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Risky Choices and Correlated Background Risk

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The analysis of a risky project should take into account not only uncertainties about the return from that project ("project risk"), but also uncertainties associated with other ongoing projects and with exogenous factors that can impact final wealth ("background risk"). The presence of background risk can change the optimal course of action with respect to a project, and ignoring such risk might lead to a poor decision. Most work on background risk assumes that project risk and background risk are independent and are additive in their impact on wealth. However, independence is often unrealistic, and background risk operates in a multiplicative manner in many situations. We relax the independence assumption and consider a model with both additive and multiplicative background risk. The optimal decisions in the correlated setting can be very different than those that would appear optimal if the correlation were ignored. The impact of correlation differs in the additive and multiplicative cases, with positive correlation being beneficial in some cases and negative correlation in others. The analytical and numerical results indicate that in analyzing decision-making problems, it is very important to understand the direction and degree of dependence between project risk and background risks.

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1. Introduction

In many situations, a decision about a risky project must be made in the presence of other important uncertainties. Pratt (1988, p. 395) makes this point very nicely:

> Most real decision makers, unlike those portrayed in our popular texts and theories, confront several uncertainties simultaneously. They must make decisions about some risks when others have been committed to but not resolved. Even when a decision is to be made about only one risk, the presence of others in the background complicates matters.

These other risks have been associated with random initial wealth (Kihlstrom et al. 1981) and have been labeled related stakes (Kadane and Winkler 1988), contextual uncertainties (Bell 1995a), exogenous risks (as opposed to the endogenous risks associated with the project), and most commonly background risks. We will use the term "project risk" to refer to any uncertainty associated with the project under consideration and the term "background risk" to refer to any other (exogenous) uncertainty that impacts the decision maker's final wealth.

Much work on background risk and multiple risks has appeared in the economics literature; see Gollier (2001) for an excellent review. A key focus is on conditions under which the presence of the background risk (a) preserves comparative risk aversion and (b) increases the decision maker's risk aversion concerning the project risk. Toward that end, various preference conditions have been studied, such as decreasing absolute risk aversion (DARA) (Pratt 1964), comparative risk aversion in the sense of Ross (1981), proper risk aversion (Pratt and Zeckhauser 1987), standard risk aversion (Kimball 1993), and risk vulnerability (Gollier and Pratt 1996).

Virtually all of the research on background risk assumes that the project risk and the background risk are additive. For example, if a firm is already committed to a number of projects and is contemplating a new one, the profits from the projects are additive. In other cases, however, the background risk may operate in a multiplicative manner. If a firm is considering a project in a foreign country but will convert any profit back to the firm's home currency, uncertainty about the exchange rate between the two currencies serves as a multiplicative background risk. Franke et al. (2005)—who point out that examples with multiplicative background risk are at least as prevalent as those with additive background risk and express surprise at the lack of attention to the former—extend some results to the multiplicative case.

The background risk literature generally assumes that the background risk is independent of the project risk. However, even for many typical examples mentioned in this literature, some dependence between project risk and background risk may be expected. For example, in personal investment decisions, wage and proprietary income are additive background risks, and investment income represents project risk. Heaton and Lucas (2000) show that correlations of stock return with wage income and proprietary income may range considerably, with relatively large positive and negative values not uncommon. For a multinational firm, exchange-rate uncertainty represents multiplicative background risk. Empirical evidence on exchange-rate exposure of Japanese corporations (He and Ng 1998) and U.S. multinational firms (Allayannis and Ihrig 2001, Jorion 1990) demonstrates that correlations between returns and exchange rates may be positive or negative, depending on the company's profile (e.g., exporting operations versus importing operations). Thus, for many decision-making situations, the assumption that project risk and background risks are independent is quite unrealistic.

There are a few exceptions to the typical assumption of independence. For example, Pratt (1988) mentions that negative correlation between background and project risks could make the additive combination of the two less risky than the project risk alone. Arrondel and Calvo Pardo (2003) show that additive negatively correlated background risk may increase the optimal exposure to portfolio risk and find empirically that French households responded to an increase in future earnings uncertainty (the background risk) by increasing their stockholdings (taking on more project risk). However, as Gollier (2001, p. 119) states, "The analysis of interaction among dependent risks is still in its infancy."

In this paper, we study the impact of correlated background risk, considering a model with both additive and multiplicative components. Optimal decisions in the correlated background risk setting can be very different from the decisions that would be recommended if the correlation were ignored, and can be very sensitive to the sign and magnitude of the correlation. Although intuition suggests that negative correlation could be beneficial (e.g., through a diversification effect) in the sense of increasing the certainty equivalent for the project, in some cases positive correlation is beneficial and negative correlation is undesirable. The results indicate that in analyzing decision-making problems, it is very important to understand the direction and degree of dependence between the project risk and the background risk.

The organization of this paper is as follows. First, we present motivating examples in §2 to demonstrate the importance of accounting for correlation between the project risk and background risk. Then, in §3, we consider a general model with both additive and multiplicative risk and show that a major factor in a decision about a project with zero mean is whether the project risk is positively or negatively related to background risk. We summarize the results and discuss some implications in §4.

2. Examples

To demonstrate the potential importance of correlation of project risks and background risks and to illustrate some results that are developed later, we start with numerical examples using two utility functions for wealth, with monetary amounts expressed in millions of dollars:

(1) Exponential utility with absolute risk aversion 0.005: $U_E(w) = -e^{-cw}$, with $c = 0.005$. Following Howard's (1988) rule of thumb that risk tolerance (the reciprocal of absolute risk aversion) tends to be about one-sixth of equity, U_E may be reasonable for a firm with equity of about \$1.2 billion. Howard (1988, p. 689) suggests that exponential utility can "satisfactorily treat a wide range of individual and corporate risk preference," and indeed exponential utility is commonly used in decision analysis. Because exponential utility exhibits constant absolute risk aversion (CARA), the firm's wealth position, aside from current projects (risks), is irrelevant for decision-making purposes.

(2) Linear plus exponential utility with risk aversion 0.005 at $w = 0$: $U_{LE}(w) = bw - e^{-cw}$, with $b = c = 0.01$. Linear plus exponential utility exhibits DARA, which is a condition invoked in much of the literature on background risk and financial decision making in general (Gollier 2001). It is proper (Pratt and Zeckhauser 1987), has constant absolute prudence (Kimball 1990), is the only increasing, risk-averse utility function that satisfies Bell's (1988) one-switch condition, satisfies Bell's (1995a) contextual uncertainty condition, and permits risks to be summarized in terms of a return measure and a risk measure (Bell 1995b). As Bell (1995a, p. 1148) says, "Perhaps this function deserves some consideration as *the* appropriate utility function for generic analyses of financial risk taking." The parameters of U_{LE} are chosen to equate its absolute risk aversion to the (constant) risk aversion of U_E when $w = 0$ (without loss of generality because the zero point of w can be defined as the initial wealth for which risk aversion equals a specific value). The absolute risk aversion of U_{LE} is $c/[1+(b/c)e^{cw}] = (0.01)/(1+e^{0.01w})$, which approaches 0.01 as $w \to -\infty$, decreases to 0.005 at $w = 0$, and approaches zero as $w \to \infty$.

In §2.1, we consider a situation with additive background risk, where we present analytical results for exponential utility and numerical results for linear plus exponential utility. Then, in §2.2, we consider an example involving multiplicative background risk, where we present numerical results for exponential utility.

2.1. Additive Background Risk

Suppose that a firm is considering a new project, having already committed to some other projects. Denote the uncertain profit from the new project by $\tilde{x}$ and the uncertain total profit from the other projects by $\tilde{y}$. Recall that all monetary amounts are expressed in millions of dollars.

If the new project is accepted, the total profit is $\tilde{x}+\tilde{y}$, so the risk associated with the new project and the background risk are additive. Assume that the uncertainty about $\tilde{x}$ and $\tilde{y}$ can be represented by a bivariate normal distribution with means $\mu_x = \mu_y = 0$, standard deviations σ_x and σ_y, and correlation ρ. The zero means enable us to focus on the risks and relate to previous results in the literature, and they do not affect the nature of our results.

In the absence of background risk, the risk-averse firm should not take on the project with mean zero. Does the presence of the background risk change that choice? For U_E, the firm's certainty equivalent (CE) for the new project is $-c\sigma_x(2\rho\sigma_y + \sigma_x)/2$. With no background risk ($\sigma_y = 0$) or independent background risk ($\rho = 0$), the CE is negative and the project is unattractive, as it is when $\rho > 0$. If $\rho < 0$ and $\sigma_x < -2\rho\sigma_y$, however, the CE is positive and the project should be taken.

A nonzero μ_x simply changes the CE for the new project by an amount equal to μ_x. For example, if $\mu_x > 0$, the CE increases, thereby expanding the set of (ρ, σ_x) pairs for which the new project is desirable and including some pairs with positive ρ. The CE remains a decreasing function of ρ. A nonzero μ_y would not change the results at all; it can be viewed as an increase or decrease in the firm's wealth position, which under exponential utility does not influence decisions regarding the new project.

What happens with linear plus exponential utility? Figure 1 shows the CE for the new project under U_{LE} as a function of the standard deviation σ_x of the project risk for different values of ρ, with the standard deviation σ_y of the background risk equal to 150 and initial wealth equal to zero. The figure is almost the same as the corresponding graph (not shown) for the U_E case discussed above; the nonzero CEs are larger in absolute value for U_{LE}, but the maximum and zero CEs occur at the same values of σ_x. The CE curve for the case of no background risk included in Figure 1 is slightly above that for independent background risk. If $\rho \geq 0$, the CE is never positive and is a decreasing function of both ρ and σ_x. A positive ρ reduces the CE because it increases the standard deviation of $\tilde{x}+\tilde{y}$, making the project even more undesirable as ρ increases. When $\rho < 0$, however, the CE is positive if σ_x is not too large, reducing the total risk because the project risk diversifies away some of the background risk. The more negative ρ is, the greater this effect, and for a given ρ, the CE increases for $\sigma_x < -\rho\sigma_y = -150\rho$, at which point it reaches its maximum. For example, the new project with $\mu_x = 0$ and $\sigma_x = -150\,(-0.8) = 120$ is worth \$45.5 million when $\rho = -0.8$.

Much of the literature on background risk deals with the effect of independent background risk on the optimal size of the investment. Suppose that the uncertain profit from an investment in the new project is $\alpha\tilde{x}$, where $\tilde{x}$ represents the profit per unit invested and $\alpha \geq 0$ is the number of units invested, with a unit representing \$1 million. For U_E, $\alpha^* = \max\{0, (\mu_x - c\rho\sigma_x\sigma_y)/c\sigma_x^2\}$ is the optimal size of the investment. For the case of no background risk ($\sigma_y = 0$) or independent background risk ($\rho = 0$), the "optimal amount of risk to take," $\alpha^*\sigma_x$, equals the return per unit of risk (the Sharp ratio, μ_x/σ_x) divided by absolute risk aversion c. When $\mu_x = 0$, $\alpha^* = 0$. With negatively correlated background risk, however, α^* increases by $-\rho\sigma_y/\sigma_x > 0$ and is positive even when $\mu_x = 0$.

Next, we consider the optimal size of the investment under U_{LE}. Figure 2 shows α^* as a function of μ_x for different values of ρ with $\sigma_y = 150$ (as in Figure 1)

Figure 1 CE in Millions of Dollars for the New Project as a Function of the Standard Deviation of Its Return in Additive Background Risk Example

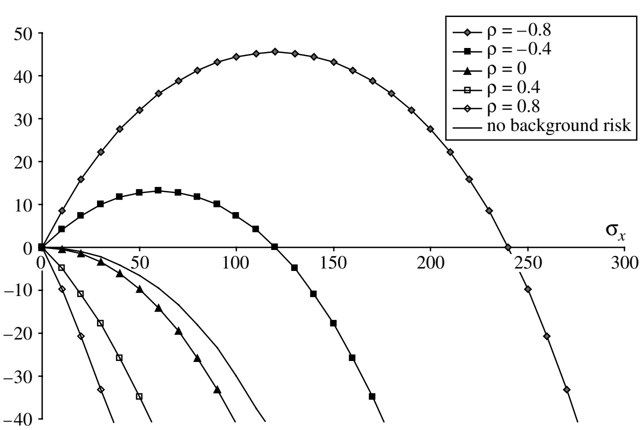
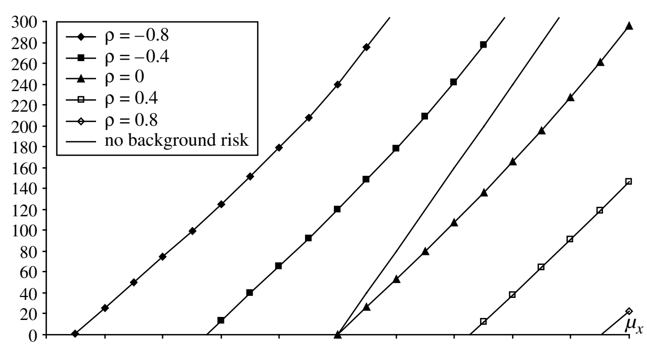


Figure 2 Optimal Investment in Millions of Dollars as a Function of the Project Mean in Additive Background Risk Example



and $\sigma_x = 0.50$. When $\mu_x = 0$, α^* is \$120 (\$240) million for $\rho = -0.4$ (-0.8). With $\rho = 0$, a mean return of more than 22% ($\mu_x = 0.22$) is required before it is optimal to invest \$120 million and a mean return of almost 42% for an investment of \$240 million. In the case of no background risk, the corresponding returns are 15% and 30%. Thus, negative values of ρ can be equivalent to substantial increases in μ_x with U_{LE}. For U_E, the tradeoff is linear, with a decrease of 0.4 in ρ being equivalent to an increase of $0.4c\sigma_x\sigma_y = 0.15$ (15%) in μ_x, smaller than the equivalent increases just noted for U_{LE}.

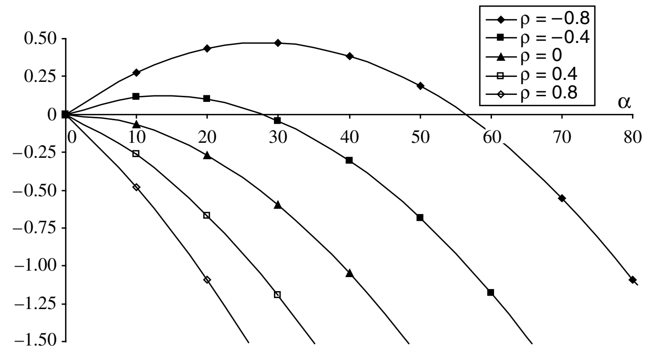
2.2. Multiplicative Background Risk

As noted in §1, there are many examples where background risk is multiplicative. In §2.2, we illustrate some effects of multiplicative background risk involving exchange-rate uncertainty. Consider a firm with a home base in the United States and some operations in Europe. The firm's total profits, assets, etc. are reported in U.S. dollars, and we consider the same CARA utility function U_E for wealth in millions of U.S. dollars as in §2.1. The firm is currently holding w_e in millions of euros and is considering a new European project that will require an investment of α million euros and will yield revenues of $\alpha\tilde{x}$ million euros. The exchange rate in dollars per euro at the time of the receipt of revenues is denoted by $\tilde{y}$. To keep the example simple, we will assume that the currency conversion will occur at a fixed time, $\tilde{x}$ and $\tilde{y}$ are independent of any uncertainties involving the company's U.S. operations, and the firm does not hedge against exchange-rate risks.

Thus, if the project is undertaken, the final wealth in millions of dollars is $[w_e + \alpha(\tilde{x} - 1)]\tilde{y}$ from the conversion of the wealth $w_e + \alpha(\tilde{x} - 1)$ in millions of euros using the exchange rate $\tilde{y}$. The risk associated with the new project and the background exchange-rate risk are multiplicative. Assume that the uncertainty about $\tilde{x}$ and $\tilde{y}$ can be represented by a bivariate normal distribution for their logarithms, $\ln\tilde{x}$ and $\ln\tilde{y}$, with means -0.125 and -0.02, standard deviations 0.5 and 0.2, and correlation ρ. This implies lognormal marginal distributions for $\tilde{x}$ and $\tilde{y}$ with means of 1 and standard deviations of 0.533 and 0.202, respectively. The means of 1 enable us to focus on the risks and do not affect the general nature of the results. Thus, the net profit in millions of euros from the new project has mean zero and standard deviation 0.533α.

As in the additive background risk example, the mean net profit of zero implies that the risk-averse firm should not take on the new project in the absence of background risk. For U_E, Figure 3 gives the CE of the new project in the presence of the multiplicative background risk as a function of the size of the project α for different values of ρ, with $w_e = 300$.

Figure 3 **CE in Millions of Euros for the New Project as a Function of the Investment α in Multiplicative Background Risk Example with $w_e = 300$**



The CE is expressed in millions of euros (the project currency). The CE curve for the case of no background risk (not shown) is virtually identical to the CE curve with independent background risk. Figure 3 is similar in nature to Figure 1: the CE is a decreasing function of ρ and the project is never desirable if $\rho \geq 0$. However, despite the expected payoff of zero, the project is desirable for some (ρ, α) pairs with $\rho < 0$ and α not too large. If the firm can choose how much to invest in the project, the optimal α is about €30 million when $\rho = -0.8$ and the corresponding CE is €0.47 million.

The results for the multiplicative background risk example are not always similar to those from the additive risk example, however. Changing w_e from 300 to 100 gives the results shown in Figure 4. The overall picture looks the same as Figure 3 at first glance, but now the CE is an increasing function of ρ, so positive correlations yield higher CEs and the project is never desirable if $\rho \leq 0$. In this situation, unlike the additive case, higher correlations lead to higher values of $E(w)$, and this can overcome the negative effect of higher variances under some circumstances (e.g., the lower value of w_e in Figure 4 as opposed to the higher value in Figure 3).

Figure 4 **CE in Millions of Euros for the New Project as a Function of the Investment α in Multiplicative Background Risk Example with $w_e = 100$**

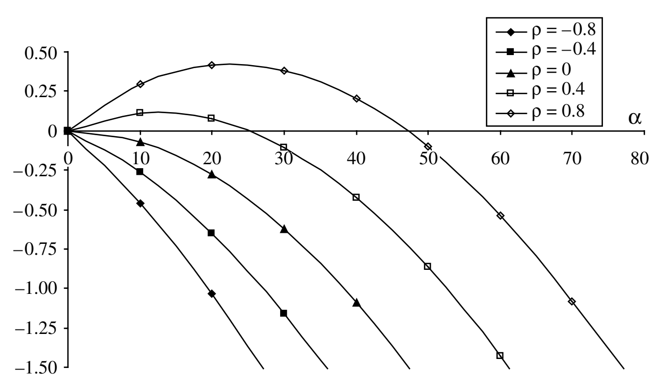
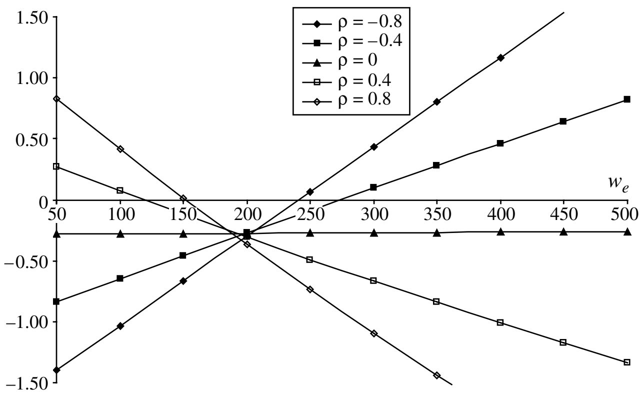


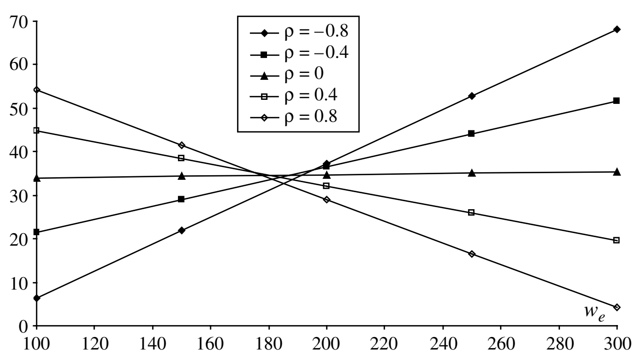
Figure 5 CE in Millions of Euros for the New Project as a Function of w_e in Multiplicative Background Risk Example with $\alpha = 20$



To see the impact of w_e in more detail, consider Figure 5, which gives the CE as a function of w_e for the case with $\alpha = 20$. Positive correlations give higher CEs for low values of w_e and negative correlations give higher CEs for high values of w_e, with the dividing line at about $w_e = €200$ million. Since $E(\tilde{y}) = 1$, €200 million corresponds approximately to the firm's risk tolerance, which is \$200 million. As §3.2 shows, this is not a coincidence. For $|\rho| < 0.8$, the CE is always negative (implying the project should not be undertaken) for intermediate values of w_e, roughly €150–250 million.

Next, we consider the effect of multiplicative background risk on the optimal size of the investment. Suppose the firm can choose how much to invest in the project (i.e., can choose $\alpha \geq 0$). We assume that $\ln \tilde{x}$ and $\ln \tilde{y}$ have a bivariate normal distribution as above, with the mean of $\ln \tilde{x}$ changed to -0.076, implying that $\tilde{x}$ has mean 1.05 and standard deviation 0.560. Figure 6 shows α^*, the optimal size of the investment, as a function of w_e for different values of ρ under U_E. This graph looks very similar in nature to Figure 5: positive (negative) correlations yield higher values of α^* for low (high) values of w_e, with the dividing line at about $w_e = €185$ million.

Figure 6 Optimal Investment in Millions of Euros as a Function of w_e in Multiplicative Background Risk Example



If $w_e = €100$ million, increasing ρ from zero to 0.4 increases α^* by €10.8 million. If $w_e = €300$ million, this increase in ρ will *reduce* α^* by €15.8 million, but decreasing ρ from zero to -0.4 increases α^* by €16.2 million. The optimal α^* with independent background risk ranges from €34.0 to €35.4 million as w_e goes from €100 to €300 million. This is comparable to the optimal investment of €34.6 million with no background risk. With the values of w_e and ρ shown in Figure 6, α^* under correlated background risk ranges from €4.2 to €68.2 million.

This example involving multiplicative background risk has assumed exponential utility. Repeating the same analysis for U_{LE} yields essentially the same results, with slightly different numbers but the same general picture.

The examples in §2 demonstrate that accounting for correlation between the project risk and the background risk can be very important in decision making. Ignoring such correlation or assessing it incorrectly might be a much more costly mistake than ignoring independent background risk. The fact that positive correlations are bad and negative correlations are good in the additive risk example is not surprising. In contrast, the results in the multiplicative risk example seem much less predictable intuitively, especially in the sense that, depending on the value of w_e, positive correlations can be good and negative correlations bad. This illustrates the importance of modeling situations with background risk carefully and paying particular attention to possible correlations between risks. Of course, the numerical results are a function of the parameters used; we turn to a more general analysis in §3.

3. General Model

Denote the decision maker's utility for wealth w by $U(w)$. We assume that U is increasing, twice differentiable, and concave (i.e., $U''(w) < 0$ for all w). Consider a project with uncertain return $\tilde{x}$ per unit invested. If α units of money are invested in this project, the uncertain wealth is given by $\tilde{w} = w_0 + \tilde{y}_1 + (w_1 + \alpha\tilde{x})\tilde{y}_2$, where $\tilde{y}_1$ and $\tilde{y}_2$ are random variables corresponding to additive and multiplicative components of background risk, respectively, and $\tilde{y}_2 > 0$. The decision maker's initial wealth consists of portions w_0 and w_1 that are, respectively, not subject to (subject to) the multiplicative background risk $\tilde{y}_2$.

Consider the standard investment problem in this setting: how much to invest in a risky project. The decision maker wants to choose $\alpha \geq 0$ to maximize expected utility,

$$V(\alpha) = E[U(\tilde{w})] = E\big[U(w_0 + \tilde{y}_1 + (w_1 + \alpha\tilde{x})\tilde{y}_2)\big].$$

The restriction $\alpha \geq 0$, which means that the decision maker cannot go short on the project, is without loss of generality. To allow for negative values of α, we could consider positive investment in the project with return $-\tilde{x}$ and use the results of this section.

Proposition 1. *For any joint distribution of $\tilde{x}$, $\tilde{y}_1$, and $\tilde{y}_2$, the optimal investment α^* is unique and has the following property: $\alpha^* > 0$ if and only if $V'(0) > 0$. If $V'(0) \le 0$, then $\alpha^* = 0$.*

Proof. Proposition 1 follows from the fact that $V(\alpha)$ is strictly concave:

$$V'(\alpha) = E\big[\tilde{x}\tilde{y}_2 U'(w_0 + \tilde{y}_1 + (w_1 + \alpha\tilde{x})\tilde{y}_2)\big]$$

and

$$V''(\alpha) = E\big[(\tilde{x}\tilde{y}_2)^2 U''(w_0 + \tilde{y}_1 + (w_1 + \alpha\tilde{x})\tilde{y}_2)\big] < 0.$$

Therefore the maximization of $V(\alpha)$ is a well-behaved problem, and the optimal unrestricted α^* is given by the unique solution of the first-order condition $V'(\alpha^*) = 0$. Furthermore, an unrestricted α^* has the same sign as $V'(0)$, and the restriction $\alpha \ge 0$ implies that $\alpha^* = 0$ when $V'(0) < 0$. □

Proposition 1 shows that $V'(0) = E[\tilde{x}\tilde{y}_2 U'(w_0 + \tilde{y}_1 + w_1\tilde{y}_2)]$ is an important characteristic of the project. If $\tilde{x}$ is independent of $\tilde{y}_1$ and $\tilde{y}_2$, then $V'(0) = E(\tilde{x})E[\tilde{y}_2 U'(w_0 + \tilde{y}_1 + w_1\tilde{y}_2)]$. Because $E[\tilde{y}_2 U'(w_0 + \tilde{y}_1 + w_1\tilde{y}_2)] > 0$, the sign of $V'(0)$ is simply the sign of $E(\tilde{x})$. Therefore the investment decision for small projects (i.e., when α is sufficiently small) is not different from the standard case of no background risk.

In general, the sign of $V'(0)$ depends on the joint distribution of $\tilde{x}$, $\tilde{y}_1$, and $\tilde{y}_2$. Below we consider two specific cases: (1) §3.1 addresses the case of dependent additive background risk, where $\tilde{x}$ and $\tilde{y}_1$ are dependent but $\tilde{y}_2$ is independent of $\tilde{x}$ and $\tilde{y}_1$ and (2) §3.2 considers the case of dependent multiplicative background risk, where $\tilde{x}$ and $\tilde{y}_2$ are dependent but $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. We will use the following notion of $\tilde{x}$ being positively (negatively) related to $\tilde{y}$.

Definition 1. If $E(\tilde{x} \mid y)$ is increasing (decreasing) in y for all y, then $\tilde{x}$ is positively (negatively) related to $\tilde{y}$.

Remark 1. Having $\tilde{x}$ positively (negatively) related to $\tilde{y}$ is a stronger condition than positive (negative) correlation between $\tilde{x}$ and $\tilde{y}$, but a weaker condition than affiliation of $\tilde{x}$ and $\tilde{y}$ $(-\tilde{y})$ (see Milgrom and Weber (1982) for a definition and discussion of affiliation). If $\tilde{x}$ and $\tilde{y}$ have a bivariate normal distribution, then correlation, relation, and affiliation are equivalent. Note also that unlike correlation and affiliation, the condition of a positive or negative relation is asymmetric: $E(\tilde{x} \mid y)$ increasing (decreasing) in y does not imply that $E(\tilde{y} \mid x)$ is increasing (decreasing) in x. The direction of concern for our purposes is whether the project risk is positively (negatively) related to the background risk.

Lemma 1, stated without a proof, will be used in §§3.1 and 3.2.

Lemma 1. *Suppose that, conditional on $\tilde{y} = y$, $\tilde{x}$ and $\tilde{t}$ are independent. If $E(\tilde{x}) = 0$, $\tilde{t} > 0$, and $E(\tilde{t} \mid y)$ is increasing in y, then $E(\tilde{x}\tilde{t}) > (<)\ 0$ if $E(\tilde{x} \mid y)$ is increasing (decreasing) in y. Similarly, if $E(\tilde{t} \mid y)$ is decreasing in y, $E(\tilde{x}\tilde{t}) > (<)\ 0$ if $E(\tilde{x} \mid y)$ is decreasing (increasing) in y.*

3.1. Dependent Additive Background Risk

This section addresses the case where the multiplicative component of background risk, $\tilde{y}_2$, is independent of $\tilde{x}$ and $\tilde{y}_1$. Proposition 2 shows that if $\tilde{x}$ is negatively related to $\tilde{y}_1$, then $V'(0) > 0$. From Proposition 1, then, it is optimal to invest some positive amount in a project with zero expected return.

Proposition 2. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_2$ is independent of $\tilde{x}$ and $\tilde{y}_1$. Then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_1)$ is decreasing (increasing) in y_1.*

Proof. Because U is concave, $E[\tilde{y}_2 U'(w_0 + y_1 + w_1\tilde{y}_2) \mid y_1] = E[\tilde{y}_2 U'(w_0 + y_1 + w_1\tilde{y}_2)]$ is decreasing in y_1. Let $\tilde{t} = \tilde{y}_2 U'(w_0 + \tilde{y}_1 + w_1\tilde{y}_2)$. Then $V'(0) = E(\tilde{x}\tilde{t})$, and Proposition 2 follows from Lemma 1. □

The example of additive background risk in §2.1 illustrates Proposition 2. From Figure 1, the project is always undesirable when $\tilde{x}$ and $\tilde{y}$ are independent or positively correlated, which for the normal model is equivalent to $\tilde{x}$ being positively related to $\tilde{y}$. However, when the correlation is negative in Figure 1, the project has a positive CE as long as σ_x is not too large. In the context of the investment problem, $\tilde{x}$ being positively (negatively) related to $\tilde{y}_1$ implies that $V'(0) < (>)\ 0$, so investing in the project is undesirable when the relationship is positive and desirable when the relationship is negative. The result for a positive relationship is not surprising because of an increasing-variance effect. The result for a negative relationship is intuitively reasonable in terms of a diversification/hedging effect and is consistent with Pratt's (1988) comment and the results of Arrondel and Calvo Pardo (2003) noted in §1.

3.2. Dependent Multiplicative Background Risk

This section addresses the case where the additive component of background risk, $\tilde{y}_1$, is independent of $\tilde{x}$ and $\tilde{y}_2$. Let $r_A(w_0 + z) = -E[U''(w_0 + \tilde{y}_1 + z)]/E[U'(w_0 + \tilde{y}_1 + z)]$, where the expectation is over $\tilde{y}_1$. Proposition 3 states the conditions under which it is optimal to invest in a project with zero expected return if $\tilde{x}$ is positively (negatively) related to $\tilde{y}_2$.

Proposition 3. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. If $w_1 y_2 r_A(w_0 + w_1 y_2) > 1$ for all y_2, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is decreasing (increasing) in y_2. Similarly, if $w_1 y_2 r_A(w_0 + w_1 y_2) < 1$ for all y_2, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is increasing (decreasing) in y_2.*

Proof. If $\tilde{t} = \tilde{y}_2 U'(w_0 + \tilde{y}_1 + w_1\tilde{y}_2)$, $E(\tilde{t} \mid y_2) = E[y_2 U'(w_0 + \tilde{y}_1 + w_1 y_2) \mid y_2] = y_2 E[U'(w_0 + \tilde{y}_1 + w_1 y_2)]$. Then $dE(\tilde{t} \mid y_2)/dy_2 = E[U'(w_0 + \tilde{y}_1 + w_1 y_2)] + w_1 y_2 E[U''(w_0 + \tilde{y}_1 + w_1 y_2)] = E[U'(w_0 + \tilde{y}_1 + w_1 y_2)] \cdot [1 - w_1 y_2 r_A(w_0 + w_1 y_2)]$. Thus, $E(\tilde{t} \mid y_2)$ is increasing (decreasing) in y_2 if $w_1 y_2 r_A(w_0 + w_1 y_2) < (>)\ 1$. But $V'(0) = E(\tilde{x}\tilde{t})$, so Proposition 3 follows from Lemma 1. □

Corollary 1. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. If $w_1 \le 0$, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is increasing (decreasing) in y_2.*

Corollary 1 is particularly interesting because of the contrast with Proposition 2. Proposition 2 shows that it is always beneficial to invest some amount if $\tilde{x}$ is *negatively* related to the additive background risk $\tilde{y}_1$ and never beneficial to do so if $\tilde{x}$ is positively related to $\tilde{y}_1$. Corollary 1 shows, on the other hand, that if the initial wealth component subject to the multiplicative background risk $\tilde{y}_2$ is negative or zero, it is always beneficial to invest some amount if $\tilde{x}$ is *positively* related to $\tilde{y}_2$.

What is the intuition behind Proposition 3? $E(\tilde{x}) = 0$, so $E(\tilde{x}\tilde{y}_2) = Cov(\tilde{x}, \tilde{y}_2)$. Therefore, if $\tilde{x}$ is positively (negatively) related to $\tilde{y}_2$, their covariance is positive (negative), and investing in $\tilde{x}$ causes expected wealth to increase (decrease) even though the expected return from the project is zero. On the other hand, $\tilde{x}$ being positively (negatively) related to $\tilde{y}_2$ may increase (decrease) the variance of wealth, compared to the case where $\tilde{x}$ and $\tilde{y}_2$ are independent. Thus, there is a potential tradeoff between increasing (decreasing) the expected value of wealth and increasing (decreasing) the variance of wealth. If $w_1 \le 0$, then the effect of the increase (decrease) in expected wealth dominates any changes in the variance of wealth, and it is always beneficial to invest some amount if $\tilde{x}$ is positively related to $\tilde{y}_2$. If $w_1 > 0$, the decision maker's attitude toward risk is relevant. If the decision maker is not too risk averse, the first effect of increasing (decreasing) the expected wealth dominates. If the decision maker is strongly risk averse, the second effect of increasing (decreasing) variance is more important. As Proposition 3 shows (see also Corollaries 2–4), a smaller (larger) w_1 tends to favor the first (second) effect: the first (second) effect dominates if $w_1 y_2 r_A(w_0 + w_1 y_2) < (>)\ 1$. This is illustrated in the multiplicative background risk example from §2.2, where Figures 3–5 show that positive (negative) correlations make some investment in the new project desirable for small (large) $w_e = w_1$.

Remark 2. Note that $r_A(w_0 + z)$ is the absolute risk aversion of the indirect, or derived, utility function $E[U(w_0 + \tilde{y}_1 + z)]$ obtained by taking expectations over $\tilde{y}_1$, which is independent of $\tilde{x}$ and $\tilde{y}_2$ (e.g., Kihlstrom et al. 1981, Nachman 1982, Gollier 2001). In turn, $r_R(w_0 + z) = (w_0 + z) r_A(w_0 + z)$ is the relative risk aversion (RRA) of the indirect utility function. Menezes and Hanson (1970) and Zeckhauser and Keeler (1970) introduce the concept of partial relative risk aversion (PRRA). The term $z r_A(w_0 + z)$, which plays an important role in Proposition 3 with $z = w_1 y_2$, is the PRRA of the indirect utility function, partial because it can be expressed in the form $[z/(w_0 + z)] r_R(w_0 + z)$.

Overall, the effect of $\tilde{x}$ being positively or negatively related to $\tilde{y}_2$ on the desirability of the project depends on the shape of the indirect utility function, the magnitudes of w_0 and w_1, and the distribution of the multiplicative background risk. Corollaries 2–4 of Proposition 3 show some conditions under which the project is desirable.

Corollary 2. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. If $w_0 > 0$ and $r_R(w_0 + z) \le 1$ for all $z > 0$, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is increasing (decreasing) in y_2.*

Corollary 3. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. If $w_0 > 0$, $w_1 > 0$, $r_R(w_0 + z) \le r^*$ for all $z > 0$ and some $r^* > 1$, and $w_1\tilde{y}_2 \le w_0/(r^* - 1)$, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is increasing (decreasing) in y_2.*

Corollary 4. *Suppose that $E(\tilde{x}) = 0$ and that $\tilde{y}_1$ is independent of $\tilde{x}$ and $\tilde{y}_2$. If $w_0 > 0$, $w_1 > 0$, $r_R(w_0 + z) \ge r^*$ for all $z > w_0/(r^* - 1)$ and some $r^* > 1$, and $w_1\tilde{y}_2 \ge w_0/(r^* - 1)$, then $V'(0) > (<)\ 0$ if $E(\tilde{x} \mid y_2)$ is decreasing (increasing) in y_2.*

Corollary 2 shows that if the RRA of the indirect utility function is no greater than 1, then $\tilde{x}$ being positively related to $\tilde{y}_2$ makes the project desirable. Corollary 3 extends this result to RRA that can be somewhat greater than one when $w_1\tilde{y}_2$ is small enough (restricting the variance effect). In contrast, Corollary 4 shows that if RRA is sufficiently above 1 and $w_1\tilde{y}_2$ is large enough (so that the variance effect dominates), then $\tilde{x}$ being negatively related to $\tilde{y}_2$ makes the project desirable. For example, if the indirect utility function exhibits constant RRA $r_R(w_0 + z) = \gamma$, which includes power and logarithmic utility, Corollary 2 applies when $\gamma \le 1$. For $\gamma > 1$, assuming that the appropriate bounds on $w_1\tilde{y}_2$ are satisfied, Corollary 3 applies if $\gamma \le r^*$ and Corollary 4 if $\gamma \ge r^*$. Note that if $\tilde{y}_1 = 0$ (i.e., there is no additive background risk), Corollaries 2–4 provide conditions in terms of the RRA of the original utility function $U(w)$.

Multiplicative background risk has not received much attention in the literature. An exception is Franke et al. (2005), which has a somewhat different focus than our work, emphasizing conditions on preferences (in particular, a condition called multiplicative risk vulnerability) that lead to more cautious behavior in the face of independent multiplicative background risk. Whether RRA is less than or greater than 1 plays a key role in many of their results, as it does in our Corollary 2, and they note more generally that many results in the literature on choice under uncertainty specify the condition that RRA is less than or greater than 1. A number of such results are sprinkled throughout Gollier (2001), for example. Thus, an important empirical question is the size of RRA for firms or individuals. Howard's (1988) informal rule of thumb for risk tolerance (noted in §2), based on extensive experience with decision analysis applications, implies an RRA of 6, which is considerably greater than 1. Bliss and Panigirtzoglou (2004) use option prices to infer the RRA of a representative agent, deriving estimates that are both less than and greater than 1, but tend toward the latter. Bliss and Panigirtzoglou (2004) also summarize a number of previous studies (their Table VII, p. 432) and conclude that their results are consistent with those studies.

The crucial condition in Proposition 3 is whether *partial* relative risk aversion is less than or greater than 1. This makes sense because only part of the total wealth, the fraction $w_1y_2/(w_0 + w_1y_2)$, is subject to the multiplicative background risk. Eeckhoudt and Gollier (1995) give some results on choice under uncertainty that turn on whether PRRA $<$ ($>$) 1. The empirical estimates noted above suggest that RRA tends to be greater than 1 but moderately so. When $w_0 > 0$, as is typically the case, PRRA may well be less than 1 even when RRA > 1 if w_1y_2 is not too large relative to w_0 (Corollary 3). Similarly, PRRA > 1 if RRA is sufficiently greater than 1 and w_1y_2 is not too small (Corollary 4).

Proposition 3 and Corollaries 1–4 indicate situations for which investing in the project is desirable. In many cases, we cannot make such a determination because $w_1y_2r_A(w_0 + w_1y_2) < 1$ for some y_2 and $w_1y_2r_A(w_0 + w_1y_2) > 1$ for other y_2. However, we expect that if $w_1y_2r_A(w_0 + w_1y_2) < (>)\ 1$ with high enough probability, as in Figure 4 (Figure 3), then Proposition 3 will correctly suggest whether investing in the project is desirable. Similarly, if $w_1y_2r_A(w_0 + w_1y_2)$ is close to 1 with high enough probability, as in Figure 5 for $w_e = 200$, we would expect that dependence between $\tilde{x}$ and $\tilde{y}_2$ is not so important and that investing in the project with zero return is not desirable.

Obtaining general results for the question of how much to invest without restrictions to specific dependence structures and utility functions is difficult in the dependent case. With additive background risk, if the risks are affiliated and utility is DARA, the optimal investment decreases as risk aversion increases (Gollier 2001, p. 120). The numerical results in §§2.1–2.2 provide some indication of the magnitude of the effect of dependence on the optimal amount to invest, and we feel that the general picture will be similar in direction for most (but not all) realistic situations.

4. Discussion

Decisions are often analyzed somewhat in isolation, although it is rare that there are no background risks regarding things such as previous commitments or exogenous variables that can impact the decision maker's final wealth. These background risks can affect the return from a project directly (e.g., uncertain exchange rates or tax rates operating on profits from the project) or indirectly (e.g., uncertain returns from other projects influencing total wealth). They can relate to the project risk in different ways; we consider additive and multiplicative background risk. Additive risks include previous commitments, but also such things as human capital, labor income, proprietary income, investment income, potential liabilities, expropriation risk in the case of multinational firms, and so on. Examples of multiplicative risks include exchange rates, tax rates, inflation, and competitive price pressure.

In some cases, such risks can safely be ignored in deciding about a project. For example, independent additive background risks can be ignored by a decision maker with an exponential utility function for wealth. Also, it may be possible to hedge against some risks through risk management actions such as trading in futures markets or purchasing insurance, in which case these risks are technically not background risks. However, many risks cannot conveniently be hedged because of incomplete markets. Moreover, even when hedging is possible, it is not without some cost, and some decision makers may be reluctant to use hedging strategies, in which case the risks are effectively background risks. For instance, He and Ng (1998) cite data showing that only 41% of Japanese multinational corporations use derivatives to hedge currency risks and note that this is comparable to results for U.S. nonfinancial firms. Somewhat surprisingly, it is also comparable to the use of derivatives by financial firms. In a large study of U.S. institutional investors, Levich et al. (1999, p. 1) report that "46% of institutions permit their asset managers to use derivatives" and claim that where derivatives are used, the positions tend to be small relative to total assets.

Previous work on background risk has primarily involved independent background risk (typically additive), emphasizing conditions on preferences under which the presence of the background risk preserves comparative risk aversion and/or increases the decision maker's risk aversion concerning the project risk. We work with a more general model with correlated additive and multiplicative background risks, focusing on how such risks can affect decisions and giving particular attention to the impact of correlation and the multiplicative case. A price we pay for the greater complexity is that it is more difficult to get strong analytical results, although Proposition 3 and its corollaries do give useful guidelines about when it is optimal to invest in a project when correlated background risk is present. The question of how much to invest has not yielded such general results, but the numerical examples in §2 underscore the importance of accounting for correlation between the project risk and the background risk.

As mentioned in §1, background risks can be correlated substantially, both positively and negatively, to project risk. Our numerical examples and analytical results show that optimal decisions with correlated background risk can be very different from the decisions that would be recommended if the correlation were ignored and very sensitive to the sign and magnitude of the correlation. Contrary to the cases of no background risk or background risk independent of project risk, it is sometimes optimal for a risk avoider to undertake a project with zero or even negative expected return. For additive background risk, negative correlation is beneficial in the sense of increasing the certainty equivalent for the project. In the case of multiplicative background risk, the decision maker is likely to face a tradeoff: positive (negative) correlation increases (decreases) expected wealth but may simultaneously increase (decrease) variance of wealth. Proposition 3 and Corollaries 1–4 indicate that if the decision maker is not too risk averse, then the expected wealth effect dominates. If the decision maker is strongly risk averse, then the variance effect is more important. In general, the certainty equivalent for a project can depend significantly on correlations between project and background risks.

Decision analysts know that it can be dangerous to ignore important uncertainties by simply assuming that a variable will equal its mean or some other summary measure. Similarly, our results indicate that it can be dangerous to ignore important correlations by assuming that project risk and background risks are independent. Thus, obtaining precise estimates of correlations is very important. When past data about relationships are available, as is the case, for example, in many financial decision-making problems, those data can be used to estimate correlations. In the absence of such data, it is necessary to assess dependence subjectively. This is not an easy judgmental task, and some experimental work has been done on assessing dependence (Gokhale and Press 1982, Kunda and Nisbett 1986, Clemen and Reilly 1999, Clemen et al. 2000). The accuracy of dependence assessments varies, improving somewhat with experience and training as well as with averaging of multiple assessments. For the subjects in Clemen et al. (2000), who had some exposure to correlation in an MBA statistics course, assessing correlations directly yielded greater accuracy than correlations calculated from assessments such as probability of concordance. Our results regarding correlated background risk highlight the importance of assessing correlations, suggesting that further research, particularly on how training and experience may lead to improved accuracy, would be useful. The difficulty in such research is finding important real situations where individuals with expertise are willing to participate and where data-based correlations are available to measure the accuracy of the assessed correlations.

An important implication of our work is related to inferred risk aversion. A decision maker may be interested in the risk attitudes of a competitor (or a partner) to be able to compete (cooperate) more effectively with her. If the decision maker fails to account appropriately for correlations in observing the competitor's (partner's) decisions, he may get a misleading notion of her attitude toward risk. For example, correlation between project and background risks can lead a risk-averse individual to accept a project with zero or negative expected return and thereby appear to be risk seeking to someone who does not take into account that correlation (possibly because he is unaware of the background risk). Similarly, observing that a decision maker does not comply with some normative rules does not necessarily imply that the decision maker is irrational in a normative sense: she may be accounting for correlations that the observer has missed.

To the extent that unrecognized correlated background risk could lead to misleading inferences about risk aversion, there may be important implications for "puzzles" in the literature on investments and asset pricing, such as the equity premium puzzle (Mehra and Prescott 1985). The observed equity premium for U.S. stocks is much higher than theory suggests, implying a relative risk aversion much higher than considered reasonable. Gollier and Eeckhoudt (2001) point out that if independent risks (e.g., other investment opportunities) are substitutes, ignoring such risks can lead to an overestimation of the demand for U.S. stocks and, therefore, to an underestimation of the equity premium. Ross (1999), on the other hand, considers Samuelson's (1963) "fallacy of large numbers" and argues that independent risks could be complements. Our results suggest that some correlated background risks could be very strong substitutes (e.g., positively correlated additive background risk), some could be very strong complements (e.g., negatively correlated additive background risk), and some could be neither. If they are very strong substitutes, a large equity premium may be reasonable.

Another related area is the delegation of risk-related decisions. For example, a firm with different subdivisions may like to give the subdivisions as much decision-making freedom as possible to enable them to react quickly to market conditions and not to miss profitable projects. If the projects of the subdivisions are independent, then they influence each other by shifting total (expected) wealth and through relatively minor effects of independent background risk. In particular, if the firm is willing to behave consistently with an exponential utility function, then each subdivision can act as a separate decision maker with the same exponential utility function. However, this is no longer the case when the project returns of different subdivisions are correlated. Developing coordinating mechanisms within a firm is an interesting challenge, and we would expect that one of their key features would be accounting for potential correlations, as is the case for the isolated decision-making problem studied here.

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