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An important managerial question is the choice of the pricing rule. We study whether this choice depends on the uncertainty about the number of participating bidders by comparing expected revenues under discriminatory and uniform pricing within an auction model with affiliated values, stochastic number of bidders, and linear bidding strategies. We show that if uncertainty about the number of bidders is substantial, then the discriminatory pricing generates higher expected revenues than the uniform pricing. In particular, the first-price auction might generate higher revenues than the second-price auction. Therefore, uncertainty about the number of bidders is an important factor to consider when choosing the pricing rule. We also study whether eliminating this uncertainty, i.e., revealing the number of bidders, is in the seller's interests, and discuss the existence of an increasing symmetric equilibrium.

Key words: discriminatory pricing; uniform pricing; auctions; demand uncertainty; stochastic number of bidders

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1. Introduction

The choice of the pricing rule is an important and complex managerial decision. The focus of this paper is on the effect of demand uncertainty, within an auction model, on expected revenues under uniform pricing (everyone pays the same price) and discriminatory pricing (price varies by customer). Specifically, we study how uncertainty about the number of bidders impacts the choice of the pricing rule.

Auctions are not only widely used to make allocation and pricing decisions in competitive environments where all involved parties act strategically in their own best interests, but auction theory also provides a framework for analytical study of pricing and allocation. Although previous research has identified a number of factors that make a particular auction format (and thus the pricing rule) more attractive to the seller, most of this work has assumed that the number of bidders participating in an auction is known at the moment of bid submission. This is not realistic, because uncertainty about the number of bidders is common: both the bid-taker at the time of making the decision on the pricing rule, and bidders at the time of submitting the bids, might not know how many bidders are participating.

The uncertainty about the number of bidders and its impact on the auction outcome was noted some

time ago in the context of auctions for oil leases (e.g., Capen et al. 1971, Engelbrecht-Wiggans et al. 1986). This uncertainty is natural in sealed-bid auctions; for example, it is an important consideration in design of spectrum auctions (Klemperer 2004, §5.6.2).[1] In auctions that are conducted electronically, bidders are also unlikely to know the number of their competitors. Arora et al. (2007) highlight the prevalence of uncertainty about the number of competitors in electronic marketplaces. Electronic auctions are widespread in business transactions: from direct sales to customers (such as millions of auctions conducted daily at eBay.com) to becoming a standard transaction format in many supply chains and distribution channels (e.g., Elmaghraby 2007 surveys current industry practice).[2] Given the prominence of auctions in business and, consequently, in management science research (e.g, see Geoffrion and Krishnan 2003, for the overview of a special issue of *Management Science*

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[1] Levin and Ozdenoren (2004, p. 230) point that "In the last spectrum auction in Great Britain that raised about $35 billion the issue of the number of competing companies was so central that the auction form was designed mainly with that in mind."

[2] Most frequently auctions are considered in the setting with a single seller and multiple buyers. In the case of procurement auctions there is a single buyer and multiple sellers. However, from a game-theoretic point of view, these two settings are equivalent.

on e-business, and Anandalingam et al. 2005, for the overview of a special issue of *Management Science* on electronic market design), it is important to understand the effect of uncertainty about the number of bidders on the decisions of auction designers and auction participants.

In this paper, we show that demand uncertainty, modeled here as the uncertainty about the number of bidders, is an important factor that cannot be overlooked. Even for the simplest auction formats, standard revenue rankings of auction pricing rules do not necessarily hold when there is uncertainty about the number of bidders.

We analyze single-round sealed-bid auctions in which every bidder demands one object. If k identical objects are allocated through the auction procedure, the top k bidders get one object each. We consider two pricing rules: discriminatory pricing (discriminatory auction), in which every winning bidder pays his bid, and uniform pricing (uniform auction), in which all winning bidders pay the same price.[3] Discriminatory and (variants of) uniform pricing are the most prevalent pricing rules in practice. For $k = 1$, i.e., one object, these two pricing rules correspond to the first-price and the second-price auctions.

A theoretical auction model specifies the information that bidders have and the structure of bidders' valuations: each bidder observes a signal about the object value and his valuation for an object depends, in general, on his signal and on signals of the other bidders. The auction literature distinguishes between two polar cases of bidders' valuations—common value and private value settings: in the former the value of the object is the same for all bidders, and in the latter it depends only upon a bidder's signal. Thus, in the common-value case, all bidders observe different estimates of the (unknown) object value; in the private-value case bidders know their own values, but not the values of others. Bidders' signals can be independent or affiliated, where affiliation implies that higher signals of any of the bidders make higher signals of other bidders more likely.

There is a large auction theory literature that deals with revenue comparisons of different auction formats in different theoretical settings (e.g., Rothkopf and Harstad 1994 provide a review). However, all of that work assumes no uncertainty about the number of bidders. A classical model of uniform and discriminatory auctions is that of Milgrom (1981). In this model, bidders are risk neutral, and bidders' signals are affiliated. Milgrom and Weber (1982) and Jackson and Kremer (2006) prove a "standard revenue ranking" result: the uniform auction yields greater expected revenue than the discriminatory auction in the setting with affiliated values and risk-neutral bidders. In the special case of valuations being independent and drawn from the same distribution (e.g., independent private values), the revenue equivalence theorem holds and the expected revenues from discriminatory and uniform auction are equal (Maskin and Riley 1989). However, discriminatory auctions could yield higher revenues than uniform auctions if the risk-neutrality assumption or the unit-demand assumption is violated. If buyers are risk averse and the seller is risk neutral, discriminatory pricing is better for the seller than uniform pricing (Holt 1980, Maskin and Riley 1984). Back and Zender (1993) show that if bidders demand more than one object, there exist collusive strategies in the uniform price auction, and argue that discriminatory auctions might therefore be more profitable for the seller.

Uncertainty about the number of bidders can be modeled as endogenous or exogenous. In the endogenous case, each bidder decides whether to participate or not. That decision would depend on, e.g., entry fee, cost of preparing the bid and of acquiring necessary information, and reservation price (Engelbrecht-Wiggans 1987, Harstad 1990, Levin and Smith 1994, Samuelson 1985, Hendricks et al. 2003, Li and Zheng 2006, and McAfee et al. 2002 also consider empirical implications). In the exogenous case, the number of bidders is drawn from a distribution specifying the probability that a particular number of bidders participate in the auction. To isolate the effect of the uncertainty about the number of bidders on revenue rankings of standard auction formats, our model eliminates other parameters such as entry fees, reserve prices, bid preparation costs, etc. Thus, we model uncertainty about the number of bidders as exogenous.

A model with an unknown (exogenous) number of bidders was introduced by Matthews (1987) and McAfee and McMillan (1987). Both of these papers focus on the case of independent private values and risk-averse bidders. Matthews (1987) also considers the case of affiliated private values, but does not compare auction revenues in first- and second-price auctions when the number of bidders is unknown. Levin and Ozdenoren (2004) consider independent private values and ambiguity-averse bidders. Harstad et al. (1990) consider the model with independent private information and risk-neutral bidders, and characterize equilibrium bidding strategies. In that case, the revenue equivalence theorem holds and the revenue in the first- and second-price auctions is the same. Furthermore, the auction revenue does not depend on whether the number of bidders is revealed or concealed.

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[3] As standard in auction theory, we assume that the uniform price is set at the value of the highest losing bid.

To study the impact of the uncertainty about the number of bidders on the revenue ranking of discriminatory and uniform pricing, we combine the classical model of Milgrom (1981) with Matthews' (1987) model for an uncertain number of bidders. Our model is presented in §2. It poses several challenges with respect to determining equilibrium behavior. Thus, we conclude §2 by introducing an assumption that will allow for analytical tractability when comparing pricing rules in §§3 and 4, and we relegate discussion about equilibrium bidding strategies to Appendix A.

If uncertainty about the number of bidders is not large in the sense that all possible numbers of bidders are sufficiently close to each other, the revenue ranking should correspond to the case where the number of bidders is known and the choice of the pricing rule will not depend on such small uncertainty. In that sense, small uncertainty can be ignored by the seller. However, if uncertainty about the number of bidders is large enough, the revenue ranking of discriminatory and uniform pricing gets reversed. Thus, if uncertainty about the number of bidders is substantial, it is an important factor to consider when choosing the pricing rule. Section 3 presents a formal theorem, an illustration, and a discussion.

In some situations, the bid-taker might be able to resolve uncertainty about the number of bidders by publicly releasing the exact number of bidders before bids are submitted (e.g., bidders might be required to register or place a deposit before the auction starts). Section 4 studies the effect of resolving uncertainty about the number of bidders on expected revenues. For example, Matthews (1987) shows that, in the case of affiliated private values, the seller benefits from concealing the number of bidders in the discriminatory auction. We confirm this result, and also show that, with common values, the seller might prefer to reveal the number of bidders. More generally, our main result in §4 is that by resolving uncertainty about the number of bidders prior to bid submission, the bid-taker benefits more in the uniform auction than in the discriminatory auction. Thus, all other things being equal, the bid-taker has a stronger incentive to reduce this uncertainty under uniform pricing than under discriminatory pricing.

Section 5 concludes. Appendix A discusses the issue of equilibrium existence. All proofs are in Appendix B.

2. Model with Linear Bidding Functions

We first generalize Milgrom's (1981) and Pesendorfer and Swinkels' (1997) common-value auction model with unit demands, by allowing the exact number of bidders to be unknown at the moment of bid submission and by allowing for a mixture of common and private valuations. Then we introduce Assumption 1, which ensures linearity of bidding functions, and state the linear bidding strategies that are used in the analysis in §§3 and 4.

We consider uniform and discriminatory sealed-bid auctions: k homogeneous objects are sold to the k highest bidders. Ties, if any, are broken randomly. In a discriminatory auction, the winning bidders pay their bids, whereas in a uniform auction, all winning bidders pay a uniform price equal to the $(k+1)$st-highest bid. When $k = 1$, discriminatory and uniform auctions reduce to first-price and second-price auctions, respectively.

The uncertainty about the number of bidders is modeled following Matthews (1987). Bidders are drawn from a pool of N potential bidders in accordance with an exogenous stochastic process, $\Omega = \{(n_1, \pi_1), \ldots, (n_M, \pi_M)\}$, specifying that n_i bidders are present with probability $\pi_i > 0$. As is standard with information about an auction setting, Ω is assumed to be common knowledge. The case with no demand uncertainty (i.e., n bidders for sure) corresponds to $\Omega = \{(n, 1)\}$. We also assume that the number of objects is always less than the number of bidders: $1 \leq k < \min(n_1, \ldots, n_M)$.

All potential bidders are risk neutral. There is one state variable V with commonly known probability density function (p.d.f.) $g(v)$ with support $[\underline{v}, \overline{v}]$. Bidder j independently observes an estimate X_j, a random scalar distributed with p.d.f. $f(x \mid v)$ and cumulative distribution function (c.d.f.) $F(x \mid v)$, conditional on $V = v$. The support of the marginal distribution of X_j is $[\underline{x}, \overline{x}]$, $-\infty < \underline{x} < \overline{x} \leq +\infty$.

As in Milgrom (1981), $f(x \mid v)$ satisfies the monotone likelihood-ratio property (MLRP)

$$\frac{f(x \mid v)}{f(x \mid v')} \geq \frac{f(x' \mid v)}{f(x' \mid v')} \quad \forall x > x', v > v', \tag{1}$$

so that the X_js and V are affiliated. For example, if X_j is normally (or uniformly) distributed with mean v, then (1) is satisfied. Furthermore, a truncated normal distribution would also satisfy MLRP.

The value of one object for bidder j is given by $V_j = u(V, X_j)$, where the function $u(\cdot)$ is continuous and increasing in both variables. That description incorporates private values and common values as special cases: If $u(V, X) = V$, then this is the common-value model (used by Milgrom 1981 and Pesendorfer and Swinkels 1997, dating back to Rothkopf 1969 and Wilson 1969), where the value of the object is the same for all bidders and unknown at the moment of bidding. If $u(V, X) = X$, then this is the affiliated private-values model, where each bidder knows his valuation for the object. Finally, if the distribution of V is degenerate, i.e., it assigns probability one to a single point, then this is the independent private-values model.

Now we derive the symmetric equilibrium bidding functions in the uniform and discriminatory auctions. It will be useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$, and to consider the order statistics associated with the signals of all other bidders. We denote the mth-highest signal of bidders $2, 3, \ldots, n_i$ (i.e., all active bidders except bidder 1) by $Y_{n_i-1}^m$, the conditional p.d.f. of Y_{n-1}^k given x by

$$f_{n-1,k}(y \mid x) = \frac{(n-1)!}{(k-1)!(n-k-1)!}$$
$$\cdot \frac{\int_{\underline{v}}^{\bar{v}} f(y \mid v) F^{n-k-1}(y \mid v)(1 - F(y \mid v))^{k-1} f(x \mid v) g(v) \, dv}{\int_{\underline{v}}^{\bar{v}} f(x \mid v) g(v) \, dv},$$
(2)

and the corresponding c.d.f. by $F_{n-1,k}(y \mid x)$.

We start by stating the symmetric equilibrium bidding function in a uniform auction. Define

$$v_{nk}(x, y) = E[V_1 \mid X_1 = x, Y_{n-1}^k = y]$$
$$= \frac{\int_{\underline{v}}^{\bar{v}} u(v, x) f(y \mid v) F^{n-k-1}(y \mid v)\left(1 - F(y \mid v)\right)^{k-1} f(x \mid v) g(v) \, dv}{\int_{\underline{v}}^{\bar{v}} f(y \mid t) F^{n-k-1}(y \mid t)\left(1 - F(y \mid t)\right)^{k-1} f(x \mid t) g(t) \, dt}$$
(3)

and

$$v(x, y) = \frac{\sum_{i=1}^{M} \pi_i n_i v_{n_i k}(x, y) f_{n_i-1, k}(y \mid x)}{\sum_{i=1}^{M} \pi_i n_i f_{n_i-1, k}(y \mid x)},$$
(4)

where $f_{n-1,k}(y \mid x)$ is defined by (2). If the number of bidders is known, then the symmetric equilibrium bidding function in a uniform auction is given by $v_{nk}(x, x)$ (Milgrom 1981).

THEOREM 2.1. *If there exists a symmetric equilibrium $b^u \colon \Re \to \Re$ in increasing strategies for a uniform auction, then it is*

$$b^u(x) = v(x, x),$$
(5)

where $v(x, x)$ is defined by (4). If b^u is increasing, then it is the unique symmetric equilibrium in increasing strategies.

Before describing the symmetric equilibrium bidding function for the discriminatory auction, denote

$$A(y, x) = \frac{\sum_{i=1}^{M} \pi_i n_i f_{n_i-1, k}(y \mid x)}{\sum_{i=1}^{M} \pi_i n_i F_{n_i-1, k}(y \mid x)}.$$
(6)

THEOREM 2.2. *If there exists a symmetric equilibrium $b^d \colon \Re \to \Re$ in increasing differentiable strategies for a discriminatory auction, then it is*

$$b^d(x) = \int_{\underline{x}}^{x} v(t, t) A(t, t) e^{\int_x^t A(s, s) \, ds} \, dt,$$
(7)

where $v(t, t)$ is defined by (4) and $A(t, t)$ is defined by (6).

To make revenue comparisons of discriminatory and uniform pricing analytically tractable, our analysis in §§3 and 4 is limited to frameworks that satisfy Assumption 1.

ASSUMPTION 1.
(i) *There exists δ, $0 \le \delta \le 1$, such that $u(v, x) = \delta v + (1 - \delta)x$.*
(ii) *$f(x \mid v)$ is location invariant, i.e., $f(x \mid v) = h(x - v)$, where $h(\cdot)$ is a p.d.f. (and the corresponding c.d.f. is denoted by H).*
(ii') *The support of $h(t)$ is $[t_1, t_2]$, and $\min_{t \in [t_1, t_2]} h(t) = \varepsilon > 0$.*
(iii) *The p.d.f. $g(v)$ is uniformly distributed on $[-T, T]$ with $T \to \infty$.*

Assumption 1(i) allows for a mixture of private-value and common-value components: $\delta = 0$ corresponds to the affiliated private-values model, and $\delta = 1$ corresponds to the common-value model. Assumption 1(ii) is equivalent to assuming that $X = V + Z$, where Z is some noise term, independent of V. Many important signal distributions satisfy (ii); (ii') is a technical assumption to simplify the proof of Theorem 3.1. Assumption 1(iii) corresponds to a diffuse prior assumption in Bayesian statistics: public information about V, specified by $g(v)$, is much less precise than private information, specified by $f(x \mid v)$. Its use in auction theory traces back to Wilson (1969, §4), and was further discussed in Rothkopf (1980a, b). The setting with a diffuse prior is also extensively used in the experimental literature (Kagel et al. 1987, Kagel and Levin 2002, Parlour et al. 2007).[4]

Note that the posterior (i.e., conditional on signal x) density of V is given by $g(v \mid x) = f(x \mid v) g(v) / \int_{\underline{v}}^{\bar{v}} f(x \mid v) g(v) \, dv$. Thus, the diffuse prior assumption (iii) implies that $g(v \mid x) = f(x \mid v) / \int_{\underline{v}}^{\bar{v}} f(x \mid v) \, dv$. Using the location invariance assumption (ii), we get $g(v \mid x) = h(x - v) / \int_{\underline{v}}^{\bar{v}} h(x - v) \, dv = h(x - v)$. Therefore, (ii) and (iii) together imply that the p.d.f. of the posterior distribution, conditional on signal x, is also location invariant.

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[4] Parlour and Rajan (2005) consider the situation where random variable V has finite support, rather than being diffuse over the real line. This situation has the advantage of making the model more realistic (e.g., V is bounded above and below), but comes at the cost of having to resort to numerical methods to calculate bidding functions near the boundary of the signal support. In the interior of the signal support, on the other hand, the bidding functions correspond to the analytically solvable case studied here.

Using the notation

$$\alpha_{nk} = \frac{(n-1)!}{(k-1)!(n-k-1)!}$$
$$\cdot \int_{-\infty}^{+\infty} t H^{n-k-1}(t)(1-H(t))^{k-1}h^2(t)\,dt,$$
$$\beta_{nk} = \frac{(n-1)!}{(k-1)!(n-k-1)!}$$
$$\cdot \int_{-\infty}^{+\infty} H^{n-k-1}(t)(1-H(t))^{k-1}h^2(t)\,dt, \tag{8}$$

we can describe the symmetric equilibrium bidding functions. In the standard case where the number of bidders n is fixed, the bidding function in the uniform auction is

$$b_{nk}^u(x) = x - \delta\frac{\alpha_{nk}}{\beta_{nk}}. \tag{9}$$

The term $-\delta(\alpha_{nk}/\beta_{nk})$ can be understood as a winner's curse correction. As Proposition 2.3 shows, if the number of bidders is unknown, the bidding function is a weighted average of the bidding functions with known numbers of bidders, where the weights correspond to the probabilities of n_i bidders conditional on the assumption that a given bidder is tied with the kth-highest rival.

PROPOSITION 2.3. *Under Assumption* 1, *the unique increasing symmetric equilibrium in a uniform auction is*

$$b^u(x) = \sum_{i=1}^{M} \frac{\pi_i n_i \beta_{n_ik}}{\left(\sum_{j=1}^{M} \pi_j n_j \beta_{n_jk}\right)} b_{n_ik}^u(x)$$
$$= x - \delta\frac{\sum_{i=1}^{M} \pi_i n_i \alpha_{n_ik}}{\sum_{i=1}^{M} \pi_i n_i \beta_{n_ik}}. \tag{10}$$

In the discriminatory auction with known number of bidders, the bidding function is

$$b_{nk}^d(x) = x - \delta\frac{\alpha_{nk}}{\beta_{nk}} - \frac{k}{n\beta_{nk}}. \tag{11}$$

The term $-k/n\beta_{nk}$ reflects underbidding in the discriminatory auction, relative to bidding function in the uniform auction. Equation (11) agrees with the equilibria described by Winkler and Brooks (1980, p. 610; they have $k=1$, $n=2$, and a normal signal distribution $h(\cdot)$); and by Klemperer (1999, p. 260; he has $k=1$ and uniform $h(\cdot)$); see also Klemperer (2004, p. 57). The next proposition describes the bidding function for a discriminatory auction.

PROPOSITION 2.4. *If there exists a symmetric equilibrium $b^d\colon \Re \to \Re$ in increasing differentiable strategies for a discriminatory auction under Assumption* 1, *then it is*

$$b^d(x) = \sum_{i=1}^{M} \frac{\pi_i n_i \beta_{n_ik}}{\left(\sum_{j=1}^{M} \pi_j n_j \beta_{n_jk}\right)} b_{n_ik}^d(x)$$
$$= x - \delta\frac{\sum_{i=1}^{M} \pi_i n_i \alpha_{n_ik}}{\sum_{i=1}^{M} \pi_i n_i \beta_{n_ik}} - \frac{k}{\sum_{i=1}^{M} \pi_i n_i \beta_{n_ik}}. \tag{12}$$

Propositions 2.3 and 2.4 provide bidding functions that will be used in §§3 and 4, and are special cases of Theorems 2.1 and 2.2.

3. Revenue Comparisons

As mentioned in the introduction, the auction theory literature has identified many important factors influencing the choice of the pricing rule. In this paper, we identify another one: uncertainty about the number of bidders. Intuitively, the uncertainty that really matters here is that both relatively large and relatively small numbers of bidders are possible. We show that sufficiently changing the number of bidders in just one of the states of the stochastic process Ω (i.e., increasing demand without changing the probability of that state, no matter how small that probability is), results in reversal of the standard revenue ranking. Our main result is the following. Consider any distribution over the number of bidders, specified by $\Omega = \{(n_1, \pi_1), \ldots, (n_M, \pi_M)\}$. Keep all parameters (i.e., $\pi_1, \ldots, \pi_M$, $n_1, \ldots, n_{M-1}$) fixed except for n_M. Then, for large enough n_M, the discriminatory auction yields greater revenues than the uniform auction. As mentioned in the introduction, the standard revenue ranking result (when there is no uncertainty about the number of bidders) implies that the uniform auction yields greater expected revenues. Thus, for any probability distribution over the number of bidders, the standard revenue ranking will be reversed provided the number of bidders is large enough in one of the states (i.e., if both relatively large and relatively small numbers of bidders are possible).

THEOREM 3.1. *For any auction setting satisfying Assumption* 1, *any number of objects k, and any $\pi_1, \ldots, \pi_M$, $n_1, \ldots, n_{M-1}$, there exists n^* such that, for any $n_M \geq n^*$, the discriminatory auction for k objects yields greater expected revenue than the uniform $(k+1)$st price auction.*

3.1. Illustration with Uniform Distribution of Signals

To understand the intuition underlying Theorem 3.1, consider an example with a uniform distribution of signals: conditional on $V = v$, each bidder's signal is drawn independently from a uniform distribution on $[v - \frac{1}{2}, v + \frac{1}{2}]$. This important example corresponds to Assumption 1 with the signal distribution $h(\cdot)$ being uniform on $[-\frac{1}{2}, \frac{1}{2}]$, and has been extensively used in the experimental literature to study first-price, second-price, and English auctions in the two polar cases of pure private ($\delta = 0$) and pure common values ($\delta = 1$); see Kagel et al. (1987) and Kagel and Levin (2002). It is the best known example in which a tractable solution can be obtained. Klemperer (2004, pp. 55–57) presents the equilibria and revenue

comparisons of first-price, second-price, and English auctions for the pure common-value case in which $\delta = 1$. With uniform signals, α_{nk} and β_{nk}, defined in (8), are $\alpha_{nk} = \frac{1}{2} - k/n$ and $\beta_{nk} = 1$.

Another advantage of this example is that, for the pure common-value case, the bidding function in the discriminatory auction does not depend on the number of bidders: it is $b^d(x) = x - \frac{1}{2}$. Thus, even though Proposition 2.4 states only the necessary condition for an increasing symmetric equilibrium in a discriminatory auction with an unknown number of bidders, the proof that it is an equilibrium is the same as for the case with a known number of bidders. Therefore, Assumption 1, with uniform signals and $\delta = 1$, provides an example of an auction setting where the increasing symmetric equilibrium, stated in Proposition 2.4, exists.

Consider a common-value auction for one object (i.e., $k = 1$ and $\delta = 1$), in which either n_1 or n_2 bidders participate with equal probability, i.e., $\Omega = \{(n_1, \frac{1}{2}), (n_2, \frac{1}{2})\}$. As Klemperer (2004, pp. 56–57) shows, when the number of bidders n is known, the bidding functions in the first-price and second-price auctions are $b_{n1}^d(x) = x - \frac{1}{2}$ and $b_{n1}^u(x) = x - \frac{1}{2} + 1/n$. When the number of bidders is unknown, the bidding function in the first-price auction is the same, $b^d(x) = x - \frac{1}{2}$. (This follows from Proposition 2.4, because the bidding function with an unknown number of bidders is a linear combination of the bidding functions with known numbers of bidders.) In the case of the second-price auction, the bidding function (10) is $b^u(x) = (\pi_1 n_1/\bar{n}) b_{n_1 1}^u(x) + (\pi_2 n_2/\bar{n}) b_{n_2 1}^u(x) = x - \frac{1}{2} + 1/\bar{n}$, where $\bar{n} = \pi_1 n_1 + \pi_2 n_2$, the expected number of bidders.

Consider the case where $n_1 = 2$ and n_2 is very large. Then $\bar{n}$ is very large as well, and therefore $b^u(x) \approx x - \frac{1}{2}$, i.e., bids in the first-price and the second-price auctions almost coincide. Now, if the realized number of bidders is n_2, the signal of the price setter, conditional on $V = v$, will be very close to $v + \frac{1}{2}$, and thus the auction price will be very close to V in both first-price and second-price auctions. However, if the realized number of bidders is $n_1 = 2$, the expected value of the price setter's signal, conditional on $V = v$, is $v + \frac{1}{6}$ in the case of the first-price auction and $v - \frac{1}{6}$ in the case of the second-price auction. Therefore, if the number of bidders is $n_1 = 2$, the expected selling price in the first-price auction is higher by $\frac{1}{3}$ than in the second-price auction.

The result above is due to the fact that although the probability of $n_1 = 2$ bidders is one-half from the seller's perspective, bidders bid as if this probability is much lower: $b^u(x) \approx x - \frac{1}{2} \approx b_{n_2 1}^u(x)$. Conditional on being active, a bidder updates the probabilities of $n_1 = 2$ and n_2 bidders and finds that the greater

number of bidders (i.e., n_2) is much more likely.[5] Indeed, in the case of n_i active bidders, the probability that a given bidder participates in the auction is given by n_i/N. Thus, by Bayes Theorem, the probability of n_i bidders from the perspective of an active bidder is $(\pi_i n_i/N)/(\sum_{i=1}^{M} \pi_i n_i/N) = \pi_i n_i/\bar{n}$.

Theorem 3.1 is due to exactly this effect. As n_M becomes large, active bidders put disproportionately large weight on n_M bidders being active, irrespective of the underlying probabilities $\pi_1, \dots, \pi_M$. That does not hurt the seller in the case of n_M bidders being active, because (for the common-value case) the selling price is very close to V in both uniform and discriminatory auctions. However, if the number of active bidders is not large, the difference between price-setting signals (for first-price versus second-price auctions, we are interested in the difference between the highest and the second-highest signals) is significant.

Note that an extreme uncertainty setting in the discussion above is convenient for understanding intuition, and for proving Theorem 3.1 for any auction setting. However, for any given signal distribution, it is easy to construct examples where the uncertainty about the number of bidders does not take such an extreme form. For the uniform distribution of signals discussed above, the difference between the expected revenues of uniform (R^u) and discriminatory (R^d) auctions is[6]

$$E(R^u - R^d) = k^2 \left(\frac{1}{\sum_{i=1}^{M} \pi_i n_i} - \frac{1 + 1/k}{2} \sum_{i=1}^{M} \frac{\pi_i}{n_i + 1} \right).$$

If the number of bidders is either n_1 or n_2 with equal probabilities, then the discriminatory auction of one object (i.e., $k = 1$) generates greater revenues than the corresponding uniform auction if and only if $n_2 > n_1 + 1 + \sqrt{4n_1 + 5}$ (e.g., $n_1 = 10$ and $n_2 = 18$, or $n_1 = 100$ and $n_2 = 122$). If the number of bidders is uniformly distributed from n_1 to n_M (i.e., $\pi_1 = \pi_2 = \cdots = \pi_M = 1/(n_M - n_1 + 1)$), then the discriminatory auction of one object generates greater revenues than the corresponding uniform auction for, e.g., $n_1 = 2$ and $n_M = 10$, $n_1 = 10$ and $n_M = 24$, $n_1 = 100$ and $n_M = 137$.

The examples above show that revenue-ranking reversal described in Theorem 3.1 occurs even with moderate levels of the uncertainty about the number of bidders. However, if the level of uncertainty about the number of bidders were low enough, the revenue ranking of uniform and discriminatory auctions would coincide with the standard revenue ranking.

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[5] The underlying intuition is also discussed in Matthews (1987) and Harstad et al. (1990).

[6] That expression follows from (B16), which appears in the proof of Theorem 3.1 in Appendix B.

4. Revealing the Number of Bidders

Depending on the auction format, neither bidders nor the bid-taker might know the exact number of bidders prior to bid submission. In some situations, the bid-taker might first register/approve bidders (e.g., by requesting that bidders deposit a considerable amount of cash, as is commonly done in government auctions such as FCC spectrum auctions), publicly release the number of participating bidders, and only then solicit bid submissions. Of course, the bid-taker might choose not to release this information.

In this section, we investigate the impact of resolving uncertainty about the number of bidders prior to bid submission by comparing auction revenues in auctions with known and unknown numbers of bidders. The auction theory literature provides answers in some special cases: if bidders' signals are independent and bidders are risk neutral, the revenue equivalence theorem holds, and therefore the expected revenue does not depend upon whether the actual numbers of bidders and objects are revealed or concealed (Harstad et al. 1990, Klemperer 1999). If bidders are risk averse and have private valuations, then, as shown by Matthews (1987), revealing the actual number of bidders decreases the seller's revenue in discriminatory auctions and does not change the expected revenue in uniform price auctions. Harstad et al. (2008) show that in the uniform auction with pure common values, for a large enough number of bidders, the seller benefits by revealing the number of bidders.

Our analysis in this section allows for affiliated and nonprivate valuations, and we show that recommendations for the special cases mentioned above cannot be generalized: even in the setting limited by Assumption 1, the effect of revealing the information about the number of bidders on discriminatory auction revenues can be positive or negative.

EXAMPLE 4.1. Consider a discriminatory auction for one object, in which either two or three bidders participate with equal probabilities, i.e., let $\Omega = \{(2, 0.5), (3, 0.5)\}$. Let $h(t) = 1 + at$, $-1/2 \le t \le 1/2$, $-2 \le a \le 2$.

Figure 1 plots the difference between the expected revenues in the discriminatory auction with unknown and known numbers of bidders ($E[R^d] - E[R_K^d]$) for different values of δ. As shown in the figure, the seller benefits from concealing the number of bidders if $\delta < 0.5$ for any a. However, in the pure common-value case (i.e., $\delta = 1$), the seller benefits from concealing the number of bidders when $a > 0$, and from revealing the number of bidders when $a < 0$.

Even though the effect of resolving uncertainty about the number of bidders is ambiguous, resolving this uncertainty under uniform pricing is always more beneficial than resolving it under discriminatory

Figure 1 **The Difference in Expected Auction Revenues with Unknown and Known Number of Bidders in a Discriminatory Auction, $E[R^d] - E[R_K^d]$ as a Function of Parameter a of the Signal Distribution, for Different Values of the Common-Value Component δ**
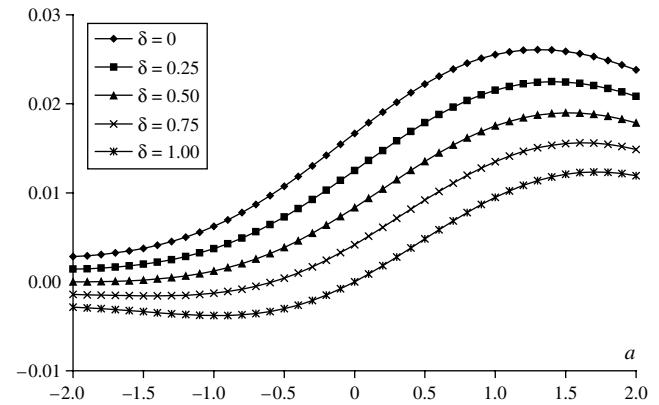
Note. Model parameters are specified in Example 4.1.

pricing. Theorem 4.2 compares the effect of revealing the number of bidders in uniform and discriminatory auctions.

THEOREM 4.2. *Consider an auction setting that satisfies Assumption 1. By making the number of bidders known prior to bid submission, the seller benefits more in the uniform auction than in the discriminatory auction.*

The following corollary is a direct consequence of the fact that for private values (i.e., for $\delta = 0$), the expected revenue in uniform auctions for known and unknown numbers of bidders is the same (because $b^u(x) = b^u_{n_i k}(x) = x$), and it corresponds to a result from Matthews (1987, Theorem 4).

COROLLARY 4.3. *For any auction setting that satisfies Assumption 1 with $\delta = 0$, the seller benefits by concealing the number of bidders in the discriminatory auction.*

The intuition behind Theorem 4.2 and Corollary 4.3 is the following. Revealing the number of bidders helps the seller if the bidding function with a known number of bidders is decreasing in n. From (11) with $\delta = 0$, $b^d_{nk}(x)$ is increasing in n, and thus, if values are private, the seller benefits by concealing the number of bidders.[7] Also, because $b^d_{nk}(x) = b^u_{nk}(x) - k/(n\beta_{nk})$, by revealing the number of bidders, the seller benefits more in the case of the uniform auction.

As Corollary 4.3 shows, in the discriminatory auction the seller benefits by concealing the number of bidders if bidders' valuations are private (and Assumption 1 is satisfied). With nonprivate values, the bidding function in the discriminatory auction with known number of bidders might be increasing

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[7] Harstad et al. (2008) prove that asymptotically (i.e., for large k and n) $n\beta_{nk}$ and α_{nk}/β_{nk} are increasing in n.

or decreasing in n, and thus Theorem 4.2 suggests that the revenue ranking is ambiguous. As shown in Example 4.1, the revenue ranking indeed depends on the signal distribution.

In summary, a decision to mitigate or induce uncertainty about the number of bidders could have impact on revenues, and the optimal policy depends on the information and valuation structure of the bidders. However, our findings indicate that the benefit of mitigating demand uncertainty is higher in the context of uniform pricing than in the context of discriminatory pricing.

5. Conclusions

We analyze the impact of uncertainty about the number of bidders on the expected revenues of uniform and discriminatory pricing rules in the single-shot unit-demand auction model that allows for an uncertain number of bidders and for affiliated values. We show that the revenue ranking gets reversed if uncertainty about the number of bidders is sufficiently large (Theorem 3.1). In other words, although uniform auctions are known to generate higher expected revenues than discriminatory auctions when there is no uncertainty about the number of bidders, the discriminatory auctions yield greater expected revenue than the uniform auctions if that uncertainty is large. Intuitively, this result follows from the fact that bidders rationally bid as if the greater number of rivals is more likely.

Section 4 analyzes the circumstances under which it is beneficial to mitigate or induce uncertainty by revealing or concealing the information about the number of bidders. There is no clear-cut answer, and it depends on whether the equilibrium bidding function with a known number of bidders is increasing in the number of bidders. If it is decreasing (as in the uniform auction), the seller should reveal the number of bidders. If it is increasing (as in the discriminatory private-values auction), the seller benefits from concealing the number of bidders.

Our approach required extending a classical auction theory model to account for uncertainty about the number of bidders. As demonstrated in Appendix A, an increasing equilibrium in such an extended model might not exist, and proving equilibria results in full generality becomes analytically intractable. These technical issues point to one weakness of our results: in the case of discriminatory auctions with an unknown number of bidders, we work only with the candidate for an increasing symmetric equilibrium. When the symmetric equilibrium fails to exist, there might be no natural equilibrium to focus on, and the revenue ordering might not be well defined, due to, e.g., existence of multiple equilibria. On the other hand, the setting with linear bidding strategies, specified in Assumption 1, is particularly amenable to experimental testing, as discussed in §3.1. Studying whether the subjects bid according to the strategies specified here, and whether model predictions hold in the laboratory setting, could lead to interesting new insights about the preferable auction format and the predictive power of Bayesian Nash equilibrium theory.

The impact of uncertainty about the number of bidders on revenue rankings, even in a setting limited by Assumption 1, implies that this uncertainty is an important factor to consider when choosing the auction pricing rule, and if that uncertainty is substantial, discriminatory pricing is preferable for the seller. To the extent that the auction setting is representative of more general competitive environments, our results suggest that, overall, the choice of uniform versus discriminatory pricing is sensitive to the level of demand uncertainty.

## Appendix A. Existence of an Increasing Symmetric Equilibrium

Theorems 2.1 and 2.2 provide the unique candidates, but do not guarantee the existence of an increasing symmetric equilibrium. The necessary and sufficient condition from Theorem 2.1 implies that in the case of a private-values model, where $u(v, x) = x$, the bidding function in a uniform auction, $b^u(x) = x$, is increasing, and thus an increasing symmetric equilibrium exists. Example A.1 shows that in the case of a common-value model, where $u(v, x) = v$, a symmetric increasing equilibrium might not exist. In the case of the discriminatory auction, an increasing symmetric equilibrium might not exist for two reasons: (i) bidding function, given by (7), might not be increasing (Example A.1); and (ii) even when it is increasing, deviations from it might be profitable (Example A.2). To simplify the exposition, we provide the examples only for $k = 1$ and for the cases where $u(v, x) = v$ or $u(v, x) = x$. However, similar examples can be constructed for $k > 1$ and for a more general form of valuation function $u(v, x)$.

EXAMPLE A.1. Consider an auction for one object (i.e., $k = 1$) with $\Omega_\mathbf{k} = \{(2, 1, 1/2), (n_2, 1, 1/2)\}$; that is, there are either 2 or n_2 bidders, each equally likely. Let $g(v)$ be uniform on $[0, \bar{v}]$, $\bar{v} > 1$, let $f(x \mid v)$ be uniform on $[v - 1/2, v + 1/2]$, and let $u(v, x) = v$. Consider $-1/2 < x < 1/2$. As shown below, $b^u(x)$, given by (5), is not increasing for $n_2 \geq 6$. Figure A1 shows that $b^d(x)$, given by (7), is not increasing for $n_2 \geq 7$. Therefore, an increasing symmetric equilibrium does not exist in the uniform auction for $n_2 \geq 6$ and in the discriminatory auction for $n_2 \geq 7$.

Figure A1 Bidding Function in a Discriminatory Auction, $b^d(x)$, for Different Values of n_2
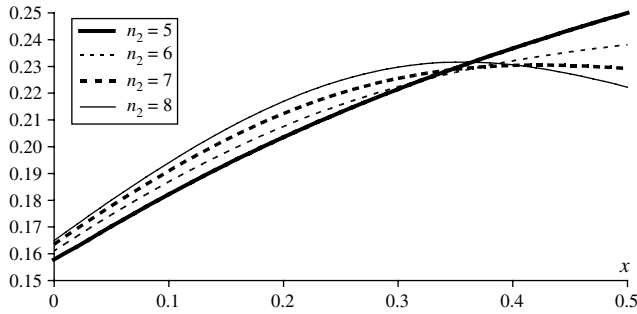
Note. Model parameters are specified in Example A.1.

Derivation of Example A.1

Note that $f(x \mid v) = 1$ if $|x - v| \le 1/2$ and $f(x \mid v) = 0$ if $|x - v| > 1/2$. Correspondingly, $F(x \mid v) = x - v + 1/2$ if $|x - v| \le 1/2$, $F(x \mid v) = 0$ if $x < v - 1/2$, and $F(x \mid v) = 1$ if $x > v + 1/2$.

The support of the posterior distribution of v is bounded by $\underline{v}_x = \max(0, x - 1/2)$, $\bar{v}_x = \min(\bar{v}, x + 1/2)$. Equation (5) becomes

$$b^u(x) = \frac{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{\underline{v}_x}^{\bar{v}_x} v\left(x - v + \frac{1}{2}\right)^{n_i - 2} dv}{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{\underline{v}_x}^{\bar{v}_x} \left(x - v + \frac{1}{2}\right)^{n_i - 2} dv}.$$

Changing variables $t = x - v + 1/2$ yields $dv = -dt$, $v = x - t + 1/2$, so

$$b^u(x) = \frac{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{x - \bar{v}_x + 1/2}^{x - \underline{v}_x + 1/2} \left(x - t + \frac{1}{2}\right) t^{n_i - 2} dt}{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{x - \bar{v}_x + 1/2}^{x - \underline{v}_x + 1/2} t^{n_i - 2} dt}. \quad \text{(A1)}$$

Consider x such that $x < 1/2 < \bar{v} - 1/2$. In that case, $x - \underline{v}_x + 1/2 = \min(x + 1/2, 1) = x + 1/2$, $x - \bar{v}_x + 1/2 = 0$, and (A1) becomes

$$b^u(x) = \frac{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{x+1/2}^0 \left(x - t + \frac{1}{2}\right) t^{n_i - 2} dt}{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_{x+1/2}^0 t^{n_i - 2} dt}$$

$$= \frac{\sum_{i=1}^M \pi_i n_i (n_i - 1)\left(\left(x + \frac{1}{2}\right)\frac{1}{n_i - 1}\left(x + \frac{1}{2}\right)^{n_i - 1} - \frac{1}{n_i}\left(x + \frac{1}{2}\right)^{n_i}\right)}{\sum_{i=1}^M \pi_i n_i (n_i - 1) \frac{1}{n_i - 1}\left(x + \frac{1}{2}\right)^{n_i - 1}}$$

$$= x + \frac{1}{2} - \frac{\sum_{i=1}^M \pi_i (n_i - 1)\left(x + \frac{1}{2}\right)^{n_i}}{\sum_{i=1}^M \pi_i n_i \left(x + \frac{1}{2}\right)^{n_i - 1}}. \quad \text{(A2)}$$

Consider $x < 1/2$. Using $n_1 = 2$,

$$b^u(x) = x + \frac{1}{2} - \frac{\pi_1\left(x + \frac{1}{2}\right)^2 + (1 - \pi_1)(n_2 - 1)\left(x + \frac{1}{2}\right)^{n_2}}{\pi_1 2\left(x + \frac{1}{2}\right) + (1 - \pi_1) n_2 \left(x + \frac{1}{2}\right)^{n_2 - 1}}$$

$$= x + \frac{1}{2} - \frac{\pi_1\left(x + \frac{1}{2}\right) + (1 - \pi_1)(n_2 - 1)\left(x + \frac{1}{2}\right)^{n_2 - 1}}{2\pi_1 + (1 - \pi_1) n_2 \left(x + \frac{1}{2}\right)^{n_2 - 2}}.$$

As $x \nearrow 1/2$, i.e., for x close enough to, but less than, $1/2$, this function is increasing for $n_2 \le 5$ and is decreasing for $n_2 \ge 6$. Therefore, by Theorem 2.1, an increasing symmetric equilibrium exists for $n_2 \le 5$, but no such equilibrium exists if $n_2 \ge 6$. (This part of Example A.1 is also discussed in Harstad et al. 2008.)

We now turn to the discriminatory auction and the derivation of b^d.

Consider $-1/2 < x < 1/2$. Then, by (A2),

$$v(x, x) = b^u(x) = x + \frac{1}{2} - \frac{\sum_{i=1}^M \pi_i (n_i - 1)(x + 1/2)^{n_i}}{\sum_{i=1}^M \pi_i n_i (x + 1/2)^{n_i - 1}}.$$

By (6), using $k = 1$,

$$A(y, x) = \frac{\sum_{i=1}^M \pi_i n_i f_{n_i - 1, k}(y \mid x)}{\sum_{i=1}^M \pi_i n_i F_{n_i - 1, k}(y \mid x)}$$

$$= \frac{\sum_{i=1}^M \pi_i n_i \int_{\underline{v}}^{\bar{v}} (n_i - 1) f(x \mid t) f(y \mid t) F^{n_i - 2}(y \mid t) g(t) \, dt}{\sum_{i=1}^M \pi_i n_i \int_{\underline{v}}^{\bar{v}} f(x \mid t) F^{n_i - 1}(y \mid t) g(t) \, dt}.$$

For $-1/2 < x < 1/2$,

$$A(x, x) = \frac{\sum_{i=1}^M \pi_i n_i (n_i - 1) \int_0^{x + 1/2} (x - t + 1/2)^{n_i - 2} \, dt}{\sum_{i=1}^M \pi_i n_i \int_0^{x + 1/2} (x - t + 1/2)^{n_i - 1} \, dt}$$

$$= \frac{\sum_{i=1}^M \pi_i n_i (x + 1/2)^{n_i - 1}}{\sum_{i=1}^M \pi_i (x + 1/2)^{n_i}}.$$

Then,

$$b^d(x) = \int_{-1/2}^x v(t, t) A(t, t) \exp\left(-\int_t^x A(s, s)\right) dt$$

$$= \int_{-1/2}^x \left(t + 1/2 - \frac{\sum_{i=1}^M \pi_i (n_i - 1)(t + 1/2)^{n_i}}{\sum_{i=1}^M \pi_i n_i (t + 1/2)^{n_i - 1}}\right)$$

$$\cdot \frac{\sum_{i=1}^M \pi_i n_i (t + 1/2)^{n_i - 1}}{\sum_{i=1}^M \pi_i (t + 1/2)^{n_i}}$$

$$\cdot \exp\left(-\int_t^x \frac{\sum_{i=1}^M \pi_i n_i (s + 1/2)^{n_i - 1}}{\sum_{i=1}^M \pi_i (s + 1/2)^{n_i}} \, ds\right) dt$$

$$= \int_{-1/2}^x \left(\frac{\sum_{i=1}^M \pi_i n_i (t + 1/2)^{n_i}}{\sum_{i=1}^M \pi_i (t + 1/2)^{n_i}} - \frac{\sum_{i=1}^M \pi_i (n_i - 1)(t + 1/2)^{n_i}}{\sum_{i=1}^M \pi_i (t + 1/2)^{n_i}}\right)$$

$$\cdot \exp\left(-\int_t^x \frac{\sum_{i=1}^M \pi_i n_i (s + 1/2)^{n_i - 1}}{\sum_{i=1}^M \pi_i (s + 1/2)^{n_i}} \, ds\right) dt$$

$$= \int_{-1/2}^x \exp\left(-\int_t^x \frac{\sum_{i=1}^M \pi_i n_i (s + 1/2)^{n_i - 1}}{\sum_{i=1}^M \pi_i (s + 1/2)^{n_i}} \, ds\right) dt.$$

Substituting $M = 2$, $n_1 = 2$, $\pi_1 = \pi_2 = 1/2$ yields bidding functions depicted in Figure A1. As indicated by the figure, $b^d(x)$ is increasing in x with $n_2 = 5$ and $n_2 = 6$, and decreasing for some x with $n_2 = 7$ and $n_2 = 8$. In general, it can be shown that $b^d(x)$ is increasing in x when $n_2 \le 6$ and decreasing for some x when $n_2 \ge 7$. $\quad \square$

Example A.1 builds on the fact that $v(x, x)$ might not be increasing. Note that, as shown in Milgrom and Weber (1982), if $v(x, x)$ is increasing (e.g., in the private-value setting, where $v(x, x) = v_{n_i k}(x, x) = x$), then $b^d(x)$ is increasing. (As stated in Theorem 2.1, an increasing $v(x, x)$ is sufficient to guarantee the existence of a symmetric equilibrium in the uniform auction.) However, even if $b^d(x)$ given by (7) is increasing, an increasing symmetric equilibrium still might not exist in a discriminatory auction. The reason is that deviations from (7) could be profitable:

Example A.2. Consider an auction for one object (i.e., $k = 1$). Let $g(v)$ be uniform on $[0; \bar{v}]$, $\bar{v} > 1$, let $f(x \mid v)$ be

uniform on $[v-1/2; v+1/2]$, and let $u(v, x) = x$. Then, for large enough $\bar{v}$, an increasing symmetric equilibrium in a discriminatory auction does not exist if $\sum_{i=1}^{M} \pi_i n_i^2 > 2\bar{n}^2 + \bar{n}$, where $\bar{n} = \sum_{i=1}^{M} \pi_i n_i$, which is equivalent to variance of the number of bidders being greater than $(\bar{n}^2 + \bar{n})$.

Derivation of Example A.2

Denote $y = (b^d)^{-1}(b)$. Then, using (6), the derivative of the expected profit, given by Equation (B6), is

$$\frac{\sum_{i=1}^{M} \pi_i n_i f_{n_i-1, k}(y \mid x)}{\bar{n}(b^d)'(y)} [v(x, y) - b^d(y) - (b^d)'(y)/A(y, x)],$$

and $v(y, y) - b^d(y) - (b^d)'(y)/A(y, y) = 0$. Thus, a necessary condition for the existence of an increasing symmetric equilibrium is that $v(x, y) - b^d(y) - (b^d)'(y)/A(y, x)$ is nondecreasing in x when x is close enough to y. (Otherwise, a bidder with signal y would increase her expected payoff by bidding either slightly higher or slightly lower than $b^d(y)$.) Note that for the case of a known number of bidders this is always the case: $v(x, y)$ is increasing in x and $A(y, x)$ is increasing in x by Lemma 1 of Milgrom and Weber (1982). In the case of an unknown number of bidders, $v(x, y)$ is still increasing in x, but $A(y, x)$ might be decreasing in x. Consider numerical Example A.2. We will show that, for large enough y, $v(x, y) - b^d(y) - (b^d)'(y)/A(y, x)$ might be decreasing in x when x is close enough to y.

For $1/2 < y \leq x < \bar{v} - 1/2$, using $k = 1$, we have

$$A(y, x) = \frac{\sum_{i=1}^{M} \pi_i n_i f_{n_i-1, k}(y \mid x)}{\sum_{i=1}^{M} \pi_i n_i F_{n_i-1, k}(y \mid x)}$$

$$= \frac{\sum_{i=1}^{M} \pi_i n_i \int_{\underline{v}}^{\bar{v}} (n_i-1) f(x \mid t) f(y \mid t) F^{n_i-2}(y \mid t) g(t)\, dt}{\sum_{i=1}^{M} \pi_i n_i \int_{\underline{v}}^{\bar{v}} f(x \mid t) F^{n_i-1}(y \mid t) g(t)\, dt}$$

$$= \frac{\sum_{i=1}^{M} \pi_i n_i (n_i-1) \int_{x-1/2}^{y+1/2} (y - t + 1/2)^{n_i-2}\, dt}{\sum_{i=1}^{M} \pi_i n_i \int_{x-1/2}^{y+1/2} (y - t + 1/2)^{n_i-1}\, dt}$$

$$= \frac{\sum_{i=1}^{M} \pi_i n_i (y - x + 1)^{n_i-1}}{\sum_{i=1}^{M} \pi_i (y - x + 1)^{n_i}}. \tag{A3}$$

Let $1/2 < y < \bar{v} - 1/2$. Because $u(v, x) = x$, $v(x, y) = x$. Then, by (7),

$$b^d(y) = \int_{-1/2}^{1/2} t A(t, t) \exp\left(-\int_t^y A(s, s)\, ds\right) dt$$

$$+ \int_{1/2}^{y} t A(t, t) \exp\left(-\int_t^y A(s, s)\, ds\right) dt$$

$$= \exp\left(-\int_{1/2}^{y} A(s, s)\, ds\right) \int_{-1/2}^{1/2} t A(t, t)$$

$$\cdot \exp\left(-\int_t^{1/2} A(s, s)\, ds\right) dt$$

$$+ \int_{1/2}^{y} t A(t, t) \exp\left(-\int_t^y A(s, s)\, ds\right) dt.$$

Note that, by (A3), $A(s, s) = \bar{n}$ for $s > 1/2$. Denoting

$$I = \int_{-1/2}^{1/2} t A(t, t) \exp\left(-\int_t^{1/2} A(s, s)\, ds\right) dt,$$

we have

$$b^d(y) = \exp\left(-\int_{1/2}^{y} \bar{n}\, ds\right) I + \int_{1/2}^{y} t \bar{n} \exp\left(-\int_t^y \bar{n}\, ds\right) dt$$

$$= I \exp((1/2 - y)\bar{n}) + \int_{1/2}^{y} t\bar{n} \exp((t - y)\bar{n})\, dt$$

$$= I \exp((1/2 - y)\bar{n}) + \exp(-y\bar{n})$$

$$\cdot [(y - 1/\bar{n}) \exp(y\bar{n}) - (1/2 - 1/\bar{n}) \exp(\bar{n}/2)]$$

$$= y - 1/\bar{n} + \exp(-y\bar{n})$$

$$\cdot [I \exp(\bar{n}/2) - (1/2 - 1/\bar{n}) \exp(\bar{n}/2)]. \tag{A4}$$

Consider the limit $\bar{v} \to \infty$ and $y \to \infty$. Then, by (A4), $b^d(y)$ approaches $y - 1/\bar{n}$, so $(b^d)'(y) \to 1$ as $y \to \infty$. Thus, using $v(x, y) = x$ and $A(y, x)$ from (A3),

$$d[v(x, y) - b^d(y) - (b^d)'(y)/A(y, x)]/dx$$

$$= d\left[x - \frac{\sum_{i=1}^{M} \pi_i (y - x + 1)^{n_i}}{\sum_{i=1}^{M} \pi_i n_i (y - x + 1)^{n_i-1}}\right] \bigg/ dx$$

$$= 1 + \frac{\sum_{i=1}^{M} \pi_i n_i (y - x + 1)^{n_i-1}}{\sum_{i=1}^{M} \pi_i n_i (y - x + 1)^{n_i-1}}$$

$$- \frac{\left(\sum_{i=1}^{M} \pi_i (y - x + 1)^{n_i}\right) \sum_{i=1}^{M} \pi_i n_i (n_i-1)(y - x + 1)^{n_i-2}}{\left(\sum_{i=1}^{M} \pi_i n_i (y - x + 1)^{n_i-1}\right)^2}.$$

For $x = y$, the above equation becomes

$$d[v(x, y) - b^d(y) - (b^d)'(y)/A(y, x)]/dx \,|_{x=y}$$

$$= 2 - \frac{\sum_{i=1}^{M} \pi_i n_i (n_i-1)}{\left(\sum_{i=1}^{M} \pi_i n_i\right)^2} = 2 + \frac{1}{\bar{n}} - \frac{\sum_{i=1}^{M} \pi_i n_i^2}{\bar{n}^2}.$$

Therefore, if $\sum_{i=1}^{M} \pi_i n_i^2 > 2\bar{n}^2 + \bar{n}$, the quantity above is negative and a symmetric increasing equilibrium does not exist. □

Appendix B. Proofs

PROOF OF THEOREM 2.1. Harstad et al. (2008) prove the theorem for the common-value case $u(v, x) = v$. The proof for the general form of $u(v, x)$ is analogous.

Assume that an increasing function b_0 is a symmetric equilibrium bid function. Hence, the bid $b_0(x)$ maximizes expected profit for a bidder who observes the signal x when all rivals use b_0. Consider such a bidder observing signal x.

By assumption, $b_0(x)$ is increasing but does not have to be continuous. Define $b_0^{-1}(b) = \sup\{x: b_0(x) \leq b\}$, i.e., $b_0^{-1}(b)$ is the largest x such that $b_0(x) \leq b$. Then b_0^{-1} is defined for all b and, for all x, $x = b_0^{-1}(b_0(x))$. Conditional on the numbers of bidders and objects (n_i, k_i), the expected profit $\Pi_i(b, x)$ of a bidder who observes x and bids b when all rivals use the function b_0 is

$$\Pi_i(b, x) = \int_{\underline{x}}^{b_0^{-1}(b)} (v_{n_i k_i}(x, y) - b_0(y)) f_{n_i-1, k_i}(y \mid x)\, dy.$$

Using Matthews (1987), the probability of n_i bidders, given that a bidder is active, is $\pi_i n_i / \bar{n}$, where $\bar{n} = \sum_{i=1}^{M} \pi_i n_i$. The ex ante unconditional expected profit for a bidder who

observes signal x and bids b (i.e., taking the expectation over all pairs (n_i, k_i)), is

$$\mathbf{\Pi}(b, x) = \sum_{i=1}^{M} \frac{\pi_i n_i}{\bar{n}} \Pi_i(b, x)$$
$$= \sum_{i=1}^{M} \frac{\pi_i n_i}{\bar{n}} \int_{\underline{x}}^{b_0^{-1}(b)} (v_{n_i k_i}(x, y) - b_0(y)) f_{n_i-1, k_i}(y \mid x) \, dy. \quad \text{(B1)}$$

Denote by f_Y the density of the pivotal rival signal, i.e., the k_ith highest of the remaining $n_i - 1$ signals where the probability of (n_i, k_i) pair is $\pi_i n_i / \bar{n}$:

$$f_Y(y \mid x) = \sum_{i=1}^{M} \frac{\pi_i n_i}{\bar{n}} f_{n_i-1, k_i}(y \mid x).$$

Then (B1) can be expressed as

$$\mathbf{\Pi}(b, x) = \int_{\underline{x}}^{b_0^{-1}(b)} (v(x, y) - b_0(y)) f_Y(y \mid x) \, dy, \quad \text{(B2)}$$

where, obviously, $f_Y(y \mid x)$ is nonnegative.

Also note that $v(x, y)$ is increasing in its first argument, and is continuous. To show this, rewrite (4):

$$v(x, y) = \frac{\int_{\underline{v}}^{\bar{v}} u(v, x) f(x \mid v) g_y(v) \, dv}{\int_{\underline{v}}^{\bar{v}} f(x \mid t) g_y(t) \, dt},$$

where $g_y(v) = \sum_{i=1}^{M} \pi_i n_i f_{n_i-1, k_i}(y \mid v) g(v) / \sum_{i=1}^{M} \pi_i n_i \cdot \int_{\underline{v}}^{\bar{v}} f_{n_i-1, k_i}(y \mid t) g(t) \, dt$. Because $f(\cdot \mid v)$ satisfies MLRP (Equation (1)), by Theorem 2.1 of Milgrom (1981), for every nondegenerate prior distribution G with p.d.f. g and every x_L and x_U in the support of X_1 such that $x_L < x_U$, $G(\cdot \mid X = x_U)$ dominates $G(\cdot \mid X = x_L)$ in the sense of first-order stochastic dominance. In particular, for G_y with p.d.f. g_y

$$x_L < x_U \;\Rightarrow\; v(x_L, y) = \mathrm{E}[u(V, x_L) \mid x_L] \leq \mathrm{E}[u(V \mid x_L) \mid x_U]$$
$$\leq \mathrm{E}[u(V \mid x_U) \mid x_U] = v(x_U, y), \quad \text{(B3)}$$

so the claim follows.

To show that b^u, given by (5), is a symmetric equilibrium if it is increasing, first note that setting $x = y$ in (4) yields $b^u = v(x, x)$ (and b^u is continuous). Suppose $v(x, x)$ is increasing. Then, using (B2),

$$\mathbf{\Pi}(b, x) - \mathbf{\Pi}(b^u(x), x)$$
$$= \int_{x}^{b^{u-1}(b)} (v(x, y) - v(y, y)) f_Y(y \mid x) \, dy. \quad \text{(B4)}$$

Thus, because $v(x, y)$ is increasing in its first argument, (B4) is nonpositive for all $b \neq b^u(x)$. Hence, the bid $b^u(x)$ maximizes expected profit for a bidder who observes the signal x when all rivals use b^u, i.e., b^u is a symmetric equilibrium bid function.

Finally, suppose there exists an increasing symmetric equilibrium bid function b_0, $b_0 \neq b^u$, i.e., suppose that there exists x_0 such that $b_0(x_0) \neq v(x_0, x_0)$. Consider the case $b_0(x_0) < v(x_0, x_0)$ (the case $b_0(x_0) > v(x_0, x_0)$ is treated similarly). If b_0 is continuous from the right at x_0, then there exist

$x_1 < x_2$ in the neighborhood of x_0 such that for all y, $x_1 \leq y \leq x_2$, $b_0(y) < v(x_1, y)$ and $f_Y(y \mid x_1) > 0$. Then, using (B2),

$$\mathbf{\Pi}(b_0(x_2), x_1) - \mathbf{\Pi}(b_0(x_1), x_1)$$
$$= \int_{x_1}^{x_2} (v(x_1, y) - b_0(y)) f_Y(y \mid x) \, dy > 0,$$

so a bidder observing signal x_1 benefits by deviating from bidding $b_0(x_1)$ to bidding $b_0(x_2)$. If $b_0(x_0)$ is discontinuous from the right at x_0, then there exists b_1, $b_0(x_0) < b_1 < v(x_0, x_0)$. Note that by definition $b_0^{-1}(b_1) = b_0^{-1}(b_0(x_0)) = x_0$. A bidder who observes signal x_0 benefits by bidding b_1 instead of $b_0(x_0)$ in the case of a tie with the pivotal bid (recall that ties are settled randomly): conditional on a pivotal bidder having signal x_0, the expected value is $v(x_0, x_0)$ and the price is $b(x_0) < v(x_0, x_0)$.

Therefore, if b^u given by (5) is increasing, it is the unique symmetric equilibrium in increasing strategies. $\square$

PROOF OF THEOREM 2.2. Assume that bidder j, observing signal x_j, bids $b^d(x_j)$. Consider a bidder who observes signal x. The bidding function $b^d(x)$ is a symmetric Nash equilibrium if and only if the expected profit of the bidder who observes signal x is maximized at $b = b^d(x)$. Conditional on the number of bidders n_i, the expected profit $\Pi_i(b)$ of the bidder who observes signal x and bids b is

$$\Pi_i(b) = \int_{\underline{x}}^{(b^d)^{-1}(b)} (v_{n_i k}(x, y) - b) f_{n_i-1, k}(y \mid x) \, dy.$$

As shown by Matthews (1987), from the bidder's point of view, the probability of n_i bidders given the observed signal x is $\pi_i n_i / \bar{n}$, where

$$\bar{n} = \sum_{i=1}^{M} \pi_i n_i.$$

Taking the expectation over all n_i, the total expected profit of the bidder who observes signal x and bids b is

$$\Pi(b) = \sum_{i=1}^{M} \pi_i \frac{n_i}{\bar{n}} \Pi_i(b)$$
$$= \sum_{i=1}^{M} \pi_i \frac{n_i}{\bar{n}} \int_{\underline{x}}^{(b^d)^{-1}(b)} (v_{n_i k}(x, y) - b) f_{n_i-1, k}(y \mid x) \, dy. \quad \text{(B5)}$$

Differentiating (B5), we get

$$\frac{d\Pi(b)}{db} = \frac{1}{(b^d)'((b^d)^{-1}(b))}$$
$$\cdot \sum_{i=1}^{M} \pi_i \frac{n_i}{\bar{n}} (v_{n_i k}(x, (b^d)^{-1}(b)) - b) f_{n_i-1, k}((b^d)^{-1}(b) \mid x)$$
$$- \sum_{i=1}^{M} \pi_i \frac{n_i}{\bar{n}} \int_{\underline{x}}^{(b^d)^{-1}(b)} f_{n_i-1, k}(y \mid x) \, dy. \quad \text{(B6)}$$

Substituting $b = b^d(x)$ into (B6), we get the following differential equation for $b^d(x)$, where the condition $(d\Pi(b)/db)|_{b=b^d(x)} = 0$ is satisfied:

$$(b^d)'(x) = \frac{\sum_{i=1}^{M} \pi_i n_i (v_{n_i k}(x, x) - b^d(x)) f_{n_i-1, k}(x \mid x)}{\sum_{i=1}^{M} \pi_i n_i \int_{\underline{x}}^{x} f_{n_i-1, k}(y \mid x) \, dy}$$
$$= \frac{\sum_{i=1}^{M} \pi_i n_i (v_{n_i k}(x, x) - b^d(x)) f_{n_i-1, k}(x \mid x)}{\sum_{i=1}^{M} \pi_i n_i F_{n_i-1, k}(x \mid x)}. \quad \text{(B7)}$$

Equation (B7) generalizes Equation (7) of Milgrom and Weber (1982, p. 1107) to the case of multi-item auctions with an unknown number of bidders.

Recalling definitions (4) and (6), Equation (B7) can be rewritten as

$$(b^d)'(x) = (v(x, x) - b^d(x))A(x, x).$$

As in Milgrom and Weber (1982), the boundary condition is $b^d(\underline{x}) = v(\underline{x}, \underline{x})$. Then the solution of (B7) is

$$b^d(x) = \int_{\underline{x}}^x v(t, t)A(t, t)e^{\int_x^t A(s,s)\,ds}\,dt. \quad \square$$

PROOF OF PROPOSITION 2.4. By (6),

$$A(x, x) = \frac{\sum_{i=1}^M \pi_i n_i f_{n_i-1,k}(x \mid x)}{\sum_{i=1}^M \pi_i n_i F_{n_i-1,k}(x \mid x)}. \tag{B8}$$

Substituting $f(x \mid v) = h(x - v)$, $F(x \mid v) = H(x - v)$, and diffuse prior $g(v)$ into (2) yields

$$f_{n-1,k}(x \mid x) = \beta_{nk}, \tag{B9}$$

$$F_{n-1,k}(x \mid x) = \frac{(n-1)!}{(k-1)!(n-k-1)!} \int_{-\infty}^{+\infty} \int_{-\infty}^x h(y-v)H^{n-k-1}(y-v)$$
$$\cdot (1 - H(y-v))^{k-1}h(x-v)\,dy\,dv. \tag{B10}$$

Applying the change of variables $H(y - v) = t$, and then $H(x - v) = z$, to the right-hand side of (B10) we have

$$F_{n-1,k}(x \mid x) = \frac{(n-1)!}{(k-1)!(n-k-1)!}$$
$$\cdot \int_0^1 \int_0^z t^{n-k-1}(1-t)^{k-1}\,dt\,dz. \tag{B11}$$

Note that the right-hand side of (B11) is the probability that a random variable z drawn from a uniform distribution on $[0, 1]$ is greater than t, which is the kth order statistic out of $n - 1$ independent draws from the same uniform distribution on $[0, 1]$. By symmetry, this probability is k/n, i.e.,

$$F_{n-1,k}(x \mid x) = \frac{k}{n}. \tag{B12}$$

Thus, by (B8), (B9), and (B12),

$$A(x, x) = \frac{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}{\sum_{i=1}^M \pi_i n_i (k/n_i)} = \frac{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}{\bar{k}}. \tag{B13}$$

Substituting $v(t, t)$ from Proposition 2.3 and $A(t, t)$ from (B13) into (7), with $\underline{x} = -\infty$, yields

$$b^d(x) = \int_{\underline{x}}^x v(t, t)A(t, t)e^{\int_x^t A(s,s)\,ds}\,dt$$
$$= \int_{-\infty}^x \left(t - \delta \frac{\sum_{i=1}^M \pi_i n_i \alpha_{n_i k}}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}\right)$$
$$\cdot \frac{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}{\bar{k}} e^{(t-x)((\sum_{i=1}^M \pi_i n_i \beta_{n_i k})/\bar{k})}\,dt.$$

Applying the identity (that holds for any B and any $C > 0$)

$$\int_{-\infty}^x (t + B)Ce^{tC}\,dt = \left. \left(te^{tC} + \left(B - \frac{1}{C}\right)e^{tC}\right) \right|_{t=-\infty}^{t=x}$$
$$= \left(x + B - \frac{1}{C}\right)e^{Cx}$$

with $B = -\delta(\sum_{i=1}^M \pi_i n_i \alpha_{n_i k}/\sum_{i=1}^M \pi_i n_i \beta_{n_i k})$ and $C = (\sum_{i=1}^M \pi_i n_i \beta_{n_i k})/\bar{k}$ yields (12). $\quad \square$

REMARK. Levin and Smith (1991, Equation (3)) identify a continuum of increasing symmetric equilibria in a diffuse prior setting. In the proof of Proposition 2.4 we consider the setting with a diffuse prior as a limit of the distributions $g(v)$ with the support bounded from below (i.e., $\underline{x} > -\infty$), and we assume that the boundary condition $b^d(\underline{x}) = v(\underline{x}, \underline{x})$ holds. This approach yields linear bidding strategies, as the bidding function in Example A.2 for large enough y.

PROOF OF THEOREM 3.1. Before comparing revenues, we first show that $\beta_{nk} \geq \varepsilon$. By (8), changing the integration variable to $H(t) = z$, and recalling that $((n - 1)!/(k-1)!(n-k-1)!) \int_0^1 z^{n-k-1}(1-z)^{k-1}\,dz = 1$, we have

$$\beta_{nk} = \frac{(n-1)!}{(k-1)!(n-k-1)!} \int_{t_1}^{t_2} H^{n-k-1}(t)(1-H(t))^{k-1}h^2(t)\,dt$$
$$= \frac{(n-1)!}{(k-1)!(n-k-1)!} \int_0^1 h(H^{-1}(z))z^{n-k-1}(1-z)^{k-1}\,dz$$
$$= h(H^{-1}(z^*))$$

for some z^*, $t_1 \leq H^{-1}(z^*) \leq t_2$. Because $h(H^{-1}(z^*)) \geq \varepsilon$ for all z^*, we get $\beta_{nk} \geq \varepsilon$ for all n, k.

Let Y_m^j denote the jth order statistic from a set of m signals. In the case of a uniform auction, when the realized number of bidders is n_i, the price per object is $b^u(Y_{n_i}^{k+1})$, and thus, using (10), the auction revenue in a uniform auction is

$$R^u = \sum_{i=1}^M \pi_i k b^u(Y_{n_i}^{k+1}) = \sum_{i=1}^M \pi_i k \left(Y_{n_i}^{k+1} - \delta \frac{\sum_{j=1}^M \pi_j n_j \alpha_{n_j k}}{\sum_{j=1}^M \pi_j n_j \beta_{n_j k}}\right). \tag{B14}$$

In the case of a discriminatory auction, when the realized number of bidders is n_i, k objects are sold at prices $b^d(Y_{n_i}^1)$, $b^d(Y_{n_i}^2), \ldots, b^d(Y_{n_i}^k)$. Using (12), the auction revenue in a discriminatory auction is

$$R^d = \sum_{i=1}^M \pi_i \sum_{j=1}^k b^d(Y_{n_i}^j)$$
$$= \sum_{i=1}^M \pi_i \sum_{j=1}^k \left(Y_{n_i}^j - \delta \frac{\sum_{l=1}^M \pi_l n_l \alpha_{n_l k}}{\sum_{l=1}^M \pi_l n_l \beta_{n_l k}} - \frac{k}{\sum_{l=1}^M \pi_l n_l \beta_{n_l k}}\right)$$
$$= \sum_{i=1}^M \pi_i \sum_{j=1}^k Y_{n_i}^j - k \sum_{i=1}^M \pi_i$$
$$\cdot \left(\delta \frac{\sum_{j=1}^M \pi_j n_j \alpha_{n_j k}}{\sum_{j=1}^M \pi_j n_j \beta_{n_j k}} + \frac{k}{\sum_{j=1}^M \pi_j n_j \beta_{n_j k}}\right). \tag{B15}$$

The difference in expected auction revenues, using (B14) and (B15), is

$$E[R^u] - E[R^d] = \sum_{i=1}^M \pi_i \sum_{j=1}^k E[Y_{n_i}^{k+1} - Y_{n_i}^j] + \frac{k^2}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}$$
$$= \sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k E[Y_{n_i}^{k+1} - Y_{n_i}^j] + \pi_M \sum_{j=1}^k E[Y_{n_M}^{k+1} - Y_{n_M}^j]$$
$$+ \frac{k^2}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}. \tag{B16}$$

Note that the first term in (B16), $\sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k E[Y_{n_i}^{k+1} - Y_{n_i}^j]$, is strictly negative and does not depend upon n_M.

The second term, $\pi_M \sum_{j=1}^k \mathrm{E}[Y_{n_M}^{k+1} - Y_{n_M}^j]$, is negative for any n_M. Let n^* be such that $\sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k \mathrm{E}[Y_{n_i}^{k+1} - Y_{n_i}^j] + k^2/\pi_M n^* \varepsilon \le 0$. Then, for any $n_M \ge n^*$,

$$\mathrm{E}[R^u] - \mathrm{E}[R^d] < \sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k \mathrm{E}[Y_{n_i}^{k+1} - Y_{n_i}^j] + \frac{k^2}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}}$$

$$< \sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k \mathrm{E}[Y_{n_i}^{k+1} - Y_{n_i}^j] + \frac{k^2}{\pi_M n^* \beta_{n_M k}}$$

$$< \sum_{i=1}^{M-1} \pi_i \sum_{j=1}^k \mathrm{E}[Y_{n_i}^{k+1} - Y_{n_i}^j] + \frac{k^2}{\pi_M n^* \varepsilon} \le 0. \quad \square$$

PROOF OF THEOREM 4.2. If the number of bidders is unknown, bidders bid according to (10) and (12) in uniform and discriminatory auctions, respectively. However, if the number of bidders is revealed before the bids are submitted, the bidding strategies are given by (9) and (11). In the uniform auction, the price per object is $b_{n_i k}^d(Y_{n_i}^{k+1})$ for n_i bidders and k objects, and thus the uniform auction revenue when the number of bidders becomes known prior to bid submission is

$$R_K^u = k \sum_{i=1}^M \pi_i b_{n_i k}^u(Y_{n_i}^{k+1}) = k \sum_{i=1}^M \pi_i \left(Y_{n_i}^{k+1} - \delta \frac{\alpha_{n_i k}}{\beta_{n_i k}} \right). \quad (B17)$$

In a discriminatory auction, in the case of n_i bidders, k objects are sold at prices $b_{n_i k}^d(Y_{n_i}^1), b_{n_i k}^d(Y_{n_i}^2), \ldots, b_{n_i k}^d(Y_{n_i}^k)$. Using (11), the auction revenue in a discriminatory auction is

$$R_K^d = \sum_{i=1}^M \pi_i \sum_{j=1}^k b_{n_i k}^d(Y_{n_i}^j) = \sum_{i=1}^M \pi_i \sum_{j=1}^k \left(Y_{n_i}^j - \delta \frac{\alpha_{n_i k}}{\beta_{n_i k}} - \frac{k}{n_i \beta_{n_i k}} \right)$$

$$= \sum_{i=1}^M \pi_i \sum_{j=1}^k Y_{n_i}^j - k \sum_{i=1}^M \pi_i \left(\delta \frac{\alpha_{n_i k}}{\beta_{n_i k}} + \frac{k}{n_i \beta_{n_i k}} \right). \quad (B18)$$

To study the effect of revealing the number of bidders on auction revenues, we compare $\mathrm{E}[R^u]$ with $\mathrm{E}[R_K^u]$ and $\mathrm{E}[R^d]$ with $\mathrm{E}[R_K^d]$.

From (B14) and (B17),

$$\mathrm{E}[R_K^u] - \mathrm{E}[R^u] = \delta \left(k \frac{\sum_{i=1}^M \pi_i n_i \alpha_{n_i k}}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}} - k \sum_{i=1}^M \pi_i \frac{\alpha_{n_i k}}{\beta_{n_i k}} \right).$$

From (B15) and (B18),

$$\mathrm{E}[R_K^d] - \mathrm{E}[R^d]$$

$$= k \frac{\delta \sum_{i=1}^M \pi_i n_i \alpha_{n_i k} + k}{\sum_{i=1}^M \pi_i n_i \beta_{n_i k}} - k \sum_{i=1}^M \pi_i \left(\delta \frac{\alpha_{n_i k}}{\beta_{n_i k}} + \frac{k}{n_i \beta_{n_i k}} \right)$$

$$= (\mathrm{E}[R_K^u] - \mathrm{E}[R^u]) + k \sum_{i=1}^M \pi_i \frac{k \sum_{j=1}^M \pi_j}{\sum_{j=1}^M \pi_j n_j \beta_{n_j k}} - k \sum_{i=1}^M \pi_i \frac{k}{n_i \beta_{n_i k}}$$

$$= (\mathrm{E}[R_K^u] - \mathrm{E}[R^u]) - k^2 \sum_{i=1}^M \pi_i n_i \beta_{n_i k} \left(\frac{1}{n_i \beta_{n_i k}} - \frac{1}{\sum_{j=1}^M \pi_j n_j \beta_{n_j k}} \right)^2$$

$$\le \mathrm{E}[R_K^u] - \mathrm{E}[R^u]. \quad \square$$

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