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Multiattribute One-Switch Utility

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The one-switch property states that the preference between any two lotteries switches at most once as wealth increases. Working within the expected utility framework, we extend the one-switch notion to the multiattribute case and identify the families of multiattribute utility functions that are one-switch. We then show that all multiattribute one-switch utility functions can be approximated by a sum of two multiattribute exponential utilities (sumex utility). Finally, we discuss how the one-switch property, when appropriate, can simplify the assessment of multiattribute utility.

Key words: decision analysis; multiattribute utility; one-switch property; utility assessment; sumex utility

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1. Introduction

Bell (1988) introduced the notion of one-switch utility functions for wealth. A utility function for wealth is one-switch if, as a decision maker's initial wealth is varied from low to high, her preferences between any two lotteries switch at most once. The set of utility functions consistent with the one-switch notion is limited, and the one-switch property can be useful in assessing utility for wealth.

The assessment of a multiattribute utility function is more complicated because it introduces trade-offs among attributes, and the complications increase with the number of attributes. Our concern in this paper is with multiattribute situations, and our primary focus is providing an approach that might be able to simplify the assessment of a multiattribute utility function.

The objective of this paper is to define a one-switch property for the multiattribute case and to investigate its implications for the form of the multiattribute utility function and, in particular, for the assessment of multiattribute utility. The spirit of the multiattribute one-switch property is the same as in the single-attribute case: We ask whether a preference between two lotteries changes at most once when moving along any path that is (weakly) increasing in each attribute. We show that this property restricts the utility function to four families, thereby extending Proposition 2 of Bell (1988) to the multiattribute case. We then show that any one-switch multiattribute utility function can be approximated by one of these families, a sum of two multivariate exponential utilities (sumex utility), in which case we need

only assess $2N + 2$ parameters, where N is the number of attributes. This can be accomplished by fitting a sumex function to some standard preference assessments.

2. Multiattribute One-Switch Utility

We consider the situation in which N real-valued attributes x_1, \dots, x_N are of interest, with outcomes $\mathbf{x} = (x_1, \dots, x_N) \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty \leq \underline{\mathbf{x}} < \bar{\mathbf{x}} \leq \infty$. A random vector is denoted by a tilde, $\tilde{\mathbf{x}}$, and lotteries are probability distributions on the set of outcomes. A preference relation on the set of lotteries, with $\tilde{\mathbf{x}} \succ (-) \tilde{\mathbf{y}}$ indicating that $\tilde{\mathbf{x}}$ is strictly preferred to $\tilde{\mathbf{y}}$ ($\tilde{\mathbf{y}}$ is strictly preferred to $\tilde{\mathbf{x}}$), is based on expected utility. The decision maker's utility function $u(\mathbf{x}) = u(x_1, \dots, x_N)$ is a real-valued function over the N attributes, with $\tilde{\mathbf{x}} \succ \tilde{\mathbf{y}}$ if and only if $E[u(\tilde{\mathbf{x}})] > E[u(\tilde{\mathbf{y}})]$. Also, for two vectors \mathbf{x} and \mathbf{y} , $\mathbf{x} \geq \mathbf{y}$ if $x_j \geq y_j$ for all j , and $\mathbf{x} + \mathbf{y}$ denotes the componentwise sum, $(x_1 + y_1, \dots, x_N + y_N)$.

DEFINITION 1. A decision maker satisfies the one-switch property for N attributes if, for every pair of lotteries $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{z}}$ and for any vectors $\boldsymbol{\lambda}_1 \leq \boldsymbol{\lambda}_2 \leq \boldsymbol{\lambda}_3$, it cannot be that $\tilde{\mathbf{y}} + \boldsymbol{\lambda}_1 \succ \tilde{\mathbf{z}} + \boldsymbol{\lambda}_1$, $\tilde{\mathbf{y}} + \boldsymbol{\lambda}_2 \prec \tilde{\mathbf{z}} + \boldsymbol{\lambda}_2$, and $\tilde{\mathbf{y}} + \boldsymbol{\lambda}_3 \succ \tilde{\mathbf{z}} + \boldsymbol{\lambda}_3$.

Definition 1 implies that the decision maker's preference between any two lotteries switches at most once when moving along any path that is (weakly) increasing in each attribute. As in Bell (1988), zero-switch automatically qualifies as one-switch. The vector $\boldsymbol{\lambda}$ in Definition 1 can be thought of as the decision maker's initial endowment in terms of the N

attributes (the values of the attributes before the decision between \tilde{y} and \tilde{z} is made), corresponding to “initial wealth” when money is the only attribute. Alternatively, if we defined u as the utility for increments to an initial endowment, λ would be an increment to both lotteries, \tilde{y} and \tilde{z} , with a positive (negative) λ_i representing an increase (a decrease) in the amount of the i th attribute. Now we present the main result of §2. All proofs are given in the appendix.

THEOREM 1. *A utility function $u(\mathbf{x})$ satisfies the one-switch property if and only if it belongs to one of the following families:*

$$u(\mathbf{x}) = ae^{c_1x_1+\dots+c_Nx_N} + be^{d_1x_1+\dots+d_Nx_N} \quad \text{where } d_i \geq c_i \\ \text{for } i=1, \dots, N; \quad (1)$$

$$u(\mathbf{x}) = c_1x_1 + \dots + c_Nx_N + be^{d_1x_1+\dots+d_Nx_N}, \quad \text{where } d_i \geq 0 \\ \text{for } i=1, \dots, N \text{ or } d_i \leq 0 \text{ for } i=1, \dots, N; \quad (2)$$

$$u(\mathbf{x}) = (c_1x_1 + \dots + c_Nx_N + b)e^{d_1x_1+\dots+d_Nx_N}, \quad \text{where } c_i \geq 0 \\ \text{for } i=1, \dots, N \text{ or } c_i \leq 0 \text{ for } i=1, \dots, N; \quad (3)$$

$$u(\mathbf{x}) = q(c_1x_1 + \dots + c_Nx_N)^2 + d_1x_1 + \dots + d_Nx_N, \\ \text{where } q = \pm 1 \text{ and } c_i \geq 0 \\ \text{for } i=1, \dots, N \text{ or } c_i \leq 0 \text{ for } i=1, \dots, N. \quad (4)$$

Theorem 1 shows that $u(\mathbf{x})$ is one-switch if and only if it is a member of one of four families: (1) sum of two exponentials (sumex), (2) linear plus exponential, (3) linear times exponential, and (4) quadratic. This extends Proposition 2 of Bell (1988) to the multiattribute case. A similar extension, developed independently, is presented in Abbas (2007) and Abbas and Bell (2010). It is based on a weaker multiattribute one-switch condition, leading to the same four families but without any constraints on the coefficients.

3. One-Switch Utility: Is Sumex Sufficient?

If the decision maker concludes that the one-switch property is reasonable in a given situation, this narrows the set of possible utility functions to the four families in Theorem 1. Theorem 2 enables us to narrow the choice of possibilities considerably.

THEOREM 2. *Suppose that a utility function $u(\mathbf{x})$, defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \mathbf{x} < \bar{\mathbf{x}} < \infty$, satisfies the one-switch property. Then there exists a family of utilities $u_s(\mathbf{x})$ such that $u(\mathbf{x}) = \lim_{s \rightarrow 0^+} u_s(\mathbf{x})$ for all \mathbf{x} , $\underline{\mathbf{x}} \leq \mathbf{x} \leq \bar{\mathbf{x}}$, and each $u_s(\mathbf{x})$ is of the sumex form given by (1).*

From Theorem 2, any one-switch utility function can be approximated by a sumex utility function, and therefore we can fit a sumex function to preference

assessments if the one-switch property holds. Note that in Theorem 2, we specify that \mathbf{x} is bounded to guarantee convergence. This assumption is not likely to be a serious restriction for practical purposes. The sumex family in (1) is large and flexible, including, for example, decreasing, nonmonotonic, and risk-averse utility for individual attributes. In the multiattribute case, it is not surprising to have such utility functions for some attributes.

To satisfy the one-switch property with $u(w)$ when $N = 1$, it is sufficient to consider sumex utility, as in Theorem 2. The four sumex parameters a , b , c , and d being negative is sufficient (but not necessary) for utility to be increasing, risk averse, and decreasingly risk averse. These are the key conditions of Proposition 3 in Bell (1988), but Bell allows w to be arbitrarily large and adds the condition that utility will approach risk neutrality for small gambles as $w \rightarrow \infty$. This rules out sumex and results in linear plus exponential utility, $cw + be^{dw}$ with $c > 0$, $b < 0$, and $d < 0$, being the only feasible form. Linear plus exponential utility for $N = 1$ also is the only form consistent with the “strong one-switch condition” studied in Bell and Fishburn (2001). However, sumex provides added flexibility and can be arbitrarily close to linear plus exponential utility. For $N = 1$ with utility for wealth, Bell (1988) notes that the sumex function is discussed by Schlaifer (1971) as being convenient for assessment purposes.

Abbas and Bell (2011) define one-switch independence, a preference assumption that is an extension of utility independence. Like utility independence, one-switch independence permits a decomposition of $u(\mathbf{x})$ into functions of individual attributes and parameters, and these functions and parameters can then be assessed separately.

The approach suggested here is different in nature. If the one-switch property is satisfied, we can use sumex as the functional form of $u(\mathbf{x})$ and do not have to assess the form of utility functions for individual attributes, as we would in approaches relying on utility independence. One-switch allows us to follow a process similar to that illustrated in §5 of Tsetlin and Winkler (2009).

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Appendix

Proof of Theorem 1

First, we show that if $u(x)$ is given by (1)–(4), the one-switch property holds. Let \tilde{y} and \tilde{z} be two arbitrary gambles.

For (1), define

$$C = \mathbb{E}[e^{c_1\tilde{y}_1+\dots+c_N\tilde{y}_N} - e^{c_1\tilde{z}_1+\dots+c_N\tilde{z}_N}] \quad \text{and}$$

$$D = \mathbb{E}[e^{d_1\tilde{y}_1+\dots+d_N\tilde{y}_N} - e^{d_1\tilde{z}_1+\dots+d_N\tilde{z}_N}].$$

Then

$$\begin{aligned} & \mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})] \\ &= e^{c_1\lambda_1 + \dots + c_N\lambda_N} (aC + bDe^{(d_1-c_1)\lambda_1 + \dots + (d_N-c_N)\lambda_N}). \end{aligned}$$

Therefore, for $\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})]$ to change sign no more than once as $\lambda_1, \dots, \lambda_N$ increase, it is necessary and sufficient that $d_1 - c_1, \dots, d_N - c_N$ all have the same sign, and we can take $d_i \geq c_i$ without loss of generality.

For (2), define

$$\begin{aligned} C &= c_1\mathbb{E}[\tilde{y}_1 - \tilde{z}_1] + \dots + c_N\mathbb{E}[\tilde{y}_N - \tilde{z}_N] \quad \text{and} \\ D &= \mathbb{E}[e^{d_1\tilde{y}_1 + \dots + d_N\tilde{y}_N} - e^{d_1\tilde{z}_1 + \dots + d_N\tilde{z}_N}]. \end{aligned}$$

Then

$$\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})] = C + bDe^{d_1\lambda_1 + \dots + d_N\lambda_N}.$$

Therefore, for $\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})]$ to change sign no more than once as $\lambda_1, \dots, \lambda_N$ increase, it is necessary and sufficient that d_1, \dots, d_N all have the same sign.

For (3), define

$$\begin{aligned} C &= \mathbb{E}[(c_1\tilde{y}_1 + \dots + c_N\tilde{y}_N)e^{d_1\tilde{y}_1 + \dots + d_N\tilde{y}_N} \\ &\quad - (c_1\tilde{z}_1 + \dots + c_N\tilde{z}_N)e^{d_1\tilde{z}_1 + \dots + d_N\tilde{z}_N}] \quad \text{and} \\ D &= \mathbb{E}[e^{d_1\tilde{y}_1 + \dots + d_N\tilde{y}_N} - e^{d_1\tilde{z}_1 + \dots + d_N\tilde{z}_N}]. \end{aligned}$$

Then

$$\begin{aligned} & \mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})] \\ &= e^{d_1\lambda_1 + \dots + d_N\lambda_N} (C + b + D(c_1\lambda_1 + \dots + c_N\lambda_N)). \end{aligned}$$

Therefore, for $\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})]$ to change sign no more than once as $\lambda_1, \dots, \lambda_N$ increase, it is necessary and sufficient that c_1, \dots, c_N all have the same sign.

For (4), define

$$\begin{aligned} C &= 2q(c_1\mathbb{E}[\tilde{y}_1 - \tilde{z}_1] + \dots + c_N\mathbb{E}[\tilde{y}_N - \tilde{z}_N]) \quad \text{and} \\ D &= q\mathbb{E}[(c_1\tilde{y}_1 + \dots + c_N\tilde{y}_N)^2 - (c_1\tilde{z}_1 + \dots + c_N\tilde{z}_N)^2] \\ &\quad + d_1\mathbb{E}[\tilde{y}_1 - \tilde{z}_1] + \dots + d_N\mathbb{E}[\tilde{y}_N - \tilde{z}_N]. \end{aligned}$$

Then

$$\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})] = C(c_1\lambda_1 + \dots + c_N\lambda_N) + D.$$

Therefore, for $\mathbb{E}[u(\tilde{\mathbf{y}} + \boldsymbol{\lambda}) - u(\tilde{\mathbf{z}} + \boldsymbol{\lambda})]$ to change sign no more than once as $\lambda_1, \dots, \lambda_N$ increase, it is necessary and sufficient that c_1, \dots, c_N all have the same sign.

Now we show that if $u(x)$ satisfies the one-switch property, then it belongs to one of the families (1)–(4). First, Definition 2 and Lemma 1 take Definition 2 and Theorem 1 of Abbas and Bell (2011) and replace their first attribute x with a set of $N - 1$ attributes $\{x_1, \dots, x_{N-1}\}$ and their second attribute y with x_N . Lemma 1 is stated only as a one-way result because that is sufficient for our purposes. The proof of Lemma 1 follows the proof of Theorem 1 of Abbas and Bell (2011) and thus is not presented here.

DEFINITION 2. A set of attributes $\{x_1, \dots, x_{N-1}\}$ is one-switch independent of attribute x_N if preference between any pair of gambles on x_1, \dots, x_{N-1} can switch at most once as the level of x_N increases.

LEMMA 1. If a set of attributes x_1, \dots, x_{N-1} is one-switch independent of attribute x_N , then $u(x_1, \dots, x_N) = g_0(x_N) + f_1(x_1, \dots, x_{N-1})g_1(x_N) + f_2(x_1, \dots, x_{N-1})g_2(x_N)$.

For the “only if” part of our Theorem 1, we proceed by induction. For $N = 1$, this is proven in Bell (1988, Proposition 2). Suppose it holds for $N = k$. From Definition 1, the one-switch property for $N = k + 1$ implies that $\{x_1, \dots, x_k\}$ is one-switch independent of attribute x_{k+1} . Therefore, by Lemma 1, $u(x_1, \dots, x_{k+1}) = g_0(x_{k+1}) + f_1(x_1, \dots, x_k)g_1(x_{k+1}) + f_2(x_1, \dots, x_k)g_2(x_{k+1})$. The one-switch property for $N = k + 1$ also implies that $u(x_1, \dots, x_k, x_{k+1}^*)$, as a function of x_1, \dots, x_k , has the one-switch property for any fixed value of x_{k+1}^* , and that $u(x_1^*, \dots, x_k^*, x_{k+1})$, as a function of x_{k+1} , has the one-switch property for any fixed value of x_1^*, \dots, x_k^* . Therefore, by the induction assumption, the functions g_0, f_1, g_1, f_2 , and g_2 are of the forms given in (1)–(4). Finally, the one-switch property for $u(x_1, \dots, x_{k+1})$ implies that $u(\alpha x_1^*, \dots, \alpha x_{k+1}^*)$, as a function of α , is one-switch for any $x_i^* \geq 0, i = 1, \dots, k + 1$, and thus is of the form (1)–(4) with $N = 1$ (as in Bell 1988, Proposition 2). Then $u(x_1, \dots, x_{k+1})$ is of the form (1)–(4). This is because if $u(x_1, \dots, x_{k+1})$ has more than two summands, the number of summands would be more than two in $u(\alpha x_1^*, \dots, \alpha x_{k+1}^*)$, as a function of α , for some choice of $x_i^* \geq 0, i = 1, \dots, k + 1$. Similarly, if $u(x_1, \dots, x_{k+1})$ has two summands but is not in the form of (1)–(4) (e.g., $u(x_1, \dots, x_{k+1}) = (c_1x_1 + \dots + c_{k+1}x_{k+1})^2 + e^{d_1x_1 + \dots + d_{k+1}x_{k+1}}$), then $u(\alpha x_1^*, \dots, \alpha x_{k+1}^*)$, as a function of α , would not be of the form (1)–(4) for some choice of $x_i^* \geq 0, i = 1, \dots, k + 1$. \square

Proof of Theorem 2

By Theorem 1, $u(x)$ belongs to one of the families (1)–(4). If $u(x)$ belongs to family (1), let $u_s(x) = u(x)$. If $u(x)$ belongs to family (2) with $d_i \geq 0$ for $i = 1, \dots, N$, define $M = \max_{i=1, \dots, N} |c_i|$, and let $u_s(x) = be^{(d_1+sM)x_1 + \dots + (d_N+sM)x_N} + (e^{s(c_1x_1 + \dots + c_Nx_N)} - 1)/s$ with $s > 0$. The case where $d_i \leq 0$ is similar. If $u(x)$ belongs to family (3), let $u_s(x) = [(e^{s(c_1x_1 + \dots + c_Nx_N)} - 1)/s + b]e^{d_1x_1 + \dots + d_Nx_N}$. If $u(x)$ belongs to family (4), assume $c_i \geq 0$ for $i = 1, \dots, N$ without loss of generality; define $M_i = \max(0, qd_i)$ and $m_i = \min(0, qd_i), i = 1, \dots, N$; and let $u_s(x) = q(e^{s((c_1+sM_1)x_1 + \dots + (c_N+sM_N)x_N)} + e^{-s((c_1-sm_1)x_1 + \dots + (c_N-sm_N)x_N)} - 2)/s^2$, with $s > 0$, so that $s(c_i + sM_i) \geq -s(c_i - sm_i), i = 1, \dots, N$. By Taylor series expansions with respect to s ,

$$\begin{aligned} & e^{s((c_1+sM_1)x_1 + \dots + (c_N+sM_N)x_N)} \\ &= 1 + s((c_1 + sM_1)x_1 + \dots + (c_N + sM_N)x_N) \\ &\quad + s^2(c_1x_1 + \dots + c_Nx_N)^2/2 + o(s^2) \end{aligned}$$

and

$$\begin{aligned} & e^{-s((c_1-sm_1)x_1 + \dots + (c_N-sm_N)x_N)} \\ &= 1 - s((c_1 - sm_1)x_1 + \dots + (c_N - sm_N)x_N) \\ &\quad + s^2(c_1x_1 + \dots + c_Nx_N)^2/2 + o(s^2), \end{aligned}$$

where $\lim_{y \rightarrow 0} o(y)/y = 0$.

Thus,

$$u_s(x) = q[s^2(c_1x_1 + \dots + c_Nx_N)^2 + s^2((M_1 + m_1)x_1 + \dots + (M_N + m_N)x_N) + o(s^2)]/s^2.$$

Note that $M_i + m_i = qd_i$, $i = 1, \dots, N$, and therefore $\lim_{s \rightarrow 0^+} u_s(\mathbf{x}) = u(\mathbf{x})$. \square

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