lambda+\delta)\left(-\lambda t^* e^{-t^*\delta} + e^{t^*\lambda}\right) - \left(\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}\right) - \frac{1}{\delta}\left(\frac{\lambda e^{-t^*\delta} + \delta e^{t^*\lambda}}{\lambda+\delta} - 1\right)(\lambda+\delta)^2 \\ &= -\frac{1}{\delta}\left(\delta^2 e^{t^*\lambda} - \delta^2 - \lambda^2 + \lambda^2 e^{-t^*\delta} - 2\lambda\delta + 2\lambda\delta e^{-t^*\delta} + t^*\lambda\delta^2 e^{-t^*\delta} + t^*\lambda^2\delta e^{-t^*\delta}\right).\end{aligned}$$

Denote $H(t) = \delta^2 e^{t\lambda} - \delta^2 - \lambda^2 + \lambda^2 e^{-t\delta} - 2\lambda\delta + 2\lambda\delta e^{-t\delta} + t\lambda\delta^2 e^{-t\delta} + t\lambda^2\delta e^{-t\delta}$. We need to show that, for $t > 0$, $H(t) > 0$. Observe that $H(0) = 0$, and $H'(t) = \lambda\delta^2\left(e^{t\lambda} - e^{-t\delta} - t\lambda e^{-t\delta} - t\delta e^{-t\delta}\right)$ is positive because

$e^{t\lambda} - e^{-t\delta} - t\lambda e^{-t\delta} - t\delta e^{-t\delta} = e^{-t\delta}\left(e^{t(\lambda+\delta)} - (1 + t(\lambda+\delta))\right) > 0 \Leftrightarrow e^{t(\lambda+\delta)} > 1 + t(\lambda+\delta)$.

∎

Establishing that $t^{1*} = t^*_\infty$.

The fixed point equation for π^1 similar to (11) is

$$\pi^1 = \frac{\lambda_0}{\lambda_0+\delta} e^{-\delta\Delta} p \frac{\lambda}{\lambda+\delta}\left[(R+\pi_0^1) + \pi^1 \frac{1-p}{p}\left(\frac{1-p}{p}\frac{\pi^1\delta}{R\lambda - \pi^1(\delta+\lambda)}\right)^{\delta/\lambda}\right]. \qquad (1)$$

Solve (15) for $(R+\pi_0^1)\lambda$:

$$t^{1*} = -\frac{1}{\lambda}\ln\left(\frac{1-p}{p}\frac{\pi^1\delta}{(R+\pi_0^1)\lambda - \pi^1(\delta+\lambda)}\right) \Leftrightarrow (R+\pi_0^1)\lambda = \frac{1-p}{p} e^{\lambda t^{1*}} \pi^1\delta + \pi^1(\delta+\lambda).$$

Substitute for $(R+\pi_0^1)\lambda$ in (1):

$$\pi^1 = \frac{\lambda_0}{\lambda_0+\delta} e^{-\delta\Delta} p \frac{\lambda}{\lambda+\delta}\left[\left(\frac{1-p}{p} e^{\lambda t^{1*}}\right)\pi^1\delta\frac{1}{\lambda} + \pi^1\frac{\delta+\lambda}{\lambda} + \pi^1\frac{1-p}{p} e^{-\delta t^{1*}}\right].$$

Divide through by π^1 and rearrange terms to yield:

$$\frac{1}{\lambda} e^{\lambda t^{1*}} + \frac{1}{\delta} e^{-\delta t^{1*}} = \frac{\lambda+\delta}{\lambda\delta(1-p)}\left(\frac{\lambda_0+\delta}{\lambda_0} e^{\delta\Delta} - p\right),$$

which matches (12), the equation for the optimal stopping time in the case of many rewards, exactly. ∎

Derivation of equation (16)

$$
\begin{aligned}
\delta \frac{1}{\lambda} e^{\lambda t^*} + e^{-\delta t^*} &= \frac{\lambda + \delta}{\lambda (1-p)} \left(\frac{\lambda_0 + \delta}{\lambda_0} e^{\delta \Delta} - p \right) \\
\delta \frac{1}{\lambda} e^{\lambda t^*} + 1 - \delta t^* &= \frac{1}{1-p} \left(1 + \delta \frac{1}{\lambda} \right) \left(\left(1 + \delta \frac{1}{\lambda_0} \right) (1 + \delta \Delta) - p \right) \\
1 + \delta \left(\frac{1}{\lambda} e^{\lambda t^*} - t^* \right) &= \frac{1}{1-p} \left(1 + \delta \frac{1}{\lambda} \right) \left(1 + \delta \left(\frac{1}{\lambda_0} + \Delta \right) - p \right) \\
1 + \delta \left(\frac{1}{\lambda} e^{\lambda t^*} - t^* \right) &= \left(1 + \delta \frac{1}{\lambda} \right) \left(1 + \delta \frac{1}{1-p} \left(\frac{1}{\lambda_0} + \Delta \right) \right) \\
\frac{1}{\lambda} e^{\lambda t^*} - t^* &= \frac{1}{\lambda} + \frac{1}{1-p} \left(\frac{1}{\lambda_0} + \Delta \right),
\end{aligned}
$$

yielding

$$
e^{\lambda t^*} - \lambda t^* - 1 = \frac{\lambda}{1-p} \left(\frac{1}{\lambda_0} + \Delta \right).
$$

∎

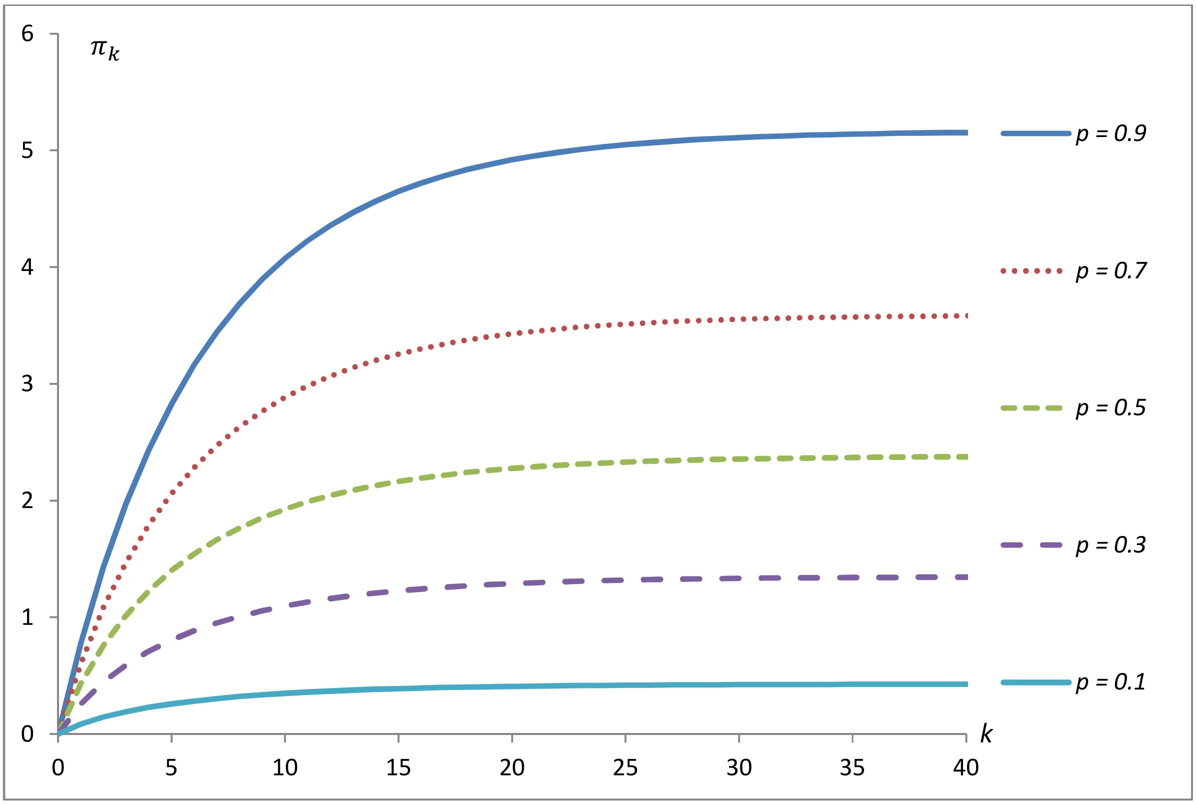

Figure 1. Graph of π_k as a function of k for different values of p. Other parameters are: $\lambda = 1$, $\lambda_0 = 2$, $\delta = 0.1$, $\Delta = 0$, $R = 1$, $\pi_0 = 0$.

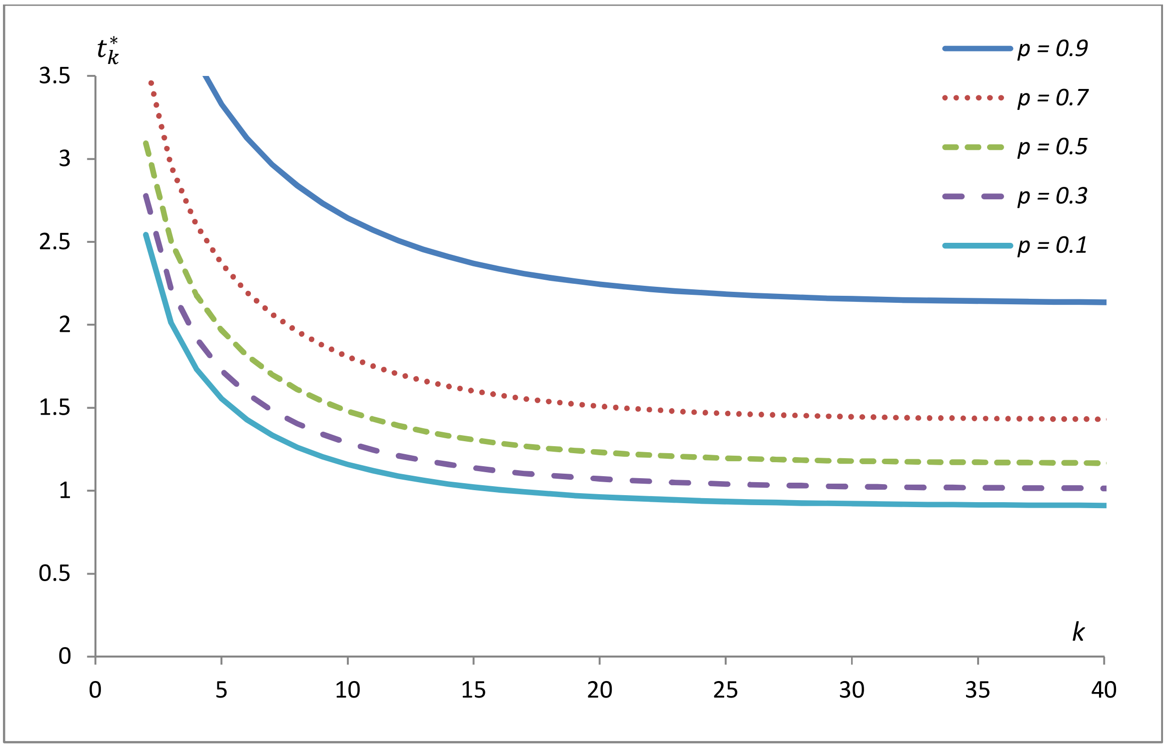

Figure 2. Graph of t_k^* as a function of k for different values of p. Other parameters are: $\lambda = 1$, $\lambda_0 = 2$, $\delta = 0.1$, $\Delta = 0$, $\pi_0 = 0$.

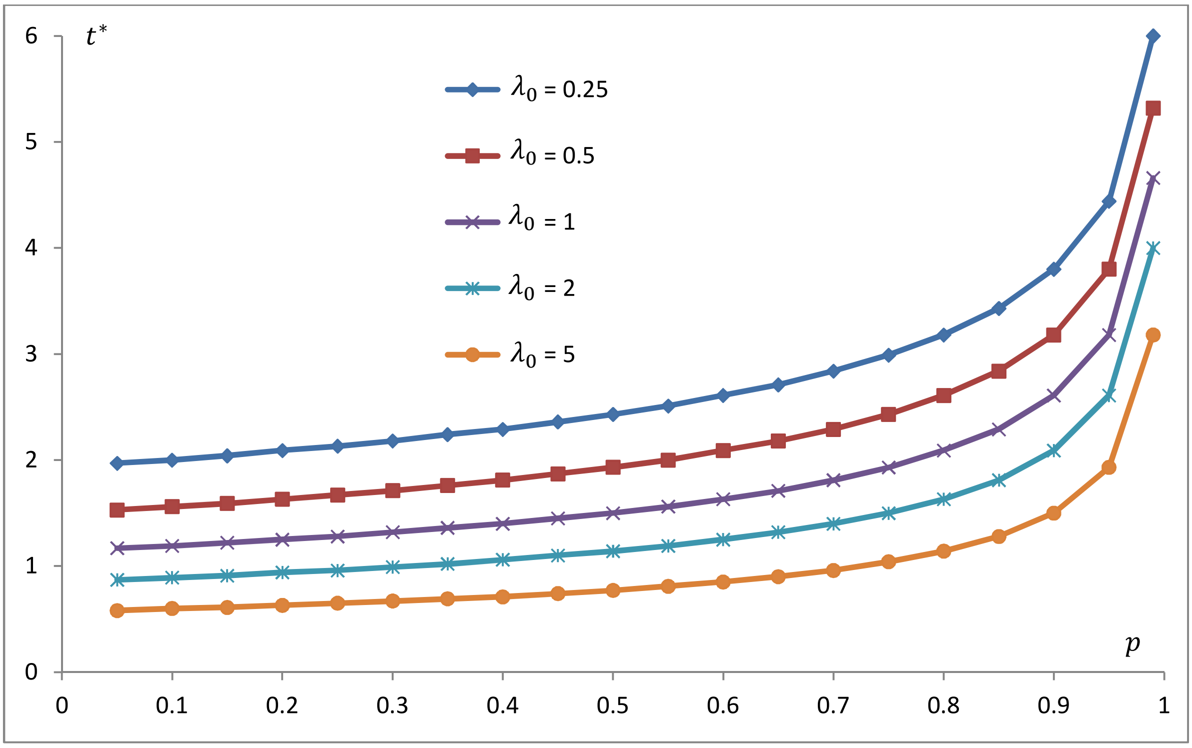

Figure 3. Graph of t^* as a function of p for different values of λ_0. Other parameters are: $\lambda = 1$, $\delta = 0$, $\Delta = 0$.

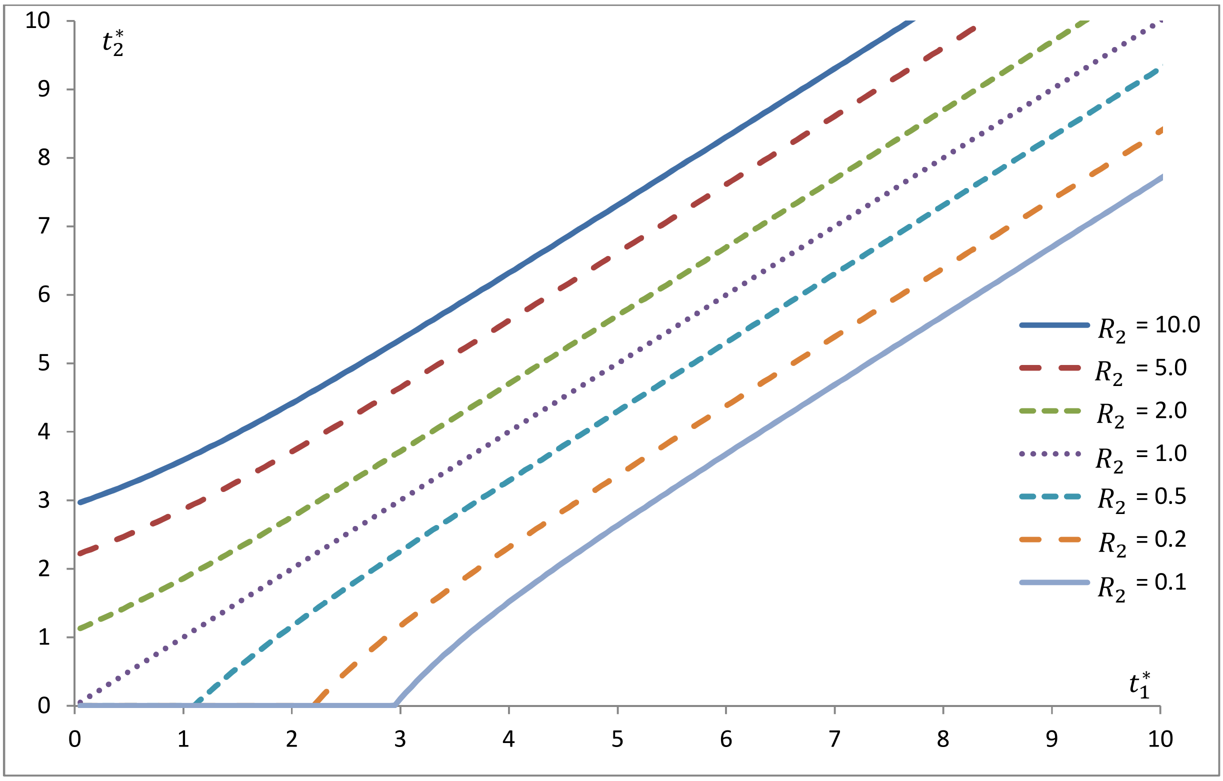

Figure 4. Graph of t_2^* as a function of t_1^* for different values of R_2. Other parameters are: $\boldsymbol{\lambda}_1 = \lambda_2 = 1$, $p_1 = p_2 = 0.5$, $\Delta_1 = \Delta_2 = 0$, $\delta = 0.1$, $R_1 = 1$.