Rev. Econ. Design (2006) 10:63–71
DOI 10.1007/s10058-006-0005-0

ORIGINAL PAPER

On the existence of an increasing symmetric equilibrium in $(k + 1)$-st price common value auctions

Ilia Tsetlin · Aleksandar Saša Pekeč

Received: 25 August 2005 / Accepted: 17 April 2006 /
Published online: 4 July 2006

Abstract In a classical result, Milgrom (1981a) established that the Monotone Likelihood Ratio Property (MLRP) is a sufficient condition for the existence of an increasing symmetric equilibrium in $(k + 1)$-st price common value auctions. We show: (1) If MLRP is violated, then for any number of bidders and objects there exists a distribution of the common value such that no increasing symmetric equilibrium exists; (2) If MLRP is violated, then for any distribution of the common value there exist infinitely many pairs of the number of bidders and the number of objects such that an increasing symmetric equilibrium does not exist; (3) There are examples where an increasing symmetric equilibrium exists even when the signal distribution violates MLRP.

Keywords Auctions · Symmetric equilibrium · Common value · Auction theory

JEL Classification Numbers D44 · C62 · C72 · D41

1 Introduction

Auction models provide an important tool for studying competitive markets. A critical property of an auction model is the existence of equilibrium. As Jackson and Swinkels (2005) note, "Much of what is known about existence of equilib-

I. Tsetlin(✉)
INSEAD, 1 Ayer Rajah Avenue, 138676 Singapore, Singapore
e-mail: ilia.tsetlin@insead.edu

A. S. Pekeč
The Fuqua School of Business, Duke University, Durham, NC 27708-0120, USA
e-mail: pekec@duke.edu

rium in auctions comes from ... relying on monotonicity arguments (e.g., Athey (2001) or Maskin and Riley (2000)).” Jackson (2005) shows that equilibria might fail to exist when bidders’ types are two-dimensional, i.e., when it is impossible to order bids of agents by their types.

Suppose that bidders’ types are one-dimensional. Would then an increasing symmetric equilibrium (where bids are increasing in bidders’ types) exist? We address this question in the standard common value auction model of Milgrom (1981a), where each bidder independently observes a signal about the common value. Milgrom (1981a) established that the Monotone Likelihood Ratio Property (MLRP) is a *sufficient* condition for the existence of an increasing symmetric equilibrium in $(k+1)$-st price common value auctions. We consider whether MLRP is a *necessary* condition for the existence of an increasing symmetric equilibrium.

Our results are the following: MLRP is indeed a necessary condition, in the sense that for any signal distribution that violates MLRP one can either choose a distribution of the common value or choose the number of bidders and objects so that no increasing symmetric equilibrium exists. However, we also provide examples where an increasing symmetric equilibrium exists even though the signal distribution violates MLRP.

2 The model

Consider a standard model of $(k+1)$-st price common value auctions. There are k identical objects for sale to n risk neutral bidders, with $1 \leq k < n$. Each object has the same common value V to all bidders, drawn from the distribution with the probability density function (hereafter, pdf) $g(v)$ and support $[\underline{v}, \overline{v}]$. Bidder $i, i = 1, \ldots, n$, independently observes signal X_i from a distribution with pdf $f(x|v)$, cumulative distribution function (hereafter, cdf) $F(x|v)$, and bids for one object. The objects are allocated to the k highest bidders and each of them pays the value of the $(k+1)$-st (highest rejected) bid. We assume that $g(v)$ is continuous and that fourth-order partial derivative of $f(x|v), f_{xvvv}(x|v)$, exists. The latter is a technical assumption used in the proof of Lemma 3.1 to create bounds via third-order Taylor expansion.

To derive a symmetric equilibrium, it is useful to take the point of view of one of the bidders, say bidder 1 with signal $X_1 = x$. Let Y^k_{n-1} denote the k-th order statistic from the set of the remaining $n-1$ signals. Assume that $b^* : \Re \to \Re$ is a (strictly) increasing symmetric equilibrium bidding function, so that $b^{*-1}()$ is well defined.

The following expression plays a key role in deriving an increasing symmetric equilibrium:

$$v(x, y) = E\left[V | X_1 = x, Y^k_{n-1} = y\right]. \tag{1}$$

If bidder 1 bids b, her expected payoff, conditional on observing signal $X_1 = x$, is

$$\int_{-\infty}^{b^{*-1}(b)} \big(v(x,y) - b^*(y)\big) f_{Y^k_{n-1}}(y|x)\mathrm{d}y, \tag{2}$$

where $f_{Y^k_{n-1}}(y|x)$ is the conditional density of Y^k_{n-1} given $X_1 = x$. If b^* is an increasing symmetric equilibrium, then bidder 1's conditional expected profit should be maximized by setting $b = b^*(x)$, which implies that the unique candidate for an increasing symmetric equilibrium is $b^*(x) = v(x,x)$. Expression (2) also implies two observations:

Observation 2.1 *If $v(x,y)$ is increasing in x and y for all x and y in the support of X_1, then $v(x,x)$ is the increasing symmetric equilibrium.*

Observation 2.2 *If $v(x,y)$ is decreasing in x at $x = y$ for some y in the support of X_1, then an increasing symmetric equilibrium does not exist.*

For Observation 2.2, note that the unique candidate for an increasing symmetric equilibrium, $b^*(x) = v(x,x)$, is not an equilibrium if $\int_{x-\varepsilon}^{x} (v(x,y) - v(y,y)) f_{Y^k_{n-1}}(y|x)\mathrm{d}y < 0$ for some ε. (Otherwise, bidder 1 with signal x would increase her conditional expected payoff (2) by bidding $v(x-\varepsilon, x-\varepsilon)$ instead of $v(x,x)$.) If $v(x,y)$ is decreasing in x at $x = y$, by continuity of $v(x,y)$ there exists $\varepsilon > 0$ such that $v(x,y) < v(y,y)$ for all y such that $x - \varepsilon < y < x$.

For further analysis, it is convenient to present (1) as [cf. Milgrom 1981a, eq. (3.5)]

$$v(x,y) = \frac{\int_{\underline{v}}^{\overline{v}} v f(x|v) g(v|y,n,k)\mathrm{d}v}{\int_{\underline{v}}^{\overline{v}} f(x|t) g(t|y,n,k)\mathrm{d}t}, \quad \text{where} \tag{3}$$

$$g(v|y,n,k) = \frac{f(y|v)(1 - F(y|v))^{k-1} F^{n-k-1}(y|v) g(v)}{\int_{\underline{v}}^{\overline{v}} f(y|t)(1 - F(y|t))^{k-1} F^{n-k-1}(y|t) g(t)\mathrm{d}t}. \tag{4}$$

Note that $g(v|y,n,k)$ defined above is the pdf of the posterior distribution of the common value conditional on the k-th order statistic out of $n-1$ signals.

3 Existence of an increasing symmetric equilibrium

By definition, $f(x|v)$ satisfies the Monotone Likelihood Ratio Property (MLRP) if

$$\frac{f(x|v)}{f(x|v')} \geq \frac{f(x|v)}{f(x|v')} \quad \text{for all } x > x', v > v'.$$

If $f(x|v)$ is twice differentiable, then MLRP is equivalent to (Milgrom and Weber 1982; Topkis 1978)

$$\frac{\partial^2 \ln f(x|v)}{\partial x \partial v} \geq 0 \quad \text{for all } x, v \text{ such that } f(x|v) > 0. \tag{5}$$

If (5) holds then $v(x,y)$ is increasing in both x and y, and in a $(k+1)$-st price common value auction there exists one (Milgrom 1981a) and only one (Pesendorfer and Swinkels 1997) symmetric equilibrium $b^*(x) = v(x,x)$. We explore the existence of an increasing symmetric equilibrium in the case where (5) is violated, i.e., when there exist x^*, v^*, such that $[\partial^2 \ln f(x|v)]/[\partial x \partial v]|_{x=x^*, v=v^*} < 0$, and $0 < F(x^*|v^*) < 1$.

Theorems 3.2 and 3.3 below provide two ways of constructing an auction setting with no increasing symmetric equilibrium. They rely on Lemma 3.1, proven in the Appendix. Lemma 3.1, which can be viewed as an alternative proof of Proposition 1 from Milgrom (1981b, p. 383), shows that if the prior distribution of the common value [i.e., $g(v)$] is concentrated around a point at which MLRP is violated (i.e., v^*), then the expected value of the objects conditional on the signal is decreasing (as a function of the signal x) at $x = x^*$.

Lemma 3.1 *Suppose the fourth-order partial derivative of $f(x|v)$, $f_{xvvv}(x|v)$, exists, and let $f(x|v)$ violate MLRP at x^*, v^*, i.e., $[\partial^2 \ln f(x|v)]/[\partial x \partial v]|_{x=x^*, v=v^*} < 0$. Consider a sequence of distributions such that along the sequence the expectation converges to v^*, variance converges to zero, and the absolute central third moment (i.e., $E[|V - E(V)|^3]$) converges to zero faster than variance. For a prior distribution with pdf $g(\cdot)$, denote $w\big(x; g(\cdot)\big) = \Big[\int_{\underline{v}}^{\overline{v}} v f(x|v) g(v) \mathrm{d}v\Big] / \Big[\int_{\underline{v}}^{\overline{v}} f(x|t) g(t) \mathrm{d}t\Big]$. Then, for $g(\cdot)$ with a small enough variance, $w\big(x; g(\cdot)\big)$ is decreasing in x for $x = x^*$.*

First we show that if MLRP (5) is violated then, for any number of bidders and objects, there exists a distribution of the common value such that an increasing symmetric equilibrium does not exist.

Theorem 3.2 *Let $f(x|v)$ violate MLRP at x^*, v^*, i.e., $[\partial^2 \ln f(x|v)]/[\partial x \partial v]|_{x=x^*, v=v^*} < 0$. Then for any k, n, $1 \leq k < n$, there exists $g(v)$ such that an auction for k objects with n bidders does not have an increasing symmetric equilibrium.*

Proof Consider a sequence of uniform densities $\{g_\varepsilon(v)\}_{\varepsilon \to 0}$ with supports $[v^* - \varepsilon, v^* + \varepsilon]$. As ε goes to zero, distribution of V with density $g_\varepsilon(v|x^*, n, k)$, defined by (4), converges to uniform distribution $[v^* - \varepsilon, v^* + \varepsilon]$ by continuity of $f(x^*|v)$ in v. Therefore, sequence of distributions with densities $\{g_\varepsilon(v|x^*, n, k)\}_{\varepsilon \to 0}$ satisfies conditions of Lemma 3.1. Note that $v(x, x^*) = w\big(x; g_\varepsilon(\cdot|x^*, n, k)\big)$, according to the notation from Lemma 3.1 applied to (3) with $g_\varepsilon(v|x^*, n, k)$. Then, by Lemma 3.1, $v(x, x^*)$ is decreasing in x at $x = x^*$ for small enough ε, and by Observation 2.2 an increasing symmetric equilibrium does not exist. □

Intuition behind Theorem 3.2 is rather straightforward. If prior distribution $g(v)$ is concentrated around a point at which MLRP is violated then, for any

k, n, posterior distribution conditional on k-th order statistic out of $n-1$ signals, given by (4), is also concentrated around this point v^*. Then by Lemma 3.1, the expected value of an object is decreasing in bidder 1's signal x, and by Observation 2.2 no increasing symmetric equilibrium exists.

We now show that if MLRP is violated, then for any given distribution of the common value there are infinitely many auction settings for which no increasing symmetric equilibrium exists. We will assume that, although MLRP is violated, higher signal values are more likely when the common value is higher: $F(x|v)$ is decreasing in v. For example, if $X_1 = V + Z$, where Z is some noise with cdf $H(z)$, then $F(x|v) = H(x - v)$ is decreasing in v.

Theorem 3.3 *Let $f(x|v)$ violate MLRP at x^*, v^*, i.e., $[\partial^2 \ln f(x|v)]/[\partial x \partial v]|_{x=x^*, v=v^*} < 0$, and let $\frac{\partial F(x^*|v)}{\partial v} < 0$ for all v. Then for any continuous $g(v)$, such that $g(v^*) > 0$, there exist infinitely many k, n pairs such that an auction for k objects with n bidders does not have an increasing symmetric equilibrium.*

Proof Let $s = F(x^*|v^*)$ and consider a sequence of auctions with $n \to \infty$ and $k/n \to 1 - s$, such that $k/n - (1 - s) = o(n^{-1/2})$. (For example, set $k = \lfloor n(1-s) \rfloor$.) Denote by $\xi_s(v)$ the population quantile of order s, given by equation $F(\xi_s(v)|v) = s = F(x^*|v^*)$.

Since $0 < s < 1$, Y^k_{n-1} is a central order statistic, and therefore its distribution, conditional on v, is asymptotically (as $n \to \infty$) normal with mean $\xi_s(v)$ and standard deviation $n^{-1/2}[s(1-s)]/[f(\xi_s(v)|v)]$. (David and Nagaraja (2003) state that result for asymptotic distribution of central order statistic in their Theorem 10.3, and also provide discussion and related references.) Denote by $\phi()$ the standard normal pdf. Then, as $n \to \infty, g(v|x^*, n, k)$, defined by (4), converges to

$$g(v|x^*, n, k) = \frac{\phi\left(n^{1/2}[(x^* - \xi_s(v))f(\xi_s(v)|v)]/[s(1-s)]\right) g(v)}{\int_{\underline{v}}^{\overline{v}} \phi\left(n^{1/2}[(x^* - \xi_s(t))f(\xi_s(t)|t)]/[s(1-s)]\right) g(t)\mathrm{d}t}. \tag{6}$$

Since $F(x|v)$ is decreasing in $v, \xi_s(v)$ is increasing in v. Since $\xi_s(v^*) = x^*$, expectation of distribution with pdf $g(v|x^*, n, k)$ given by (6) converges to v^*, variance converges to zero as n^{-1}, and absolute central third moment converges to zero as $n^{-3/2}$. Thus sequence of distributions with densities $\{g(v|x^*, n, k)\}_{n\to\infty}$ satisfies conditions of Lemma 3.1, and therefore $v(x, x^*) = w\big(x; g(\cdot|x^*, n, k)\big)$ is decreasing in x at $x = x^*$ for large enough n. By Observation 2.2, an increasing symmetric equilibrium does not exist. □

The intuition behind Theorem 3.3 is similar to the one behind Theorem 3.2. We can choose the number of objects k and the number of bidders n in such a way that the posterior distribution conditional on k-th order statistic out of $n - 1$ signals, given by (4), is concentrated around the point at which MLRP is violated. Similar to Theorem 3.2, Lemma 3.1 and Observation 2.2 imply no increasing symmetric equilibrium.

Remark 3.4 Note that if $[\partial^2 \ln f(x|v)]/[\partial x \partial v]|_{x=\hat{x},v=\hat{v}} \geq 0$ for some $\hat{x}, \hat{v}$, Theorem 3.3 does not imply that an increasing symmetric equilibrium exists for large enough n in a sequence of auctions with $n \to \infty$ and $k/n \to 1 - F(\hat{x}|\hat{v})$. This is because $v(x, \hat{x})$ being increasing in x at $x = \hat{x}$ is a necessary (Observation 2.2), but not a sufficient (as Observation 2.1) condition for the existence of an increasing symmetric equilibrium.

Theorems 3.2 and 3.3 rely on a setting where conditional distribution given by (4) is concentrated around a particular point v^*, and do not rule out the possibility of an auction model in which an increasing symmetric equilibrium exists in spite of a signal distribution that violates MLRP. Such a model would require that prior distribution is not concentrated around a point at which MLRP is violated (so that Theorem 3.2 does not apply) and that the number of bidders and the number of objects are such that Theorem 3.3 does not apply. Theorem 3.5 below provides an example with one object and two bidders. Remark 3.6 illustrates that point further, using the notion of a diffuse prior distribution.

Theorem 3.5 *There exist auction settings in which MLRP is violated, but an increasing symmetric equilibrium exists.*

Proof Consider $f(x|v) = (1/4) + (3/4)(x - v)^2, v - 1 < x < v + 1$. For that distribution, $[\partial^2 \ln f(x|v)]/[\partial x \partial v] = 6[6(x-v)^2-1]/[\left(1 + 3(x - v)^2\right)^2]$, so MLRP (5) is violated for $|x^* - v^*| < 1/\sqrt{6}$. Let $k = 1$ and $n = 2$, and consider $g(v) = (1/2)\alpha e^{-\alpha|v|}, \alpha > 0$. As α goes to zero, $v(x, y)$, given by (3), uniformly converges to $(x + y)/2$. Then, by Observation 2.1, for a small enough α an increasing symmetric equilibrium exists. □

Remark 3.6 To understand the intuition behind the example in Theorem 3.5, consider the following abstract setting. Let $f(x|v) = h(x - v)$, where $h(t)$ is pdf and is symmetric about zero, i.e., $h(t) = h(-t)$ for all t. Let $g(v)$ be diffuse relative to the signal distribution (this setting is used in Klemperer 1999, Winkler and Brooks 1980, Levin and Smith 1991). Let $k = 1$ and $n = 2$. Then $g(v|y, n, k)$, given by (4), becomes $g(v|y, n, k) = [h(y - v)]/[\int_{-\infty}^{+\infty} h(y - t)\mathrm{d}t] = h(y - v)$, and $v(x, y)$, given by (3), becomes $v(x, y) = \left[\int_{-\infty}^{+\infty} vh(x - v)h(y - v)\mathrm{d}v\right] / \left[\int_{-\infty}^{+\infty} h(x - t)h(y - t)\mathrm{d}t\right]$. Since $h(t)$ is symmetric around zero, $v(x, y) = (x + y)/2$. Therefore, in this setting the increasing symmetric equilibrium exists for all signal distributions with pdf of the form $h(x - v)$, including the distributions that violate MLRP, as in Theorem 3.5.

4 Conclusions

We have shown that if a signal distribution violates MLRP, then for any number of bidders and objects one can choose a distribution of the common value so that the $(k + 1)$-st price common-value auction model does not have an increasing symmetric equilibrium (Theorem 3.2). Furthermore, for any distribution of the common value one can choose the number of bidders and the number of objects so that an increasing symmetric equilibrium does not exist (Theorem 3.3). In that sense, MLRP is a necessary condition for the existence of an increasing symmetric equilibrium. However, there are examples where an increasing symmetric equilibrium exists even when the signal distribution violates MLRP (Theorem 3.5).

5 Appendix

Proof of Lemma 3.1 Consider the distribution with pdf $g(v)$, denote its expectation by v_0, variance by σ^2, and the absolute central third moment by M. Consider the first derivative of $w\big(x; g(\cdot)\big)$ with respect to x.

$$w'_x\big(x; g(\cdot)\big) = \left(\frac{\int_{\underline{v}}^{\overline{v}} v f(x|v)g(v)\mathrm{d}v}{\int_{\underline{v}}^{\overline{v}} f(x|v)g(v)\mathrm{d}v}\right)'_x$$
$$= \frac{\int_{\underline{v}}^{\overline{v}} v f_x(x|v)g(v)\mathrm{d}v \int_{\underline{v}}^{\overline{v}} f(x|v)g(v)\mathrm{d}v - \int_{\underline{v}}^{\overline{v}} v f(x|v)g(v)\mathrm{d}v \int_{\underline{v}}^{\overline{v}} f_x(x|v)g(v)\mathrm{d}v}{\left(\int_{\underline{v}}^{\overline{v}} f(x|v)g(v)\mathrm{d}v\right)^2}. \quad (7)$$

By third-order Taylor expansion around v_0, there exists $z(v)$, $|z(v) - v_0| < |v - v_0|$, such that $f(x|v) = f(x|v_0) + f_v(x|v_0)(v - v_0) + (1/2)f_{vv}(x|v_0)(v - v_0)^2 + (1/6)f_{vvv}(x|z(v))(v - v_0)^3$. Denote $A = \max_v(f_{vvv}(x|z(v)))$. Observe that

$$\left|\int_{\underline{v}}^{\overline{v}} f_{vvv}(x|z(v))(v - v_0)^3 g(v)\mathrm{d}v\right| \le A \int_{\underline{v}}^{\overline{v}} |v - v_0|^3 g(v)\mathrm{d}v = A * M = O(M),$$

where $\limsup_{t\to 0} |O(t)/t| < \infty$. Therefore,

$$\int_{\underline{v}}^{\overline{v}} f(x|v)g(v)\mathrm{d}v = f(x|v_0) + \frac{1}{2} f_{vv}(x|v_0)\sigma^2 + O(M).$$

By the assumption, $M = o(\sigma^2)$, where $\lim_{t\to 0} o(t)/t = 0$. Therefore,

$$\int_{\underline{v}}^{\overline{v}} f(x|v)g(v)\mathrm{d}v = f(x|v_0) + \frac{1}{2} f_{vv}(x|v_0)\sigma^2 + o(\sigma^2).$$

Similarly,

$$\int_{\underline{v}}^{\overline{v}} f_x(x|v)g(v)\mathrm{d}v = f_x(x|v_0) + \tfrac{1}{2} f_{xvv}(x|v_0)\sigma^2 + o(\sigma^2),$$

$$\int_{\underline{v}}^{\overline{v}} vf(x|v)g(v)\mathrm{d}v = v_0 f(x|v_0) + \tfrac{1}{2}\left[2f_v(x|v_0) + v_0 f_{vv}(x|v_0)\right]\sigma^2 + o(\sigma^2),$$

$$\int_{\underline{v}}^{\overline{v}} vf_x(x|v)g(v)\mathrm{d}v = v_0 f_x(x|v_0) + \tfrac{1}{2}\left[2f_{xv}(x|v_0) + v_0 f_{xvv}(x|v_0)\right]\sigma^2 + o(\sigma^2).$$

Therefore the denominator in (7) is $f^2(x|v_0) + O(\sigma^2)$. The numerator is (for brevity, below we denote $f(x|v_0)$ by f)

$$\begin{aligned}
&\left(v_0 f_x + \frac{1}{2}\left[2f_{xv} + v_0 f_{xvv}\right]\sigma^2\right)\left(f + \frac{1}{2} f_{vv}\sigma^2\right) \\
&\quad -\left(v_0 f + \frac{1}{2}\left[2f_v + v_0 f_{vv}\right]\sigma^2\right)\left(f_x + \frac{1}{2} f_{xvv}\sigma^2\right) + o(\sigma^2) \\
&= \frac{1}{2}\Big(\left[2f_{xv} + v_0 f_{xvv}\right] f + v_0 f_x f_{vv} - \left[2f_v + v_0 f_{vv}\right] f_x - v_0 f f_{xvv}\Big)\sigma^2 + o(\sigma^2) \\
&= \Big(f_{xv} f - f_v f_x\Big)\sigma^2 + o(\sigma^2).
\end{aligned}$$

Therefore, (7) equals to

$$\frac{f_{xv}(x|v_0)f(x|v_0) - f_v(x|v_0)f_x(x|v_0)}{f^2(x|v_0)}\sigma^2 + o(\sigma^2) = \frac{\partial^2 \ln f(x|v_0)}{\partial x \partial v}\sigma^2 + o(\sigma^2).$$

By the assumption, $[\partial^2 \ln f(x|v)]/[\partial x \partial v]$ is continuous in v, so $[\partial^2 \ln f(x^*|v_0)]/[\partial x \partial v] < 0$ if v_0 is close enough to v^*. Therefore, for small enough variance $\sigma^2, w\big(x; g(\cdot)\big)$ is decreasing in x for $x = x^*$. □

Acknowledgements We are grateful to Matthew Jackson, Paul Klemperer, Claudio Mezzetti, Paul Milgrom, and Jeroen Swinkels for helpful comments.

References

Athey S (2001) Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica 69: 861–890

David H, Nagaraja H (2003) Order statistics, 3rd edn. Wiley, New York

Jackson M (2005) Non-existence of equilibrium in Vickrey, second-price, and English auctions. California Institute of Technology, Mimeo

Jackson M, Swinkels J (2005) Existence of equilibrium in single and double private value auctions. Econometrica 73: 93–139

Klemperer P (1999) Auction theory: a guide to the literature. J Econ Surveys 13: 227–286

Levin D, Smith JL (1991) Some evidence on the winner's curse: Comment. Am Econ Rev 81: 370–375
Maskin E, Riley J (2000) Equilibrium in sealed high bid auctions. Rev Econ Stud 67: 439–452
Milgrom PR (1981a) Rational expectations, information acquisition, and competitive bidding. Econometrica 49: 921–943
Milgrom PR (1981b) Good news and bad news: representation theorems and applications. Bell J Econ 12: 380–391
Milgrom PR, Weber RJ (1982) A theory of auctions and competitive bidding. Econometrica 50: 1089–1122
Pesendorfer W, Swinkels JM (1997) The loser's curse and information aggregation in common value auctions. Econometrica 65: 1247–1281
Topkis DM (1978) Minimizing a submodular function on a lattice. Oper Res 26: 305–321
Winkler RL, Brooks DG (1980) Competitive bidding with dependent value estimates. Oper Res 28: 603–613