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Multiattribute Utility Satisfying a Preference for Combining Good with Bad

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An important challenge in multiattribute decision analysis is the choice of an appropriate functional form for the utility function. We show that if a decision maker prefers more of any attribute to less and prefers to combine good lotteries with bad, as opposed to combining good with good and bad with bad, her utility function should be a weighted average (a mixture) of multiattribute exponential utilities ("mixex utility"). In the single-attribute case, mixex utility satisfies properties typically thought to be desirable and encompasses most utility functions commonly used in decision analysis. In the multiattribute case, mixex utility implies aversion to any multivariate risk. Risk aversion with respect to any attribute decreases as that attribute increases. Under certain restrictions, such risk aversion also decreases as any other attribute increases, and a multivariate one-switch property is satisfied. One of the strengths of mixex utility is its ability to represent cases where utility independence does not hold, but mixex utility can be consistent with mutual utility independence and take on a multilinear form. An example illustrates the fitting of mixex utility to preference assessments.

Key words: multiattribute utility; combining good with bad; mixex utility; risk aversion; multivariate one-switch property; sumex utility; multivariate Laplace transform
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1. Introduction

A group of senior managers has worked hard to define the important attributes of interest in a complex decision-making problem. One of the managers, who once took a decision analysis course in an MBA program, suggests that they attempt to assess utility functions for each of the attributes. Working in fits and starts, they are able to accomplish this task with reasonable success. However, they find it difficult to think about their preferences for multiattribute combinations. Finally, someone proposes taking a weighted average of the single-attribute utility functions, and the others agree. They manage to assess weights to come up with their overall utility function. Using an additive utility approach like this is a common default option, but is it likely to be a good representation of the managers' preferences?

In multiattribute decision analysis, assessing a utility function and representing it with an appropriate functional form can be much more complex than in the case of a single attribute. One must consider not just preferences regarding each attribute individually, but also preferences involving trade-offs among attributes. Some conditions constraining the form of the multiattribute utility function are needed to simplify the process. Keeney and Raiffa (1976) moved the field forward tremendously in this regard, largely by delineating different types of utility independence and studying their implications for the form of the utility function. These types include additive independence as well as other conditions (e.g., mutual utility independence) that allow for a richer class of utility functions.

The objective of this paper is to present an alternative framework for choosing a functional form for a multiattribute utility function, using a basic preference condition. The main result is that if a decision maker prefers more of any attribute to less and prefers to combine good lotteries with bad lotteries as opposed to combining good lotteries with good and bad lotteries with bad, then her utility function should be a weighted average (a mixture) of multiattribute exponential utilities, which we call "mixex utility." The preference for combining good lotteries with bad lotteries in the multivariate case can be thought of as a type of risk aversion, so it is similar in spirit to the assumption of risk aversion that is frequently used to constrain the form of single-attribute utility functions. A preference for combining good with bad is related to risk apportionment, studied in the univariate case by Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009), and to concepts such as bivariate risk aversion and correlation aversion (Epstein and Tanny 1980, Scarsini 1988, Eeckhoudt et al. 2007, Denuit et al. 2008) and multivariate risk aversion (Richard 1975). Of course, decision makers will not always prefer to combine good lotteries with bad, just as they will not always have preferences consistent with risk aversion for a single attribute. Combining good lotteries with bad is a stronger condition, encompassing single-attribute risk aversion and going beyond it. Nonetheless, the condition seems appealing enough to have reasonably widespread applicability.

Mixex utility comprises a rich class of functions that satisfy many properties often thought to be desirable. In the single-attribute case, these properties include risk aversion, decreasing risk aversion, prudence, and temperance. By definition, exponential and sumex utility functions belong to the mixex family, which also includes logarithmic and power utility (Brockett and Golden 1987). In this paper we show that mixex utility is rich and satisfies appealing properties in the multivariate case as well. Interestingly, the commonly used additive utility is inconsistent with a strict preference for combining good lotteries with bad lotteries that involve two or more attributes, implying indifference instead, and is therefore not mixex except in a weak limiting sense.

This paper is organized as follows. In §2, we illustrate utility implications of preferring to combine good lotteries with bad, considering the cases of one and two attributes. We formalize these ideas in the N-attribute case in §3, showing that our basic preference condition is related to alternating signs of partial derivatives of successive orders for the utility function, which is consistent with a preference for combining good with bad if and only if utility is of the mixex form. Some properties of mixex utility and connections with utility independence are explored in §4. An advantage of mixex is its ability to represent cases where preferences do not satisfy utility independence. We present an illustration of fitting mixex utility to preference assessments in §5, followed by concluding comments in §6.

2. Combining Good with Bad vs. Good with Good and Bad with Bad

In this section, we discuss and illustrate the notion of preferring to combine good lotteries with bad ones as opposed to combining good lotteries with good ones and bad lotteries with bad ones. Of particular interest are the implications of this type of preference for an individual's utility function. We begin with the single-attribute case and then consider the two-attribute case, setting the stage for the general development in §3. Each attribute is assumed to be defined so that more of the attribute is preferred to less.

First, suppose that wealth, denoted by x, is the only attribute of interest, with nondecreasing utility for wealth: $u'(x) \geq 0$. That means, for example, that $200 \succsim 100$, where $\succsim$ represents a preference ordering. Let $[A, B]$ denote a lottery with equal chances of getting A or B, and consider $[100, 200]$. Here 200 is good and 100 is bad because $200 \succsim 100$; good and bad are relative terms in this context. Suppose that we combine two such lotteries by adding outcomes. This can be done by combining good with good and bad with bad, yielding $[200, 400]$, or by combining good with bad, yielding $[300, 300]$, which is 300 for sure. For the choice between 300 for sure and an uncertain outcome with a mean of 300, a preference for combining good with bad is consistent with risk aversion: $u''(x) \leq 0$.

Now we can say that 300 is good and $[200, 400]$ is bad. Combining these two lotteries with an independent $[100, 200]$ lottery gives $[[300, 500], 500]$ if we combine good with good and bad with bad, and $[[400, 600], 400]$ if we combine good with bad. Combining good with bad is preferred if $0.75u(400) + 0.25u(600) > 0.25u(300) + 0.75u(500)$, or $[u(600) - 2u(500) + u(400)] - [u(500) - 2u(400) + u(300)] > 0$. This holds if $d[u(x+100) - 2u(x) + u(x-100)]/dx = u'(x+100) - 2u'(x) + u'(x-100) > 0$, which is consistent with convex u': $u'''(x) \geq 0$. This condition is called prudence (Kimball 1990). If we carried this out yet another step, we would find that a preference for combining good with bad is consistent with $u''''(x) \leq 0$, which is called temperance (Kimball 1992, Gollier and Pratt 1996).

This recursive approach shows that a preference for combining good with bad is consistent with alternating signs for successive derivatives of u and with often-assumed properties such as nondecreasing utility, risk aversion, and prudence. Eeckhoudt and Schlesinger (2006) develop a similar notion, risk apportionment, adding "basic harms" (a negative amount or a zero-mean risk) at each step and assuming a preference for adding one harm to each outcome in a 50–50 lottery over adding both harms to one outcome and no harm to the other.

Next, suppose that an MBA student is interviewing for jobs and has a utility function $u(x_1, x_2)$ over two attributes: a financial attribute measured by income over a fixed time horizon (x_1) and a quality of life attribute measured in terms of the amount of time available to spend with family (x_2). On each dimension, assume that more is preferred to less, holding the other attribute constant. Letting $u_i(x_1, x_2) = \partial u(x_1, x_2)/\partial x_i$, $u_{ij}(x_1, x_2) = \partial^2 u(x_1, x_2)/\partial x_i \partial x_j$, and so on, $u_i \geq 0$ for $i = 1, 2$. Thus, $(x_1 + c, x_2) \succsim (x_1, x_2)$ and $(x_1, x_2 + d) \succsim (x_1, x_2)$ for any $c, d > 0$. A preference for combining good with bad means that $[(x_1 + c, x_2), (x_1, x_2 + d)] \succsim [(x_1 + c, x_2 + d), (x_1, x_2)]$. Expressing this in terms of utilities and rearranging,

$u(x_1+c, x_2)-u(x_1, x_2) \geq u(x_1+c, x_2+d)-u(x_1, x_2+d)$, which in turn implies that $u_1(x_1, x_2) - u_1(x_1, x_2+d) \geq 0$, or $u_{12}(x_1, x_2) \leq 0$. Similarly, we can show that a preference for combining good with bad implies that $u_{ii} \leq 0$ for $i = 1, 2$: the student is risk averse for each attribute.

In the two-attribute case, a preference for combining good with bad is therefore consistent not just with $u_{11}(x_1, x_2) \leq 0$ and $u_{22}(x_1, x_2) \leq 0$ (risk aversion on each attribute individually), but also with $u_{12}(x_1, x_2) \leq 0$. A bivariate utility function exhibiting this nonpositive cross-partial derivative everywhere is called multivariate risk averse by Richard (1975). The "risk averse" terminology makes sense when we consider that $(x_1+c, x_2+d) \succsim (x_1+c, x_2) \succsim (x_1, x_2)$ and $(x_1+c, x_2+d) \succsim (x_1, x_2+d) \succsim (x_1, x_2)$. Combining good with good and bad with bad leads to a lottery with the extreme outcomes, whereas combining good with bad leads to a lottery with the intermediate outcomes. Note that the preference between the two intermediate outcomes cannot be determined from our preference assumptions and does not matter for our combination of lotteries. Another apt label for utility with $u_{12}(x_1, x_2) \leq 0$ everywhere is correlation averse (Epstein and Tanny 1980, Eeckhoudt et al. 2007).

As in the single-attribute case, a preference for combining good with bad in the two-attribute case is consistent with alternating signs for successive partial derivatives. First-order partials are nonnegative by the definition of the attributes, and we have illustrated the nonpositive nature of second-order partials. This extends to higher orders, with all third-order partials of u being nonnegative and all fourth-order partials being nonpositive. Eeckhoudt et al. (2007) introduce the notions of cross-prudence (nonnegative third-order cross-partials) and cross-temperance (nonpositive fourth-order cross-partials).

3. Implications for Utility: Alternating Signs for Partial Derivatives and Mixtures of Exponential Utilities

Now we formally extend the ideas of §2 to the case where N real-valued attributes $x_1, \ldots, x_N$ are of interest, and the utility function $u(\mathbf{x}) = u(x_1, \ldots, x_N)$ is a real-valued function over these N attributes. First, the connection between a preference for combining good with bad and alternating signs for successive partial derivatives of u is formalized. Next, we show that the signs for successive partial derivatives of u alternate if and only if u is a mixture of multivariate exponential utilities.

We begin by defining some notation. A random vector is denoted by a tilde, $\tilde{\mathbf{x}}$, and $\mathbf{0}$ is a vector of zeroes. For two vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} \gneqq \mathbf{y}$ if $x_j \geq y_j$ for all j and $\mathbf{x} \neq \mathbf{y}$. Also, $\mathbf{x}+\mathbf{y}$ denotes the componentwise sum, $(x_1+y_1, \ldots, x_N+y_N)$. As in §2, we let $[\mathbf{x}, \mathbf{y}]$ denote a lottery with equal chances of getting $\mathbf{x}$ or $\mathbf{y}$.

To examine the implications of a preference for combining good with bad over combining good with good and bad with bad, hereafter simplified to "a preference for combining good with bad," we need to be more precise about the notions of "good" and "bad." We assume that each attribute is defined such that more of the attribute is preferred to less, and we let $(\mathbf{a}_i, \mathbf{b}_i)$ be a series of n pairs of any N-dimensional vectors with $\mathbf{a}_i \gneqq \mathbf{b}_i$, $i = 1, \ldots, n$. Then we combine these vectors to create a set of recursive pairs of good and bad lotteries, $(\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i)$, $i = 1, \ldots, n$, assuming a preference for combining good with bad.

We start by letting $\mathbf{x}_1 = \mathbf{a}_1$ and $\mathbf{y}_1 = \mathbf{b}_1$. Then $\mathbf{x}_1$ is better than $\mathbf{y}_1$ because more of each attribute is preferred to less. In considering the pair $(\mathbf{x}_1, \mathbf{y}_1)$, we call $\mathbf{x}_1$ good and $\mathbf{y}_1$ bad at Level 1, denoted by $\mathbf{x}_1 \succsim_{L1} \mathbf{y}_1$. Now we combine $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{a}_2, \mathbf{b}_2)$ to define

$$\tilde{\mathbf{x}}_2 = [\mathbf{x}_1+\mathbf{b}_2, \mathbf{y}_1+\mathbf{a}_2] \quad \text{and} \quad \tilde{\mathbf{y}}_2 = [\mathbf{x}_1+\mathbf{a}_2, \mathbf{y}_1+\mathbf{b}_2]. \tag{1}$$

In $(\mathbf{a}_2, \mathbf{b}_2)$, $\mathbf{a}_2$ is good and $\mathbf{b}_2$ is bad because $\mathbf{a}_2 \gneqq \mathbf{b}_2$. Thus, $\tilde{\mathbf{x}}_2$ combines good with bad, whereas $\tilde{\mathbf{y}}_2$ combines good with good and bad with bad. From a preference for combining good with bad, we call $\tilde{\mathbf{x}}_2$ good and $\tilde{\mathbf{y}}_2$ bad at Level 2, denoted by $\tilde{\mathbf{x}}_2 \succsim_{L2} \tilde{\mathbf{y}}_2$. This is a comparison of random vectors, and we call it a Level 2 comparison because it involves combining two Level 1 comparisons.

Proceeding recursively, we define

$$\tilde{\mathbf{x}}_i = [\tilde{\mathbf{x}}_{i-1}+\mathbf{b}_i, \tilde{\mathbf{y}}_{i-1}+\mathbf{a}_i] \quad \text{and} \quad \tilde{\mathbf{y}}_i = [\tilde{\mathbf{x}}_{i-1}+\mathbf{a}_i, \tilde{\mathbf{y}}_{i-1}+\mathbf{b}_i] \quad \text{for } i = 3, \ldots, n. \tag{2}$$

For each pair $(\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i)$, $\tilde{\mathbf{x}}_i$ combines good with bad and $\tilde{\mathbf{y}}_i$ combines good with good and bad with bad. Moreover, $(\tilde{\mathbf{x}}_i, \tilde{\mathbf{y}}_i)$ involves combining a Level $i-1$ comparison with a Level 1 comparison, so we call $\tilde{\mathbf{x}}_i$ good and $\tilde{\mathbf{y}}_i$ bad at Level $(i-1)+1 = i$, denoted by $\tilde{\mathbf{x}}_i \succsim_{Li} \tilde{\mathbf{y}}_i$. We use these recursive lotteries to define a preference for combining good with bad.

Definition 1. A decision maker prefers to combine good with bad up to Level n if, for lotteries $\tilde{\mathbf{x}}_i$ and $\tilde{\mathbf{y}}_i$ based on any $\mathbf{a}_i \gneqq \mathbf{b}_i$, $i = 1, \ldots, n$, as defined recursively in (1) and (2) starting with $\mathbf{x}_1 = \mathbf{a}_1$ and $\mathbf{y}_1 = \mathbf{b}_1$, she prefers $\tilde{\mathbf{x}}_i$ to $\tilde{\mathbf{y}}_i$.

Next, we formalize the notion of alternating signs for partial derivatives of u:

Definition 2. $\mathbb{U}_n^N = \{u \mid (-1)^{k-1}\partial^k u(\mathbf{x})/\partial x_{i_1} \cdots \partial x_{i_k} \geq 0$ for $k = 1, \ldots, n$ and any $i_j \in \{1, \ldots, N\}, j = 1, \ldots, k\}$.

$\mathbb{U}_n^N$ consists of all N-dimensional real-valued functions for which all partial derivatives of a given order up to order n have the same sign, and that sign alternates, being positive for odd orders and negative for even orders. For the single-attribute case, $\mathbb{U}_n^1$ corresponds to all u satisfying risk apportionment of order n (Eeckhoudt and Schlesinger 2006). Note that if $u \in \mathbb{U}_n^N$, then $u \in \mathbb{U}_k^N$ for any $k < n$. Also, if $u \in \mathbb{U}_n^N$, then for any $k < n$ and $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$, $(-1)^k \partial^k u(\mathbf{x})/\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k} \in \mathbb{U}_{n-k}^N$. Now we are ready to state the first main result of this section. All proofs are given in the appendix.

Theorem 1. *A decision maker with a utility function $u(\mathbf{x})$ that is differentiable up to order n prefers to combine good with bad up to Level n if and only if $u \in \mathbb{U}_n^N$.*

Theorem 1 formalizes the connection between our preference assumptions and $\mathbb{U}_n^N$. The essence of the theorem is that an individual who wants to be consistent with preferring more of each attribute to less and with preferring to combine good with bad should have a utility function with alternating signs for successive partial derivatives. Note that we developed good and bad lotteries for Level i by combining good and bad lotteries for Level $i-1$ with good and bad lotteries for Level 1. The results would be the same if, for any $j \in \{1, \ldots, i-1\}$, we combined good and bad lotteries for Level $i-j$ with good and bad lotteries for Level j. This is because the operator defined by (2) for combining good and bad lotteries of different levels satisfies associativity.

Having built the connection between our preference assumptions and alternating signs for partial derivatives of the multiattribute utility function, we can extend the alternating-signs condition to hold for any n in Definition 3 and extend Theorem 1 accordingly in Corollary 1. Then, in Definition 4, we extend the definition of completely monotone utility (Feller 1971) to the multiattribute case, which in turn will enable us to connect our preference assumptions to mixtures of multiattribute utilities.

Definition 3. $\mathbb{U}_\infty^N = \{u \mid (-1)^{k-1} \partial^k u(\mathbf{x})/\partial x_{i_1} \cdots \partial x_{i_k} \geq 0$ for $k = 1, 2, \ldots,$ and any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k\}$.

Corollary 1. *A decision maker with a utility function $u(\mathbf{x})$ that is infinitely differentiable prefers to combine good with bad for any Level $n = 1, 2, \ldots$ if and only if $u \in \mathbb{U}_\infty^N$.*

Remark 1. Alternatively, we could define $\mathbb{U}_n^N$ and $\mathbb{U}_\infty^N$ via difference operators (e.g., Definition 3.3.14 in Müller and Stoyan 2002). In this case, we would not need to assume differentiability of u in Theorem 1 and Corollary 1.

Definition 4. A function $\varphi(\mathbf{x})$ on $(\mathbf{0}, \infty)$ is multivariate completely monotone if it possesses partial derivatives of all orders and $(-1)^k \partial^k \varphi(\mathbf{x})/\partial x_{i_1} \cdots \partial x_{i_k} \geq 0$ for $k = 0, 1, \ldots$ and any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$.

A univariate completely monotone function can be expressed as a Laplace transform, and therefore as a mixture of exponential functions (Feller 1971). Some implications of this result for utility theory are discussed in Brockett and Golden (1987), Pratt and Zeckhauser (1987), and Caballe and Pomansky (1996). Theorem 2 extends the result to the multivariate case.

Theorem 2. *A function $\varphi(\mathbf{x})$ on $(\mathbf{0}, \infty)$ is multivariate completely monotone if and only if it is the N-variate Laplace transform of a (not necessarily finite) measure F on $[\mathbf{0}, \infty)$:*

$$\varphi(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty e^{-(r_1 x_1 + \cdots + r_N x_N)}\, dF(r_1, \ldots, r_N).$$

Remark 2. If $u \in \mathbb{U}_\infty^N$ for $\mathbf{x} > \underline{\mathbf{x}}$, then $\partial u(\mathbf{x} - \underline{\mathbf{x}})/\partial \mathbf{x_i}$ is multivariate completely monotone for $i = 1, \ldots, N$. If, in addition, $u(\mathbf{x}) \leq 0$ for $\mathbf{x} > \underline{\mathbf{x}}$, then $-u(\mathbf{x} - \underline{\mathbf{x}})$ is multivariate completely monotone.

We now can relate a preference for combining good with bad to multiattribute exponential utility. Building on a result developed for the univariate case by Brockett and Golden (1987), we show that a utility function $u(\mathbf{x})$ satisfies a preference for combining good with bad (i.e., $u \in \mathbb{U}_\infty^N$) if and only if it is a mixture of multiattribute exponential utilities.

Theorem 3. *Consider a function $u(\mathbf{x})$, $\mathbf{x} > \underline{\mathbf{x}}$, and fix any point $\mathbf{x}^* = (\mathbf{x}_1^*, \ldots, x_N^*)$, $x^* > \underline{\mathbf{x}}$. Then $u(x) \in \mathbb{U}_\infty^N$ if and only if there exists a (not necessarily finite) measure F on $[\mathbf{0}, \infty)$ and constants $b_1, \ldots, b_N$ with $b_i \geq 0$, $i = 1, \ldots, N$, such that*

$$\begin{aligned} u(\mathbf{x}) = {} & u(\mathbf{x}^*) \\ & + \int_0^\infty \cdots \int_0^\infty \left(1 - e^{-(r_1(x_1 - x_1^*) + \cdots + r_N(x_N - x_N^*))}\right) dF(r_1, \ldots, r_N) \\ & + \sum_{i=1}^N b_i (x_i - x_i^*). \end{aligned} \tag{3}$$

The linear terms in (3) reflect the situation where $r_i \to 0$ with $r_j = 0$ for $j \neq i$. In this case, $b_i(1 - e^{-(r_i(x_i - x_i^*))})/r_i \to b_i(x_i - x_i^*)$ as $r_i \to 0$, and b_i is the coefficient of this linear term. Thus, viewing the linear terms as limiting forms of exponential utilities, we call (3) a mixture of multiattribute exponential utilities. Under certain reasonable conditions, as illustrated in Remark 3, (3) can be simplified in the sense that the linear terms disappear (i.e., $b_i = 0$, $i = 1, \ldots, N$), and the proof that $u(x)$ satisfies a preference for combining good with bad if and only if it is a mixture of multiattribute exponential utilities is also simpler.

Remark 3. If $u(\mathbf{x}) \in \mathbb{U}_\infty^N$, $x > \underline{\mathbf{x}}$, is bounded from above, it can be rescaled such that $u(\mathbf{x}) \leq 0$ for $\mathbf{x} > \underline{\mathbf{x}}$. Then, $-u(\mathbf{x} - \underline{\mathbf{x}})$ is multivariate completely monotone by Remark 2 and is the multivariate Laplace transform of a measure F on $[\mathbf{0}, \infty)$ by Theorem 2.

In terms of the application of mixex utility, the assumption that u is bounded is not restrictive for practical purposes, and we will also assume that the mixing distribution is a proper probability distribution. Moreover, it is convenient to rescale $u(\mathbf{x})$ so that $u(\mathbf{0}) = 0$ and $u(\mathbf{x}) \leq 1$. Thus, we will express mixex utility in the form

$$u(x) = 1 - \int_0^\infty \cdots \int_0^\infty e^{-(r_1 x_1 + \cdots + r_N x_N)}\, dF(r_1, \ldots, r_N), \quad (4)$$

where F is a cumulative distribution function. F can be continuous or discrete (or both), and the corresponding density or mass function will be denoted by f. In §4 we present some results concerning mixex utility and discuss its properties.

4. Mixex Utility

To interpret mixex utility, it is helpful to start by considering the univariate case. A utility function $u(x) = 1 - e^{-rx}$ with $r > 0$ is the standard risk-averse exponential utility used often in decision analysis, with constant risk aversion $r(x) = -u''(x)/u'(x) = r$. The limiting case as r approaches zero corresponds to linear utility. Mixex utility includes all of these utility functions, because the mixing distribution F can be degenerate, placing probability one on some r. When the mixing distribution is not degenerate, a natural interpretation is to view it as representing uncertainty about the risk-aversion coefficient. However, we should be careful about this interpretation, because F is really just a convenient and useful modeling device to develop a utility function corresponding to the types of preferences assumed in §§3 and 4.

The mixex form can also be thought of as extending exponential utility to provide more flexibility in utility modeling. Among other things, when F is not degenerate, mixex does not inherit the constant risk aversion of exponential utility. Instead, it exhibits decreasing risk aversion as well as prudence and temperance. As Brockett and Golden (1987) note, mixex includes logarithmic and risk-averse power utility. It also includes sumex utility and linear plus exponential utility (corresponding to two-point mixing distributions, with the latter a limiting case of the former), the only two increasing, risk-averse, and decreasingly risk-averse forms to satisfy the one-switch property (Bell 1988, Bell and Fishburn 2001). In addition to providing such flexibility for practical applications, mixex restricts the class of utility functions to a form that can often be used to generate analytical results for modeling purposes.

The risk aversion $r(x)$ for mixex utility with $N = 1$ satisfies $r_{\min} \leq r(x) \leq r_{\max}$, where $r_{\min}$ and $r_{\max}$ are the lower and upper bounds of the support of F. Theorem 4 shows the limiting behavior of $r(x)$.

Theorem 4. *If $r_{\min}$ and $r_{\max}$ are the lower and upper bounds of the support of F, then $r(x) \to r_{\max}$ as $x \to -\infty$ and $r(x) \to r_{\min}$ as $x \to \infty$.*

Thus, for large positive (negative) values of x, mixex utility can be approximated by a single exponential utility with $r = r_{\min}(r_{\max})$. Furthermore, if a particular action in a decision-making problem is optimal under an exponential utility with any $r \in [r_{\min}, r_{\max}]$, then it is optimal under the mixex utility.

In the multivariate case, things get more complex. If the mixing distribution is degenerate, however, the mixex form simplifies to multiattribute exponential utility, $u(\mathbf{x}) = 1 - e^{-(r_1 x_1 + \cdots + r_N x_N)}$. This is a special case of the situation treated in Theorem 5 where the mixing distribution exhibits independence.

Theorem 5. *For a mixex utility of the form (4), if $F(r_1, \ldots, r_N) = \prod_{i=1}^N F_i(r_i)$, then*

$$u(\mathbf{x}) = 1 - \prod_{i=1}^N (1 - u^{(i)}(x_i))$$
$$= \sum_{k=1}^N \sum_{\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, N\}} (-1)^{k-1} \left(\prod_{j=1}^k u^{(i_j)}(x_{i_j}) \right), \quad (5)$$

where $u^{(i)}(x_i) = 1 - \int_0^\infty e^{-r_i x_i}\, dF_i(r_i)$ and the inside sum is over all combinations of size k from $\{1, \ldots, N\}$.

Theorem 5 shows that when $u(\mathbf{x})$ is mixex and F is such that the risk aversion coefficients $r_1, \ldots, r_N$ are independent, $u(\mathbf{x})$ can be expressed as a function of the single-attribute utilities $u^{(i)}(x_i), i = 1, \ldots, N$, which are themselves mixex. Moreover, $u(\mathbf{x})$ has a multiplicative/multilinear form, a special case of (6.12) in Keeney and Raiffa (1976) with scaling constants $k_i = 1, i = 1, \ldots, N$, and $k = -1$ (using notation from Keeney and Raiffa 1976). For example, $u(\mathbf{x}) = u^{(1)}(x_1) + u^{(2)}(x_2) - u^{(1)}(x_1) u^{(2)}(x_2)$ when $N = 2$. The signs of the derivatives associated with combining good with bad carry over to (5), with the coefficients equal to 1 (−1) for products of odd (even) numbers of single-attribute utilities. With mixex utility, then, stochastic independence in the mixing distribution F is consistent with mutual utility independence of the attributes $x_1, \ldots, x_N$. Thus, if we would like $u(\mathbf{x})$ to reflect mutual utility independence and a preference for combining good with bad, the mixex form in (5) is appropriate.

The decreasing risk aversion for mixex utility with $N = 1$ extends to $N > 1$. However, the behavior of risk aversion for one attribute as another attribute increases depends on how the attributes are related in F. With independence in F, the risk aversion with respect to one attribute is not affected by changes in another attribute. This separability typically does not hold when the risk aversion coefficients are dependent, however. We consider a particular form of dependence, called affiliation (Milgrom and Weber 1982). Affiliation is sometimes called log-supermodularity (Gollier 2001) or total positivity (Karlin and Rinott 1980). As Theorem 6 shows, when $r_1, \ldots, r_N$ are affiliated, risk aversion with respect to one attribute is a decreasing function of the other attribute.

Definition 5. If $f(\mathbf{r}^{\max})f(\mathbf{r}^{\min}) \geq f(\mathbf{r})f(\mathbf{r}')$ for all $\mathbf{r}, \mathbf{r}'$, where f is the density or mass function corresponding to the cdf F of $r_1, \ldots, r_N$ and $\mathbf{r}^{\max}$ and $r^{\min}$ are the component-by-component maximum and minimum of $\mathbf{r}$ and $\mathbf{r}'$, then $r_1, \ldots, r_N$ are affiliated with respect to F.

Theorem 6. *If $u(\mathbf{x})$ is mixex, then risk aversion with respect to one attribute is decreasing as that attribute increases: $r(x_i \mid \mathbf{x}_{-i}) = -u_{ii}(\mathbf{x})/u_i(\mathbf{x})$ is decreasing in x_i, $i = 1, \ldots, N$. Furthermore, if $r_1, \ldots, r_N$ are affiliated with respect to the mixing distribution F, then $r(x_i \mid \mathbf{x}_{-i})$ is decreasing in x_j for any $j \neq i$.*

Note that substituting a weaker form of dependence for affiliation in the second part of Theorem 6 will not necessarily lead to decreasing risk aversion. For example, consider $u(\mathbf{x}) = 1 - 0.25e^{-x_1-x_2} - 0.25e^{-2x_1-2x_2} - 0.25e^{-10x_1-x_2} - 0.25e^{-11x_1-2x_2}$, which is mixex utility for $N = 2$ with a discrete mixing distribution: $f(1,1) = f(2,2) = f(10,1) = f(11,2) = 0.25$. Letting $r_1 = 10, r_1' = 2, r_2 = 2$, and $r_2' = 1$, we have $f(10,2)f(2,1) = 0 < f(2,2)f(10,1) = 0.0625$, so although r_1 and r_2 are positively correlated, they are not affiliated. Numerical calculations show that risk aversion for x_1 is decreasing as x_2 increases for most x_1, but it is increasing for values of x_1 roughly between 0.10 and 0.34.

Intuitively, the result in Theorem 6 makes sense. We might think of it as falling under the general idea of becoming less risk averse as the situation improves. As x increases in the single-attribute case, the situation improves because more of the attribute is preferred to less, and any mixex utility exhibits decreasing risk aversion. When $N > 1$, the situation improves if any x_i increases. Here mixex guarantees that risk aversion for x_i decreases as x_i increases, but also requires affiliation to assure that it decreases as x_j increases. The message is that mixex will guarantee some properties often considered desirable, but other properties may require constraining the mixing distribution.

Although affiliation is generally defined in terms of a positive relationship between variables, we can consider r_i and r_j to be negatively affiliated if r_i and $-r_j$ are affiliated. Theorem 6 covers only positive relationships, but if, say, r_i is negatively affiliated with the other elements of $\mathbf{r}$, then we can say that $r_1, \ldots, r_{i-1}, -r_i, r_{i+1}, \ldots, r_N$ are affiliated. Negative relationships can be tricky in terms of affiliation just as they are in a correlation matrix, which must be positive definite. In Corollary 2, we extend Theorem 6 to the cases of negative affiliation and independence.

Corollary 2. *If $u(\mathbf{x})$ is mixex and if r_i and r_j are affiliated (negatively affiliated, independent) with respect to the mixing distribution F, then risk aversion with respect to attribute x_i is decreasing (increasing, constant) as attribute x_j, $j \neq i$, increases.*

To illustrate Corollary 2, we consider another discrete $N = 2$ example, with $f(2,2) = f(1,1) = 0.5p$ and $f(1,2) = f(2,1) = 0.5(1-p), 0 \leq p \leq 1$. With this mixing distribution, $u(\mathbf{x}) = 1 - 0.5pe^{-2x_1-2x_2} - 0.5(1-p)e^{-2x_1-x_2} - 0.5(1-p)e^{-x_1-2x_2} - 0.5pe^{-x_1-x_2}$, and r_1 and r_2 are affiliated (negatively affiliated, independent) when $p > (<, =)$ 0.5. Risk aversion for one attribute is decreasing as the other attribute increases if $p > 0.5$, increasing if $p < 0.5$, and constant if $p = 0.5$.

As discussed earlier, some mixex utility functions are consistent with mutual utility independence and a multiplicative/multilinear utility form. What about an additive utility function, which is arguably the most widely used multiattribute utility form in practice? Mixex utility is additive only in very special limiting cases when the preference for combining good lotteries with bad is not strict, and key properties of mixex utility therefore do not carry over except in a weak sense. For the attributes to be additively separable in (4), each $(r_1, \ldots, r_N)$ for which $f(r_1, \ldots, r_N) > 0$ can have only one nonzero element. This means that the support of F is concentrated on the axes, a property called mutual exclusivity by Dhaene and Denuit (1999).

For example, let $N = 2$ with $f(1,0) = f(0,2) = 0.5$. The corresponding mixex utility is additive, $u(\mathbf{x}) = 1 - 0.5e^{-x_1} - 0.5e^{-2x_2}$. Although r_1 and r_2 are negatively affiliated, risk aversion for each attribute is constant as the other attribute increases. Moreover, the additive utility implies that x_1 and x_2 are additive independent, which in turn implies indifference between combining good with bad and combining good with good and bad with bad across the attributes. In this sense, additive utility is at odds with the basic spirit that motivates mixex and might be viewed as a pathological limiting case of mixex utility.

Another property often considered desirable in the single-attribute case is the one-switch property (Bell 1988). The intuition for the one-switch property is that if the preference between two alternatives switches as things get better (e.g., wealth increases), then the preference should not switch back if things continue to get even better. With multiple attributes, "things get better" translates into "along any increasing path." The single-attribute result that mixex utility functions with two-point mixing distributions satisfy the one-switch property is extended to the multiattribute case in Theorem 7. As in Bell (1988), zero-switch automatically qualifies as one-switch.

Definition 6. A decision maker satisfies the one-switch property for N attributes if, for every pair of alternatives $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{z}}$ for which her preference between $\tilde{\mathbf{y}}+\boldsymbol{\lambda}$ and $\tilde{\mathbf{z}}+\boldsymbol{\lambda}$ is not independent of the vector $\boldsymbol{\lambda}$, there exists a vector $\boldsymbol{\lambda}_0$ such that $\tilde{\mathbf{y}}+\boldsymbol{\lambda} \succ (\sim, \prec)\ \tilde{\mathbf{z}}+\boldsymbol{\lambda}$ for $\boldsymbol{\lambda} \lneqq (=, \gneqq)\ \boldsymbol{\lambda}_0$. That is, her preference between the alternatives switches at most once when moving along any path that is (weakly) increasing in each attribute.

Theorem 7. *If $u(\mathbf{x})$ is mixex, it satisfies the one-switch property if and only if the mixing distribution is two-point with $r_1, \ldots, r_N$ affiliated*:

$$u(\mathbf{x}) = 1 - pe^{-r_{11}x_1-\cdots-r_{N1}x_N} - (1-p)e^{-r_{12}x_1-\cdots-r_{N2}x_N}, \quad (6)$$

where $0 \le p \le 1$ and $r_{i1} \ge r_{i2} \ge 0$ for $i = 1, \ldots, N$.

Just as in the single-attribute case, one-switch utility functions are sumex utility functions based on two-point mixing distributions. Sumex now refers to a sum of two multiattribute exponential utility functions, and $r_1, \ldots, r_N$ must be affiliated. (For a two-point distribution, affiliation is equivalent to any other form of positive dependence, because positive dependence in this case simply means that the two points are ordered consistently, as in Theorem 7.)

The multiattribute one-switch property from Definition 6 is stronger than the mutual one-switch independence for multiattribute utility functions introduced by Abbas and Bell (2009). Therefore, the class of utility functions implied by mutual one-switch independence is broader than that specified by (1). Note that although the mixex utility form in (1) is a sum, it is not additive in the sense of additive multiattribute utility based on additive independence.

The discussion in this section has focused on several important properties of mixex utility. What about its use to find expected utilities in a multiattribute decision-making problem? The multivariate normal distribution is easily the most commonly encountered multivariate distribution, is very tractable, and is a reasonable representation of uncertainty in many situations. Thus, it is useful to see how expected utility behaves when utility is mixex as given by (4) and the distribution of the attributes is multivariate normal with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$ and covariance matrix $\Sigma = (\rho_{ij}\sigma_i\sigma_j)$:

$$E[u(\tilde{\mathbf{x}})] = 1 - \int_0^\infty \cdots \int_0^\infty E[e^{-(r_1\tilde{x}_1+\cdots+r_N\tilde{x}_N)}]\, dF(r_1, \ldots, r_N)$$
$$= 1 - \int_0^\infty \cdots \int_0^\infty e^{-\mathbf{r}\boldsymbol{\mu}^t + (\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t/2)}\, dF(r_1, \ldots, r_N),$$

where a superscript t denotes transposition and

$$-\mathbf{r}\boldsymbol{\mu}^t + (\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t/2) = -\sum_{i=1}^N r_i\mu_i + \left(\sum_{i=1}^N\sum_{j=1}^N r_i r_j \rho_{ij}\sigma_i\sigma_j/2\right).$$

Note that increasing any mean μ_i and decreasing any correlation ρ_{ij} increases $E[u(\tilde{\mathbf{x}})]$, the former in a stochastic dominance sense because more of any attribute is preferred to less and the latter because of a diversification effect associated with the multivariate risk aversion of u. Increasing any standard deviation σ_i decreases expected utility due to risk aversion if $\rho_{ij} \ge 0$ for all j. However, if $\rho_{ij} < 0$ for some j, things are more complicated. Increasing σ_i without changing any other standard deviations or any correlations will cause a decrease in expected utility from the terms in $\sum_{i-1}^N \sum_{j=1}^N r_i r_j \rho_{ij}\sigma_i\sigma_j$ with positive ρ_{ij} and an increase in expected utility from the terms with negative ρ_{ij}. Thus, if there are negative ρ_{ij}, $E[u(\tilde{\mathbf{x}})]$ could go either way as σ_i increases, in contrast to the simpler single-attribute case where increasing the standard deviation always decreases expected utility. Similar effects of increasing standard deviations in target-oriented situations are established in Tsetlin and Winkler (2007). In general, it is easy to find $E[u(\tilde{\mathbf{x}})]$ when u is mixex and $\tilde{\mathbf{x}}$ is multivariate normal, and the same would be true for any distribution of $\tilde{\mathbf{x}}$ yielding a convenient expression for $E[e^{-(r_1\tilde{x}_1+\cdots+r_N\tilde{x}_N)}]$.

In both the single-attribute and multiattribute cases, then, mixex utility provides a large family of utility functions satisfying many desirable properties and including many utility functions commonly used in practice. It is consistent with a simple preference assumption of preferring to combine good with bad. A utility form not compatible with mixex is not consistent with a preference for combining good with bad. The flexibility of mixex is provided by the latitude in the choice of a mixing distribution, from limiting degenerate mixing distributions to two-point distributions all the way to continuous distributions. For mixtures of multiattribute exponential utilities, the degree and direction of dependence in the mixing distribution plays an important role in the behavior of the resulting utility. A brief example involving the process of preference assessment and fitting of mixex utility is presented in §5.

5. An Example of the Assessment of Mixex Utility

We illustrate some concepts from §4 using a simple hypothetical example of a telecom company entering a new market. Many individuals are involved in the decision-making process, but the primary decision maker (DM) is the CEO. She does some value-focused thinking (Keeney 1992) concerning objectives and possible strategies, and decides to focus on two attributes. The first attribute, x_1, involves financial results during the first five years in the new market, measured by the net present value (NPV) of profit in millions of dollars. The second attribute, x_2, is the market share in percentage terms at the end of the period. The goal is to assess $u(x_1, x_2)$ for $-100 \le x_1 \le 500$ and $10 \le x_2 \le 50$.

The DM starts by thinking about her preferences over each individual attribute. She feels that for each attribute, she prefers more of the attribute to less and she is risk averse. Then, moving to preferences involving lotteries on (x_1, x_2) and thinking about the notion of combining lotteries, she finds the preference for combining good with bad attractive and consistent with her general feeling of being risk averse.

Next, the DM assesses certainty equivalents (CEs) for a few specific lotteries over one attribute when the other attribute is fixed. When $x_2 = 50$, her CE for the lottery $[-100, 500]$ on x_1 is 180, and when $x_2 = 10$, her CE for that lottery is 100. Similarly, her assessed CEs for the lottery $[10, 50]$ on x_2 are 25 when $x_1 = 500$ and 20 when $x_1 = -100$. Each of the CEs is less than the expected value of the lottery, which is consistent with the DM's claim of being risk averse. Also, the assessments indicate that risk aversion is decreasing in each attribute as the other attribute increases, which is consistent with mixex utility with r_1 and r_2 affiliated. Note that the assessments are not consistent with mutual utility independence and therefore not with multilinear utility.

The DM's qualitative statements and quantitative assessments suggest that a mixex utility might provide a good fit to her preferences. To keep the model as simple as possible, we consider a mixex utility with a two-point mixing distribution:

$$u(\mathbf{x}) = \frac{1 - pe^{-(r_{11}(x_1+100)+r_{21}(x_2-10))} - (1-p)e^{-(r_{12}(x_1+100)+r_{22}(x_2-10))}}{1 - pe^{-(600r_{11}+40r_{21})} - (1-p)e^{-(600r_{12}+40r_{22})}}. \tag{7}$$

The use of $x_1 + 100$ and $x_2 - 10$ in the numerator (adjusting for the lower bounds of -100 on x_1 and 10 on x_2) and the normalizing factor in the denominator (accounting for the fact that both attributes are bounded above) allow us to scale u such that $u(-100, 10) = 0 \le u(\mathbf{x}) \le 1 = u(500, 50)$.

To assess the five parameters in (7), we use the four equations based on the DM's CE assessments and ask her for another assessment to provide a fifth equation. The additional assessment is the probability q that makes her indifferent between (a) the middle point $(x_1 = 200, x_2 = 30)$ for sure and (b) the lottery giving the best outcome $(x_1 = 500, x_2 = 50)$ with probability q and the worst outcome $(x_1 = -100, x_2 = 10)$ with probability $1-q$. Suppose that $q = 0.75$. Then we have

$$u(180, 50) = 0.5u(-100, 50) + 0.5u(500, 50),$$
$$u(100, 10) = 0.5u(-100, 10) + 0.5u(500, 10),$$
$$u(500, 25) = 0.5u(500, 10) + 0.5u(500, 50),$$
$$u(-100, 20) = 0.5u(-100, 10) + 0.5u(-100, 50), \quad \text{and}$$
$$u(200, 30) = 0.75.$$

We fit the utility function in (7) to these five equations numerically via least squares.

Least squares yields $p = 0.126$, $r_{11} = 0.0041$, $r_{21} = 0.070$, $r_{12} = 00016$, and $r_{22} = 0.00052$, so that

$$\begin{aligned} u(\mathbf{x}) &= (1 - 0.126e^{-(0.0041(x_1+100)+0.070(x_2-10))} \\ &\quad - 0.874e^{-(0.00016(x_1+100)+0.00052(x_2-10))}) \\ &\quad \cdot (1 - 0.0126^{-(0.0041(600)+0.070(40))} \\ &\quad - 0.202e^{-(0.00016(600)+0.00052(40))})^{-1} \\ &= 0.566(1 - e^{-(0.0041(x_1+100)+0.070(x_2-10))}) \\ &\quad + 3.917(1 - e^{-(0.00016(x_1+100)+0.00052(x_2-10))}). \end{aligned}$$

Note that $r_{11} > r_{12} > 0$ and $r_{21} > r_{22} > 0$. Therefore, $u(\mathbf{x})$ is mixex with r_1 and r_2 affiliated, which is consistent with the DM's preference for combining good with bad and the implication from her CEs that her risk aversion in each attribute is decreasing as the other attribute increases. Furthermore, the two-point mixing distribution provides a very close least-squares fit. A discussion with the DM verifies that she feels that if things get better (in terms of higher NPV and/or higher market share), her preference between any two strategies should switch at most once. From Theorem 7, this provides additional support for a two-point mixing distribution. We considered only five assessments for simplicity. In practice, we recommend making more assessments to check for consistency of the best fit utility with the DM's preferences.

The example illustrates how a mixex utility form can be fit to preference assessments. It also suggests how the notion of combining good with bad can be part of the discussion during the assessment process. In particular, when a decision maker appears to be risk averse on the individual attributes of interest, a preference for combining good with bad might have considerable appeal. A notion such as combining good with bad offers an alternative to a standard utility-independence-based approach.

6. Summary and Discussion

The choice of an appropriate yet manageable functional form for a multiattribute utility function can be difficult. We develop multiattribute mixex utility starting from a preference condition over basic lotteries: a preference for combining good with bad. This condition is similar in spirit to risk aversion in the sense that it reflects an aversion to the combination of bad with bad. The resulting mixex family of utility functions satisfies some appealing properties. For example, it exhibits decreasing risk aversion for each attribute. When the mixing distribution is affiliated, risk aversion for any attribute is also decreasing as any *other* attribute increases. When the mixing distribution is affiliated and is a two-point distribution, mixex satisfies a multivariate extension of Bell's (1988) one-switch property, thereby generalizing the sumex one-switch utility developed by Bell (1988).

In general, one of the strengths of mixex utility is its ability to represent cases where utility independence does not hold. However, mixex utility can be consistent with mutual utility independence and take on a multilinear form. Additive utility, on the other hand, is a special limiting case that does not satisfy the spirit of preferring to combine good with bad, suggesting that the group of managers who wound up with an additive utility function in the opening paragraph of §1 might want to think about whether violating a preference for combining good with bad seems reasonable.

An attractive characteristic of exponential utility is that expected utility calculations are often relatively straightforward for applications. The expected utility corresponding to any individual exponential utility in the mixture is often a simple closed-form expression, and the overall expected utility is a mixture of these expected exponential utilities. For discrete mixing distributions, this last step is particularly easy to accomplish. Thus, the potential ease of calculating expected utilities with exponential utility carries over to mixex.

With mixex utility, parameters of the utility function are not assessed directly, as they are when the assessment process focuses on utility independence and involves the assessment of scaling constants in additive and multilinear utility functions (Keeney and Raiffa 1976). Instead, the mixex form is fit to preference assessments. In §5, a two-point mixing distribution is used. To the extent that satisfying the one-switch condition is desirable, a two-point distribution is the only suitable choice, and for N attributes it has only $2N+1$ parameters to estimate in the fitting process.

In the assessment process, the preference condition underlying mixex utility should be part of the discussion, with the decision maker being given choices that highlight the issue of preferences for combining good with bad. If the decision maker expresses a preference for combining good with good and bad with bad, then mixex is not a suitable form. This is likely to be the case, for example, when attributes are complements, as opposed to being substitutes. An issue for future work is to determine when a preference for combining good with bad tends to hold and when it tends not to hold. Some work on how best to structure assessment questions once it is determined that mixex is suitable would also be useful.

The focus here has been on the theoretical properties of mixex utility and its use to represent a decision maker's utility in decision analysis, but it could also be useful in other settings. For instance, consider competitive situations, where the mixing distribution could represent uncertainty about an opponent's exponential utility. This has the potential to enrich game-theoretic models, which typically rely on the unrealistic assumption of common knowledge about everyone's utility function.

In summary, mixex provides a large family of single-attribute and multiattribute utility functions satisfying many desirable properties and including many utility functions commonly used in practice. An appealing feature of mixex is its connection to a basic preference condition that is easy to understand and discuss with decision makers as part of an assessment process. Of course, not all decision makers will exhibit a preference for combining good with bad in every context, but we feel that it has sufficient appeal to make the approach developed in this paper a useful framework for thinking about preferences and determining a multiattribute utility function through a combination of assessment and fitting.

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Appendix

Proof of Theorem 1. We proceed by induction. Observe that the claim holds for $n=1$. We need to prove that if it holds for $n=k$, $k\geq 1$, then it holds for $n=k+1$: $0.5E[u(\tilde{\mathbf{x}}_k+\mathbf{b}_{k+1})]+0.5E[u(\tilde{\mathbf{y}}_k+\mathbf{a}_{k+1})]\geq 0.5E[u(\tilde{\mathbf{x}}_k+\mathbf{a}_{k+1})]+0.5E[u(\tilde{\mathbf{y}}_k+\mathbf{b}_{k+1})]$, or

$$E[u(\tilde{\mathbf{y}}_k+\mathbf{a}_{k+1})]-E[u(\tilde{\mathbf{x}}_k+\mathbf{a}_{k+1})] \geq E[u(\tilde{\mathbf{y}}_k+\mathbf{b}_{k+1})]-E[u(\tilde{\mathbf{x}}_k+\mathbf{b}_{k+1})], \quad (8)$$

if and only if $u\in\mathbb{U}_{k+1}^N$.

For sufficiency, let $u\in\mathbb{U}_{k+1}^N$ and define $v(z)=E[u(\tilde{\mathbf{y}}_k+\mathbf{z})]-E[u(\tilde{\mathbf{x}}_k+\mathbf{z})]$, so that (8) is equivalent to $v(\mathbf{a}_{k+1})\geq v(\mathbf{b}_{k+1})$. Then, $\partial v(\mathbf{z})/\partial z_i = E[\partial u(\tilde{\mathbf{y}}_k+\mathbf{z})/\partial z_i - \partial u(\tilde{\mathbf{x}}_k+\mathbf{z})/\partial z_i]\geq 0$ because $-\partial u(\mathbf{z})/\partial z_i\in\mathbb{U}_k^N$ and by the induction assumption that the claim holds for $n=k$. Therefore, $v(\mathbf{a}_{k+1})\geq v(\mathbf{b}_{k+1})$.

For necessity, we need to prove that if (8) holds, then $(-1)^{k+1}\partial^{k+1}u(\mathbf{x})/\partial x_{i_1}\cdots\partial x_{i_{k+1}}\geq 0$ for any $i_j\in\{1,\ldots,N\}$, $j=1,\ldots,k+1$. Let $\mathbf{b}_{k+1}=\mathbf{0}$ and let all components of $\mathbf{a}_{k+1}$

equal 0 except for the i_{k+1}th component, which equals $\delta > 0$. If (8) holds, then, for $\mathbf{b}_{k+1} = \mathbf{0}$, $E[u(\tilde{\mathbf{y}}_k + \mathbf{a}_{k+1})] - E[u(\tilde{\mathbf{x}}_k + \mathbf{a}_{k+1})] \geq E[u(\tilde{\mathbf{y}}_k)] - E[u(\tilde{\mathbf{x}}_k)]$, or $E[u(\tilde{\mathbf{y}}_k + \mathbf{a}_{k+1})] - E[u(\tilde{\mathbf{y}}_k)] \geq E[u(\tilde{\mathbf{x}}_k + \mathbf{a}_{k+1})] - E[u(\tilde{\mathbf{x}}_k)]$. Taking the limit as $\delta \to 0$, $E[\partial u(\tilde{\mathbf{y}}_k)/\partial x_{i_{k+1}}] \geq E[\partial u(\tilde{\mathbf{x}}_k)/\partial x_{i_{k+1}}]$, which implies that $-\partial u(\mathbf{x})/x_{i_{k+1}} \in \mathbb{U}_k^N$ by the induction assumption. Therefore, $-(-1)^{k-1}\partial^{k+1}u(\mathbf{x})/\partial x_{i_1}\cdots\partial x_{i_{k+1}} \geq 0$. □

Proof of Corollary 1. Theorem 1 holds for $n = 1, 2, \ldots$ if and only if $u \in \mathbb{U}_\infty^N$. □

Proof of Theorem 2. For sufficiency, differentiating $\varphi(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty e^{-(r_1x_1+\cdots+r_Nx_N)}\, dF(r_1, \ldots, r_N)$ with respect to $x_{i_1}x_{i_2}\cdots x_{i_k}$ for $k = 1, 2, \ldots$ and any $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, k$, yields $(-1)^k \partial^k \varphi(\mathbf{x})/\partial x_{i_1}\cdots\partial x_{i_k} \geq 0$. The proof of necessity builds on the one-dimensional case (Feller 1971, Theorem 1a of §XIII.4). Consider $\varphi(a_1 - a_1s_1, \ldots, a_N - a_Ns_N)$ for fixed $a_1, \ldots, a_N > 0$ as a function of $(s_1, \ldots, s_N)$ for $0 \leq s_i < 1$, $i = 1, \ldots, N$. All partial derivatives with respect to $(s_1, \ldots, s_N)$ are positive, and the Taylor expansion is

$$\varphi(a_1 - a_1s_1, \ldots, a_N - a_Ns_N) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty} \frac{(-a_1)^{n_1}\cdots(-a_N)^{n_N}}{n_1!\cdots n_N!} \cdot \frac{\partial^{n_1+\cdots+n_N}\varphi(a_1, \ldots, a_N)}{\partial x_1^{n_1}\cdots\partial x_N^{n_N}} s_1^{n_1}\cdots s_N^{n_N}.$$

This Taylor expansion is valid by consecutively applying Theorem 2 of Feller (1971, §VII.2) N times: to $\phi(a_1 - a_1s_1, a_2, \ldots, a_N)$ as a function of s_1, to $\phi(a_1 - a_1s_1, a_2 - a_2s_2, a_3, \ldots, a_N)$ as a function of $s_2, \ldots,$ and to $\varphi(a_1 - a_1s_1, \ldots, a_N - a_Ns_N)$ as a function of s_N. Thus, defining

$$\varphi_{a_1,\ldots,a_N}(\lambda_1, \ldots \lambda_N) = \varphi(a_1 - a_1e^{-\lambda_1/a_1}, \ldots, a_N - a_Ne^{-\lambda_N/a_N}),$$

$$\varphi_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) = \sum_{n_1=0}^{\infty}\cdots\sum_{n_N=0}^{\infty} \frac{(-a_1)^{n_1}\cdots(-a_N)^{n_N}}{n_1!\cdots n_N!} \frac{\partial^{n_1+\cdots+n_N}\varphi(a_1, \ldots, a_N)}{\partial x_1^{n_1}\cdots\partial x_N^{n_N}} \cdot e^{-[(n_1/a_1)\lambda_1+\cdots+(n_N/a_N)\lambda_N]}$$

is the Laplace transform of an arithmetic measure $F_{a_1,\ldots,a_N}$ with mass

$$\frac{(-a_1)^{n_1}\cdots(-a_N)^{n_N}}{n_1!\cdots n_N!} \frac{\partial^{n_1+\cdots+n_N}\varphi(a_1, \ldots, a_N)}{\partial x_1^{n_1}\cdots\partial x_N^{n_N}}$$

at $(n_1/a_1, \ldots, n_N/a_N)$ for $n_1, \ldots, n_N = 0, 1, \ldots$. Now $\varphi_{a_1,\ldots,a_N}(\lambda_1, \ldots, \lambda_N) \to \varphi(\lambda_1, \ldots, \lambda_N)$ as $a_1, \ldots, a_N \to \infty$. By the extended continuity theorem (Feller 1971, Theorem 2a of §XIII.1), there exists a measure $F_{a_1,\ldots,a_{N-1}}$ such that $F_{a_1,\ldots,a_N} \to F_{a_1,\ldots,a_{N-1}}$ as $a_N \to \infty$ and $\varphi(a_1 - a_1e^{-\lambda_1/a_1}, \ldots, a_{N-1} - a_{N-1}e^{-\lambda_{N-1}/a_{N-1}}, \lambda_N)$ is its Laplace transform. Applying the extended continuity theorem $N-1$ more times, as $a_{N-1} \to \infty, a_{N-2} \to \infty, \ldots,$ and $a_1 \to \infty$, there exists a measure F such that $F_{a_1,\ldots,a_N} \to F$ as $a_1, \ldots, a_N \to \infty$, and φ is its Laplace transform. □

Proof of Theorem 3. For sufficiency, differentiating (3) with respect to $x_{i_1}x_{i_2}\cdots x_{i_k}$ for $k = 1, 2, \ldots$ and any $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, k$, yields $(-1)^{k-1}\partial^k u(\mathbf{x})/\partial x_{i_1}\cdots\partial x_{i_k} \geq 0$. For necessity, suppose that $u \in \mathbb{U}_\infty^N$ and define $v(\mathbf{y}) = u(\mathbf{x}^* + \mathbf{y}) - u(\mathbf{x}^*)$. Then, $v(\mathbf{0}) = 0$ and $v \in \mathbb{U}_\infty^N$. By Remark 2 and Theorem 2, for each $i = 1, \ldots, N$ there exists a measure F_i without mass at zero and a constant $b_i \geq 0$ such that $\partial v(\mathbf{y})/\partial y_i = \int_0^\infty \cdots \int_0^\infty e^{-(r_1y_1+\cdots+r_Ny_N)}\, dF_i(\mathbf{r}) + b_i$. Note that because $\partial^2 v(\mathbf{y})/\partial y_i \partial y_j = \partial^2 v(\mathbf{y})/\partial y_j \partial y_i$, $r_j dF_i(\mathbf{r}) = r_i dF_j(r)$ for all $i, j = 1, \ldots, N$. Also, note that $dF_i(\mathbf{r})$ does not have a mass point at $r_i = 0$ and $r_j > 0$ because $r_j dF_i(r_1, \ldots, r_i = 0, \ldots, r_N) = (0)dF_j(r_1, \ldots, r_N) = 0$. Therefore, $dF_i(\mathbf{r})/r_i = dF_j(\mathbf{r})/r_j$ for $i, j = 1, \ldots, N$, and we define $dF(\mathbf{r}) = dF_i(\mathbf{r})/r_i$ for $i = 1, \ldots, N$. Then

$$\begin{aligned} v(\mathbf{y}) &= [v(y_1, 0, \ldots, 0) - v(\mathbf{0})] \\ &\quad + [v(y_1, y_2, 0, \ldots, 0) - v(y_1, 0, \ldots, 0)] \\ &\quad + \cdots + [v(\mathbf{y}) - v(y_1, \ldots, y_{N-1}, 0)] \\ &= \int_0^\infty \frac{\partial v(t_1, 0, \ldots, 0)}{\partial y_1}\, dt_1 + \int_0^\infty \frac{\partial v(y_1, t_2, 0, \ldots, 0)}{\partial y_2}\, dt_2 \\ &\quad + \cdots + \int_0^\infty \frac{\partial v(y_1, \ldots, y_{N-1}, t_N)}{\partial y_N}\, dt_N. \end{aligned}$$

Consider, e.g., the last term:

$$\begin{aligned} &\int_0^\infty \frac{\partial v(y_1, \ldots, y_{N-1}, t_N)}{\partial y_N}\, dt_N \\ &= \int_0^{y_N} \left(\int_0^\infty \cdots \int_0^\infty e^{-\left(\sum_{i=1}^{N-1} r_iy_i\right) - r_Nt_N}\, dF_N(\mathbf{r}) + b_N \right) dt_N \\ &= \int_0^\infty \cdots \int_0^\infty \left(\int_0^{y_N} e^{-r_Nt_N}\, dt_N \right) e^{-\sum_{i=1}^{N-1} r_iy_i}\, dF_N(\mathbf{r}) + b_Ny_N \\ &= \int_0^\infty \cdots \int_0^\infty \left(\frac{1 - e^{-r_Ny_N}}{r_N} \right) e^{-\sum_{i=1}^{N-1} r_iy_i}\, dF_N(\mathbf{r}) + b_Ny_N \\ &= \int_0^\infty \cdots \int_0^\infty (1 - e^{-r_Ny_N}) e^{-\sum_{i=1}^{N-1} r_iy_i}\, dF(\mathbf{r}) + b_Ny_N. \end{aligned}$$

Going through a similar process for the other terms and adding the N terms yields $v(\mathbf{y}) = \int_0^\infty \cdots \int_0^\infty [(1 - e^{-r_1y_1}) + (1 - e^{-r_2y_2})e^{-r_1y_1} + \cdots + (1 - e^{-r_Ny_N})e^{-\sum_{i=1}^{N-1} r_iy_i}]\, dF(\mathbf{r}) + \sum_{i=1}^N b_iy_i = \int_0^\infty \cdots \int_0^\infty (1 - e^{-\sum_{i=1}^N r_iy_i})\, dF(\mathbf{r}) + \sum_{i=1}^N b_iy_i$. Substituting this expression for v in $u(\mathbf{x}) = v(\mathbf{x} - \mathbf{x}^*) + u(\mathbf{x}^*)$ completes the proof. □

Proof of Theorem 4. We have

$$\lim_{x\to\infty} r(x) = \lim_{x\to\infty} \int_0^\infty r^2e^{-rx}\, dF(r) \Big/ \lim_{x\to\infty} \int_0^\infty re^{-rx}\, dF(r).$$

In both the numerator and denominator, the term $e^{-r_{\min}x}$ dominates as $x \to \infty$, and $\lim_{x\to\infty} r(x) = r_{\min}^2/r_{\min} = r_{\min}$. Similarly, $e^{-r_{\max}x}$ dominates as $x \to -\infty$, and $r(x) \to r_{\max}$. □

Proof of Theorem 5. Under independence, $u(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty [1 - e^{-(r_1x_1+\cdots+r_Nx_N)}]\, dF(r_1, \ldots, r_N)$ simplifies to

$$\begin{aligned} u(x) &= 1 - \prod_{i=1}^N \left(\int_0^\infty e^{-r_ix_i}\, dF_i(r_i) \right) \\ &= 1 - \prod_{i=1}^N (1 - u^{(i)}(x_i)) \\ &= 1 - \left[1 - \sum_{k=1}^N \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,N\}} (-1)^{k-1} \left(\prod_{j=1}^k u^{(i_j)}(x_{i_j}) \right) \right]. \quad \square \end{aligned}$$

Proof of Theorem 6. With $u(\mathbf{x})$ given by (4),

$$r(x_i \mid \mathbf{x}_{-i}) = \frac{\int_0^\infty \cdots \int_0^\infty r_i^2 e^{-\sum_{k=1}^N r_kx_k} f(\mathbf{r})\, dr_1 \cdots dr_N}{\int_0^\infty \cdots \int_0^\infty r_i e^{-\sum_{k=1}^N r_kx_k} f(\mathbf{r})\, dr_1 \cdots dr_N}.$$

Then,

$$\frac{\partial r(x_i \mid \mathbf{x}_{-i})}{\partial x_i} = -\int_0^\infty \cdots \int_0^\infty r_i^2 f^*(\mathbf{r})\, dr_1 \cdots dr_N + \left(\int_0^\infty \cdots \int_0^\infty r_i f^*(\mathbf{r})\, dr_1 \cdots dr_N\right)^2 = -V(r_i) \le 0,$$

where the variance is taken with respect to

$$f^*(\mathbf{r}) = r_i e^{-\sum_{k=1}^N r_k x_k} f(\mathbf{r}) \Big/ \int_0^\infty \cdots \int_0^\infty r_i e^{-\sum_{k=1}^N r_k x_k} f(\mathbf{r})\, dr_1 \cdots dr_N.$$

Similarly, for any $j \ne i$,

$$\begin{aligned}\partial r(x_i \mid \mathbf{x}_{-i})/\partial x_i &= -\int_0^\infty \cdots \int_0^\infty r_i r_j f^*(\mathbf{r})\, dr_1 \cdots dr_N \\ &\quad + \left(\int_0^\infty \cdots \int_0^\infty r_i f^*(\mathbf{r})\, dr_1 \cdots dr_N\right) \\ &\quad \cdot \left(\int_0^\infty \cdots \int_0^\infty r_j f^*(\mathbf{r})\, dr_1 \cdots dr_N\right) \\ &= -\mathrm{Cov}(r_i, r_j).\end{aligned}$$

If $r_1, \ldots, r_N$ are affiliated with respect to F, then they are affiliated with respect to the distribution F^* corresponding to f^*, and any subset of $r_1, \ldots, r_N$ is therefore affiliated with respect to F^* (Karlin and Rinott 1980). Thus, $\mathrm{Cov}(r_i, r_j) \ge 0$ and $\partial r(x_i \mid \mathbf{x}_{-i})/\partial x_i \le 0$. □

Proof of Corollary 2. As in the proof of Theorem 6, r_i and r_j are affiliated (negatively affiliated, independent) with respect to F^* as well as F. Thus, $\mathrm{Cov}(r_i, r_j) \ge (\le, =) 0$ and $\partial r(x_i \mid \mathbf{x}_{-i})/\partial x_j \le (\ge, =) 0$. □

Proof of Theorem 7. First, we show that if $u(\mathbf{x})$ is given by (6), the one-switch property holds. Let $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{z}}$ be two arbitrary gambles, and define $m_i = E(e^{-r_{1i}\tilde{y}_1 - \cdots - r_{Ni}\tilde{y}_N} - e^{-r_{1i}\tilde{z}_1 - \cdots - r_{Ni}\tilde{z}_N})$, $i = 1, 2$. Then, $E[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})] = -e^{-r_{11}\lambda_1 - \cdots - r_{N1}\lambda_N}(pm_1 + (1-p)m_2 e^{(-r_{12}-r_{11})\lambda_1 - \cdots - (r_{N2}-r_{N1})\lambda_N})$. Therefore, for $E[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})]$ to change sign no more than once as $\lambda_1, \ldots, \lambda_N$ increase, it is necessary and sufficient that $r_{12} - r_{11}, \ldots, r_{N2} - r_{N1}$ all have the same sign.

Next, we show that if $u(\mathbf{x})$ is mixex and it satisfies the one-switch property, then (6) holds. For any $\boldsymbol{\alpha} \gneqq \mathbf{0}$, consider the induced single-attribute utility $u_{\boldsymbol{\alpha}}(t) = u(t\alpha_1, \ldots, t\alpha_N)$. Because $u(\mathbf{x})$ is one-switch, $u_{\boldsymbol{\alpha}}(t)$ is also one-switch. From Bell (1988), $u_{\boldsymbol{\alpha}}(t)$ is the sum of two exponential utilities. Therefore, the mixing distribution corresponding to $u(\mathbf{x})$ in (4) is such that its projection to the N-dimensional nonnegative vector $\boldsymbol{\alpha}$ consists of at most two points. Given that the choice of $\boldsymbol{\alpha}$ is arbitrary, this is the case only if the support of the mixing distribution consists of no more than two points. □

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