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METHODS

Generalized Almost Stochastic Dominance

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Almost stochastic dominance allows small violations of stochastic dominance rules to avoid situations where most decision makers prefer one alternative to another but stochastic dominance cannot rank them. While the idea behind almost stochastic dominance is quite promising, it has not caught on in practice. Implementation issues and inconsistencies between integral conditions and their associated utility classes contribute to this situation. We develop generalized almost second-degree stochastic dominance and almost second-degree risk in terms of the appropriate utility classes and their corresponding integral conditions, and extend these concepts to higher degrees. We address implementation issues and show that generalized almost stochastic dominance inherits the appealing properties of stochastic dominance. Finally, we define convex generalized almost stochastic dominance to deal with risk-prone preferences. Generalized almost stochastic dominance could be useful in decision analysis, empirical research (e.g., in finance), and theoretical analyses of applied situations.

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1. Introduction

In decision analysis within the expected utility framework, a ranking of the distributions of outcomes for different actions is given by expected utility. However, if only partial information is available about the utility function (e.g., it is increasing and risk averse), expected utilities cannot be found. Stochastic dominance (SD) was developed to provide a partial ranking of distributions in such cases by limiting the signs of the derivatives of the utility function, and it has been widely studied and applied.

Less than full information about the utility function arises when a decision maker has difficulty expressing her preferences in sufficient detail, or when there are multiple decision makers who have different preferences and thus disagree on the appropriate utility function for the problem at hand. For example, consider an analyst helping a committee decide which of two projects a firm should implement. The committee must choose one of the projects, and it is not feasible to implement both. The first project gives a sure gain of \$1 million; the other gives a gain of \$4 million with probability p, $0 < p < 1$, and zero otherwise. The committee members agree that the utility function should be increasing and risk averse but provide no further information about their preferences, although the analyst infers from the discussion that some are more risk averse than others. First-degree stochastic dominance (FSD) cannot choose between these two projects. Second- (and higher-) degree SD would prefer the sure gain if $p \leqslant 0.25$ but is silent about the choice if $p > 0.25$.

It is easy to find examples where virtually all decision makers would prefer one distribution to another, but where SD cannot rank the distributions by second-degree stochastic dominance (SSD) because the set of all risk-averse utility functions includes some "pathological" functions that conform to very few (if any) individuals' preferences (Leshno and Levy 2002). In the above example, most decision makers would choose the second project if $p = 0.9$, but the analyst cannot recommend a choice based on SSD because of the possibility of some extremely risk-averse utility functions that would lead to a preference for the first project (the sure gain). Also, it is convenient to use risk aversion as a benchmark. Upon reflection, however, many decision makers who are uncomfortable with detailed assessment of their utility functions might conclude that accepting some risks that are not too large is reasonable. In this sense, SD might be too rigid in practice, even as a screening device.

The notion of almost stochastic dominance (ASD) developed in Leshno and Levy (2002) is intended to overcome the above drawbacks. Almost first- and second-degree stochastic dominance (AFSD and ASSD) limit ratios of marginal utilities and second derivatives of the utility function, respectively, to exclude some of the pathological utility functions. Levy (2006) uses a slightly modified version, changing the set of utility functions consistent with ASSD. Both approaches use the same ASSD integral condition. The idea behind ASSD is that whereas SSD determines a partial ranking of distributions consistent with *all* risk-averse decision makers, ASSD determines a partial ranking of distributions consistent with *most* risk-averse decision makers, or with *economically relevant* risk-averse decision makers (Levy et al. 2010).

The idea behind ASD seems promising, but despite a few empirical studies that focus on applications in finance, it has not caught on. One problem was pointed out by Tzeng et al. (2013), who show that the integral condition is not sufficient for the utility function to belong to Leshno and Levy's (2002) utility class. Beyond such fundamental problems, little attention has been paid to implementation issues, including the choice of how much to relax SD when applying ASD, the feasibility of applying the ASD dominance conditions, and the presentation of ASD results to decision makers in an understandable and useful way.

Tzeng et al. (2013) modify the integral condition to correct the inconsistencies found in the earlier approaches, but their modification does not inherit many appealing properties of SD or address implementation. In addition to internal consistency, some important issues are as follows:

- Unlike ASD in Tzeng et al. (2013), SD satisfies a hierarchy property: lower-degree dominance is stronger than higher-degree dominance in the sense that nth-degree SD implies kth-degree SD for all $k > n$.
- SD is related to probability shifts such as improvements, deteriorations, and mean-preserving spreads (Rothschild and Stiglitz 1970), which have a behavioral interpretation and could provide a more intuitive way to think about ASD than via integral conditions.
- More convenient computational methods to generate numerical results for ASD are needed. Lizyayev and Ruszczyński (2012) focus on computational efficiency but use a completely different definition of ASD.
- Practical matters such as the type of information needed from the decision maker and the presentation of ASD results to the decision maker should be addressed.
- SD satisfies a preference for combining good with bad (Eeckhoudt and Schlesinger 2006, Eeckhoudt et al. 2009) and can be viewed as related to a general notion of risk aversion. It would be reasonable to ask for the same from ASD.
- The concept of nth-degree risk (Ekern 1980) is a special case of SD that isolates the nth-degree effect. It would be helpful to have a concept of almost nth-degree risk (AnR) defined in a way consistent with ASD.

In this paper, we address the above drawbacks by developing a generalized concept of ASD. We define generalized AnSD (GAnSD) and AnR for all n in such a way that GAnSD satisfies the hierarchy property, is consistent with a preference for combining good with bad, and is consistent with the definition of AnR. The basic definitions and results are given in §2, and generalized almost stochastic dominance (GASD) relations are characterized in terms of allowing some relaxation of conditions of probability shifts (e.g., mean-preserving spreads) in §3. We consider implementation issues in §4, illustrating the application of GASD dominance conditions analytically and numerically and discussing how results might be presented to a decision maker. Connections with a preference for combining good with bad are discussed in §5, and extensions to the convex case, which might be thought of as the "risk-prone version" of GASD, are considered in §6, followed by concluding remarks in §7. Proofs and other supporting materials are provided in the appendices.

2. Generalized Almost Stochastic Dominance

2.1. Previous Work

Let u denote a decision maker's utility function and $u^{(k)}$ the kth derivative of u. Also, let F and G represent cumulative distribution functions (cdfs) with support $[a, b]$, define $F^{(k)}(x) = \int_a^x F^{(k-1)}(t)\,dt$ for $k \geqslant 2$ with $F^{(1)}(x) = F(x)$, and define $G^{(k)}(x)$ similarly. Finally, define $S_1(F, G) = \{x \in [a, b] \mid F(x) \geqslant G(x)\}$ and $S_2^{LL}(F, G) = \{x \in S_1(F, G) \mid F^{(2)}(x) \geqslant G^{(2)}(x)\}$.

The initial definition of AFSD by Leshno and Levy (2002) is given in terms of an integral condition.

Definition 1. For $0 < \varepsilon_1 < \frac{1}{2}$, F dominates G by ε_1-AFSD if

$$\int_{S_1(F,G)} [F(x) - G(x)]\,dx \leqslant \varepsilon_1 \int_a^b |F(x) - G(x)|\,dx.$$

They also associate AFSD with the utility class

$$U_1(\varepsilon_1) = \left\{ u \mid u^{(1)} > 0 \text{ and } \sup\{u^{(1)}(x)\} \leqslant \inf\{u^{(1)}(x)\}\left(\frac{1}{\varepsilon_1} - 1\right) \right\}$$

by showing that for $0 < \varepsilon_1 < \frac{1}{2}$, F dominates G by ε_1-AFSD if and only if $E_F(u) \geqslant E_G(u)$ for all $u \in U_1(\varepsilon_1)$. The intuition of Definition 1 is that F dominates G by ε_1-AFSD if and only if the ratio of the area between F and G for which $F(x) \geqslant G(x)$ (i.e., the area that violates the condition of F dominating G by FSD) to the total area between F and G is less than or equal to ε_1. For FSD, this ratio must be zero (i.e., F can never be above G), but for ε_1-AFSD, it only has to satisfy the weaker condition of being less than or equal to ε_1, with the limiting case $\varepsilon_1 = 0$ corresponding to FSD.

Leshno and Levy (2002) use another integral condition to define what we label ASSD^{LL}.

Definition 2. For $0 < \varepsilon_2 < \frac{1}{2}$, F dominates G by ε_2-ASSD^{LL} if

$$\int_{S_2^{LL}(F,G)} [F(x) - G(x)]\,dx \leqslant \varepsilon_2 \int_a^b |F(x) - G(x)|\,dx$$

and $E_F(X) \geqslant E_G(X)$.

Leshno and Levy (2002) associate ε_2-ASSD^{LL} with the utility class

$$U_2(\varepsilon_2) = \left\{ u \mid u^{(1)} > 0,\ u^{(2)} < 0, \text{ and } \sup\{-u^{(2)}(x)\} \leqslant \inf\{-u^{(2)}(x)\}\left(\frac{1}{\varepsilon_2} - 1\right)\right\},$$

claiming that for $0 < \varepsilon_2 < \frac{1}{2}$, F dominates G by ε_2-ASSD^{LL} if and only if $E_F(u) \geqslant E_G(u)$ for all $u \in U_2(\varepsilon_2)$. The intuition is similar to that for Definition 1, where the values of x for which SSD is violated are of concern.

While AFSD, as discussed above, is used consistently in the literature, there are different versions of ASSD. Some applications of ASD in finance, such as Bali et al. (2013), use Leshno and Levy's (2002) ASSD^{LL}. Following Levy (2006), other applications, such as Levy et al. (2010) and Levy (2012), use the integral condition in Definition 2 but replace $U_2(\varepsilon_2)$ with

$$U_2^*(\varepsilon_1) = \left\{ u \mid u^{(1)} > 0,\ u^{(2)} < 0, \text{ and } \sup\{u^{(1)}(x)\} \leqslant \inf\{u^{(1)}(x)\}\left(\frac{1}{\varepsilon_1} - 1\right)\right\},$$

replacing the constraint on $u^{(2)}$ in $U_2(\varepsilon_2)$ with a constraint on $u^{(1)}$ like the one in $U_1(\varepsilon_1)$.

Tzeng et al. (2013) show that the integral condition in Definition 2 is not sufficient for the result that $E_F(u) \geqslant E_G(u)$ for all $u \in U_2(\varepsilon_2)$. It is also not necessary for that result, and it is neither necessary nor sufficient for the Levy (2006) claim that $E_F(u) \geqslant E_G(u)$ for all $u \in U_2^*(\varepsilon_1)$. Thus, the standard integral condition used in the literature is not consistent with either of the two utility classes to which it has been claimed to be consistent.

To fix that inconsistency, Tzeng et al. (2013) replace $S_2^{LL}(F,G)$ in Definition 2 with $S_2(F,G) = \{x \in [a,b] \mid F^{(2)}(x) \geqslant G^{(2)}(x)\}$ and replace $F(x) - G(x)$ in the integrand of the integral condition with $F^{(2)}(x) - G^{(2)}(x)$.

Definition 3. For $0 < \varepsilon_2 < \frac{1}{2}$, F dominates G by ε_2-ASSD if

$$\int_{S_2(F,G)} [F^{(2)}(x) - G^{(2)}]\,dx \leqslant \varepsilon_2 \int_a^b |F^{(2)}(x) - G^{(2)}(x)|\,dx$$

and $E_F(X) \geqslant E_G(X)$.

They show that for $0 < \varepsilon_2 < \frac{1}{2}$, F dominates G by ε_2-ASSD according to Definition 3 if and only if $E_F(u) \geqslant E_G(u)$ for all $u \in U_2(\varepsilon_2)$.

2.2. Generalized Almost Second-Degree Stochastic Dominance (GASSD)

Now, we present our generalized concept of ASD, starting with ASSD and then extending our approach to almost nth-degree SD. For AFSD, we use the same utility class and definition as Leshno and Levy (2002), discussed above. For ASSD, we first define two new utility classes:

$$U_2^R(\varepsilon_2) = \left\{ u \mid u^{(2)} < 0 \text{ and } \sup\{-u^{(2)}(x)\} \leqslant \inf\{-u^{(2)}(x)\}\left(\frac{1}{\varepsilon_2} - 1\right)\right\}, \quad \text{and}$$

$$\underline{U}_2(\varepsilon_1, \varepsilon_2) = \left\{ u \mid u^{(1)} > 0,\ u^{(2)} < 0 \text{ and } \sup\{(-1)^{k+1}u^{(k)}(x)\} \leqslant \inf\{(-1)^{k+1}u^{(k)}(x)\}\left(\frac{1}{\varepsilon_k} - 1\right),\ k = 1, 2.\right\}.$$

Definitions of almost second-degree risk (ASR), which is related to the nth-degree risk of Ekern (1980), and our GASSD are as follows.

Definition 4. For $0 \leqslant \varepsilon_2 \leqslant \frac{1}{2}$, G has more ε_2-ASR than F if $E_F(u) \geqslant E_G(u)$ for all $u \in U_2^R(\varepsilon_2)$.

Definition 5. For $0 \leqslant \varepsilon_k \leqslant \frac{1}{2}$, $k = 1, 2$, F dominates G by $(\varepsilon_1, \varepsilon_2)$-GASSD if $E_F(u) \geqslant E_G(u)$ for all $u \in \underline{U}_2(\varepsilon_1, \varepsilon_2)$.

The following theorems provide integral conditions to find partial rankings of distributions for $u \in U_2^R(\varepsilon_2)$ and $u \in \underline{U}_2(\varepsilon_1, \varepsilon_2)$, respectively.

Theorem 1. *For $0 \leqslant \varepsilon_2 \leqslant \frac{1}{2}$, G has more ε_2-ASR than F if and only if*

$$F^{(2)}(b) - G^{(2)}(b) = 0 \text{ and}$$

$$\int_{S_2(F,G)} \left[F^{(2)}(x) - G^{(2)}(x)\right] dx \leqslant \varepsilon_2 \int_a^b |F^{(2)}(x) - G^{(2)}(x)|\,dx.$$

Theorem 2. *For $0 \leqslant \varepsilon_k \leqslant \frac{1}{2}$, $k = 1, 2$, F dominates G by $(\varepsilon_1, \varepsilon_2)$-GASSD if and only if*

$$F^{(2)}(b) - G^{(2)}(b) \leqslant 0 \text{ and}$$

$$\max_C \left\{ \frac{1}{(1 - 2\varepsilon_2)|C| + \varepsilon_2(b - a)} \cdot \left[(1 - 2\varepsilon_2) \int_C (F^{(2)}(x) - G^{(2)}(x))\,dx + \varepsilon_2 \int_a^b (F^{(2)}(x) - G^{(2)}(x))\,dx \right] \right\} \leqslant \frac{\varepsilon_1}{1 - 2\varepsilon_1} [G^{(2)}(b) - F^{(2)}(b)],$$

where $C \subset [a,b]$ and $|C| = \int_C dx$.

The proof of Theorem 1 follows the proof of the main result in Tzeng et al. (2013) and is omitted. The proof of Theorem 2 is given in Appendix A.

We define more ASR in terms of a distribution having higher expected utilities than another distribution for utility functions in the utility class $U_2^R(\varepsilon_2)$, and then prove that the conditions in Theorem 1 hold if and only if G has more ε_2-ASR than F. We define GASSD similarly, in terms of its utility class, and then prove that the conditions in Theorem 2 hold if and only if F dominates G by $(\varepsilon_1, \varepsilon_2)$-GASSD. The condition $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$ in Theorem 2 is equivalent to the $E_F(X) \geqslant E_G(X)$ condition in Definitions 2 and 3.

We can also define Tzeng et al.'s (2013) ASSD in terms of its utility class, $U_2(\varepsilon_2)$, because their integral condition holds if and only if $E_F(u) \geqslant E_G(u)$ for all $u \in U_2(\varepsilon_2)$. Note that their ASSD is not implied by AFSD because $U_2(\varepsilon_2)$ is not a subset of $U_1(\varepsilon_1)$. This violates the hierarchy property that is satisfied by SD: nth-degree SD implies kth-degree SD for all $k > n$.

In contrast, GASSD does satisfy the hierarchy property because $\underline{U}_2(\varepsilon_1, \varepsilon_2) \subset \underline{U}_1(\varepsilon_1)$. Moreover, $\underline{U}_2(\varepsilon_1, \varepsilon_2) \subset U_2(\varepsilon_2) \subset U_2^R(\varepsilon_2)$, so that GASSD is implied by Tzeng et al.'s (2013) ASSD, which, in turn, is implied by ASR. The extension to GAnSD, as defined later, satisfies the hierarchy property for all n.

Theorem 1 says that, when $E_F(X) = E_G(X)$, G has more ε_2-ASR than F if and only if the ratio of the area between $F^{(2)}$ and $G^{(2)}$ that violates the condition of F dominating G by SSD dominance to the total area between $F^{(2)}$ and $G^{(2)}$ is less than or equal to ε_2. Similar to Ekern (1980), $U_2^R(\varepsilon_2)$ does not impose any constraint on $u^{(1)}$. Thus, the ASR in Theorem 2 can be isolated from the effect of $u^{(1)}$.

Theorem 2 generalizes the concept of Tzeng et al.'s (2013) ASSD. The integral condition in Theorem 2 is equivalent to

$$\begin{aligned} &\max_{h(x)} \int_a^b h(x)(F^{(2)}(x) - G^{(2)}(x))\,dx \\ &\qquad \leqslant \frac{\varepsilon_1}{1-2\varepsilon_1}[G^{(2)}(b) - F^{(2)}(b)] \\ &\text{s.t.}\quad h(x) \geqslant 0, \quad \int_a^b h(x)\,dx = 1, \quad \text{and} \\ &\qquad \sup h(x) \leqslant \inf\{h(x)\}\left(\frac{1}{\varepsilon_2} - 1\right). \end{aligned} \tag{1}$$

Note that the left-hand side of Equation (1) represents the maximum weighted loss between $F^{(2)}(x)$ and $G^{(2)}(x)$, whereas the right-hand side of Equation (1) is the adjusted gain from $E_F(X) - E_G(X)$. If the adjusted gain from the mean difference is larger than the maximum loss between $F^{(2)}(x)$ and $G^{(2)}(x)$, then all decision makers in $\underline{U}_2(\varepsilon_1, \varepsilon_2)$ prefer F to G.

A couple of boundary cases in Theorem 2 deserve some discussion because of connections to earlier ASSD approaches. If $\varepsilon_1 = 0$, the utility class $\underline{U}_2(0, \varepsilon_2)$ associated with $(0, \varepsilon_2)$-GASSD is the same as the utility class $U_2(\varepsilon_2)$ associated with Leshno and Levy's (2002) ASSD. If $\varepsilon_2 = 0$, the utility class $\underline{U}_2(\varepsilon_1, 0)$ associated with $(\varepsilon_1, 0)$-GASSD is the same as the utility class $U_2^*(\varepsilon_1)$ associated with Levy's (2006) ASSD.

Despite these connections in terms of utility classes, the necessary and sufficient conditions from Theorem 2 for $(0, \varepsilon_2)$-GASSD and $(\varepsilon_1, 0)$-GASSD are not the same as the integral condition from Definition 2 that is used for ASSD by Leshno and Levy (2002) and Levy (2006). From Theorem 2, necessary and sufficient conditions for the boundary cases $(0, \varepsilon_2)$-GASSD and $(\varepsilon_1, 0)$-GASSD can be expressed as follows:

- F dominates G by $(\varepsilon_1, 0)$-GASSD if and only if $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$ and

$$\max_{x\in[a,b]}\{F^{(2)}(x) - G^{(2)}(x)\} \leqslant \frac{\varepsilon_1}{1-2\varepsilon_1}[G^{(2)}(b) - F^{(2)}(b)]. \tag{2}$$

- F dominates G by $(0, \varepsilon_2)$-GASSD if and only if $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$ and

$$\begin{aligned} &(1-2\varepsilon_2)\int_{S_2(F,G)} (F^{(2)}(x) - G^{(2)}(x))\,dx \\ &\quad + \varepsilon_2 \int_a^b (F^{(2)}(x) - G^{(2)}(x))\,dx \leqslant 0. \end{aligned} \tag{3}$$

This condition is equivalent to Equation (9) in Tzeng et al. (2013).

Some intuition underlying the logic behind $(\varepsilon_1, \varepsilon_2)$-GASSD is that GASSD includes constraints on $u^{(1)}$ *and* $u^{(2)}$ in the utility class $\underline{U}_2(\varepsilon_1, \varepsilon_2)$. This is an important difference between the earlier ASSD approaches and GASSD. Constraining only $u^{(1)}$ for ASSD, as is done in $U_2^*(\varepsilon_1)$, leaves out higher-order effects. For example, by constraining only $u^{(1)}$, we cannot consider second-degree risk, which seems quite relevant for ASSD. Constraining only $u^{(2)}$ for ASSD, as is done in $U_2(\varepsilon_2)$, means that ASSD is not implied by AFSD. This prevents ASD from being nested (i.e., from satisfying the hierarchical property of SD). These issues do not arise with AFSD, for which there is no reason to consider constraining higher-order derivatives of u.

A potential drawback of $(\varepsilon_1, \varepsilon_2)$-GASSD as defined in Theorem 2 is that for many situations, it is not feasible (or not practical) to obtain analytical results. In such cases, GASSD can be investigated numerically. Implementation issues are discussed in §4 and illustrated via an example that is amenable to analytical results. Appendix B restates Theorem 2 for numerical implementation and provides an example of its application.

2.3. GAnSD

Next, we extend our approach to GAnSD. Define

$$\begin{aligned} U_n^R(\varepsilon_n) = \Big\{ u \mid (-1)^{n+1}u^{(n)} > 0 \text{ and } &\sup\{(-1)^{n+1}u^{(n)}(x)\} \\ &\leqslant \inf\{(-1)^{n+1}u^{(n)}(x)\}\left(\frac{1}{\varepsilon_n} - 1\right)\Big\}, \end{aligned}$$

$$U_n(\varepsilon_n) = \Big\{ u \mid (-1)^{k+1}u^{(k)} > 0,\ k = 1, 2, \ldots, n, \text{ and } \sup\{(-1)^{n+1}u^{(n)}(x)\} \leqslant \inf\{(-1)^{n+1}u^{(n)}(x)\}\left(\frac{1}{\varepsilon_n} - 1\right) \Big\}, \quad \text{and}$$

$$\underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \Big\{ u \mid (-1)^{k+1}u^{(k)} > 0,\ k = 1, 2, \ldots, n, \text{ and } \sup\{(-1)^{k+1}u^{(k)}(x)\} \leqslant \inf\{(-1)^{k+1}u^{(k)}(x)\}\left(\frac{1}{\varepsilon_k} - 1\right),\ k = 1, 2, \ldots, n \Big\}.$$

Definition 6. G has more ε_n-AnR than F if $E_F(u) \geqslant E_G(u)$ for all $u \in U_n^R(\varepsilon_n)$.

Let $S_n(F, G) = \{x \in [a, b] \mid F^{(n)}(x) \geqslant G^{(n)}(x)\}$. The following integral conditions can be used to partially rank distributions for $u \in U_n(\varepsilon_n)$. The proof is similar to the proof for AnSD in Tzeng et al. (2013).

Theorem 3. *For* $0 \leqslant \varepsilon_n \leqslant \frac{1}{2}$, *$G$ has more ε_n-AnR than F if and only if* $F^{(k)}(b) - G^{(k)}(b) = 0$, $k = 2, \ldots, n$, *and* $\int_{S_n(F,G)} [F^{(n)}(x) - G^{(n)}(x)]\, dx \leqslant \varepsilon_n \int_a^b |F^{(n)}(x) - G^{(n)}(x)|\, dx$.

Similar to the condition of nth-degree risk defined by Ekern (1980), almost nth-degree risk also requires that the kth moments, $k \leqslant n$, of the two distributions be the same. To find out whether G has more nth-degree risk than F, we need to check whether $F^{(n)}(x) \leqslant G^{(n)}(x)$ for all x. AnR relaxes the condition of nth-degree risk by allowing small violations. As long as the violation area, $\int_{S_n(F,G)} [F^{(n)}(x) - G^{(n)}(x)]\, dx$, is small enough, then G has more ε_n-almost nth-degree risk than F.

Definition 7. F dominates G by ε_n-AnSD if $E_F(u) \geqslant E_G(u)$ for all $u \in U_n(\varepsilon_n)$.

Theorem 4, the proof of which is in Tzeng et al. (2013), shows the close relationship between AnR and AnSD.

Theorem 4. *For* $0 \leqslant \varepsilon_n \leqslant \frac{1}{2}$, *$F$ dominates G by ε_n-AnSD if and only if* $F^{(k)}(b) - G^{(k)}(b) \leqslant 0$, $k = 2, \ldots, n$, *and* $\int_{S_n(F,G)} [F^{(n)}(x) - G^{(n)}(x)]\, dx \leqslant \varepsilon_n \int_a^b |F^{(n)}(x) - G^{(n)}(x)|\, dx$.

Now, we define GAnSD as follows.

Definition 8. F dominates G by $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$-GAnSD if $E_F(u) \geqslant E_G(u)$ for all $u \in \underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

As with GASSD, GAnSD satisfies the hierarchy property exhibited by SD because $\underline{U}_k(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \subset \underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for all $k > n$. Also, $\underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \subset U_n(\varepsilon_n) \subset U_n^R(\varepsilon_n)$ for all n. Thus, GAnSD is implied by AnSD and is consistent with our definition of AnR.

The following theorem uses $n = 3$ to demonstrate the corresponding integral conditions.

Theorem 5. *For* $0 \leqslant \varepsilon_k \leqslant \frac{1}{2}$, $k = 1, 2, 3$, *F dominates G by $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$-generalized almost third-degree SD if and only if*

$$F^{(2)}(b) - G^{(2)}(b) \leqslant 0,\ F^{(3)}(b) - G^{(3)}(b) + \frac{\varepsilon_1}{1 - 2\varepsilon_1}(b - a)[F^{(2)}(b) - G^{(2)}(b)] \leqslant 0, \quad \textit{and}$$

$$\int_a^b h(x)(F^{(3)}(x) - G^{(3)}(x))\, dx + \frac{\varepsilon_1}{1 - 2\varepsilon_1} \int_a^b \left(\int_x^b h(s)\, ds \right) dx [F^{(2)}(b) - G^{(2)}(b)] \leqslant \frac{\varepsilon_2}{1 - 2\varepsilon_2} \left\{ \frac{\varepsilon_1}{1 - 2\varepsilon_1}(b - a)[G^{(2)}(b) - F^{(2)}(b)] + [G^{(3)}(b) - F^{(3)}(b)] \right\} \ \textit{for all}\ h(x) \geqslant 0,$$

where $\int_a^b h(x)\, dx = 1$ *and* $\sup h(x) \leqslant \inf\{h(x)\}(1/\varepsilon_3 - 1)$.

The proof is given in Appendix A. The conditions of Theorem 5 suggest that finding partial rankings of distributions by GAnSD in practice becomes more difficult as n gets larger. As mentioned earlier, $n = 2$, the most important case for most applications, is tractable (see Appendix B). Also, in §5, we show that if preferences satisfy higher-degree GASD, which is not difficult to check, some results can be obtained by comparing the distributions using lower-order GASD and a preference for combining good with bad.

3. Almost Stochastic Dominance and Probability Shifts

Partial orders representing SD are typically characterized in terms of integral conditions. ASD, as discussed in §2.1, builds on this by allowing some limited violation of the corresponding SD integral conditions. Another way to characterize SD relations is in terms of probability shifts, which might be more intuitive and have more of a behavioral interpretation. Shifting some probability mass to the right (i.e., to higher values of x), for example, is a clear FSD improvement. A mean-preserving spread or sequence of such spreads (Rothschild and Stiglitz 1970) is an SSD deterioration. In this section, we show that ASD relations can be characterized in terms of allowing some relaxation of the conditions on probability shifts for SD relations, thereby expressing the relaxation associated with values of ε_1, ε_2, etc. in terms of probability shifts and finding conditions on the probability shifts that are inconclusive for SD but conclusive for ASD. We start with AFSD and then consider GASSD.

FSD and ε_1-AFSD. Under FSD, shifting probability mass to the right (from x_+ to $x_+ + \delta_+$ with $\delta_+ > 0$) is an improvement, and shifting probability mass to the left (from x_- to $x_- - \delta_-$ with $\delta_- > 0$) is a deterioration. FSD is silent if first-degree improvement and first-degree deterioration occur simultaneously: suppose probability mass is shifted to the right, from x_+ to $x_+ + \delta_+$, and the same amount of probability mass is shifted to the left, from x_- to $x_- - \delta_-$. FSD cannot say whether this combination is desirable as long as both δ_+ and δ_- are strictly positive. However, intuitively it seems plausible that if δ_- is much smaller than δ_+, this combination of first-degree improvement and deterioration should be desirable. This is what happens with ε_1-AFSD: the combination is an ε_1-AFSD improvement if and only if $\delta_+ \geqslant (1/\varepsilon_1 - 1)\delta_-$, and it is an ε_1-AFSD deterioration if and only if $\delta_- \geqslant (1/\varepsilon_1 - 1)\delta_+$. Thus ε_1-AFSD allows us to rank a combination of FSD improvement and FSD deterioration if a deterioration is much smaller or larger than an improvement, and ε_1 can be thought of in terms of the relative sizes of the probability shifts.

Theorem 6 states this result formally. To simplify the exposition, this theorem is presented for simple 50-50 lotteries. More general comparisons involving continuous distributions can be made via the integral condition from Definition 1.

Theorem 6. *Let G be a 50-50 lottery between x_- and x_+, and F a 50-50 lottery between $x_- - \delta_-$ and $x_+ + \delta_+$, with $\delta_- > 0$, $\delta_+ > 0$, $\delta_+ \geqslant (1/\varepsilon_1 - 1)\delta_-$, and all lottery outcomes contained in $[a, b]$. Then a decision maker with utility function u prefers F to G for all x_-, x_+, δ_-, and δ_+ satisfying the conditions above if and only if $u \in U_1(\varepsilon_1)$.*

SSD and $(\varepsilon_1, 0)$-GASSD. Under SSD, two shifts as above might also be desirable if an improvement precedes a deterioration of the same or smaller size, where "precedes" refers to the final values of x after the two shifts (Menezes et al. 1980, Chiu 2005): $x_+ + \delta_+ < x_- - \delta_-$ and $\delta_+ \geqslant \delta_-$. Note that this can be viewed as an example of combining good with bad. The precedence condition means that a good (the improvement) is combined with a bad (the lower value of x) and the corresponding bad (the deterioration) with the other good (the higher value of x). Preferences consistent with $(\varepsilon_1, 0)$-GASSD allow for that, and also for a deterioration to precede the improvement if $\delta_+ \geqslant (1/\varepsilon_1 - 1)\delta_-$.

SSD and $(0, \varepsilon_2)$-GASSD. Under SSD, a mean-preserving contraction is an improvement, a mean-preserving spread is a deterioration, and SSD is silent if a mean-preserving contraction and a mean-preserving spread occur simultaneously. Again, intuitively it seems plausible that if a mean-preserving contraction is much stronger than a mean-preserving spread, this combination should be desirable. This is what $(0, \varepsilon_2)$-GASSD assumes.

Consider a mean-preserving spread with two probability masses of equal size being shifted "outward," from x_1 to $x_1 + \delta$ and from x_2 to $x_2 - \delta$, where $x_1 \geqslant x_2$. The decrease in expected utility increases with δ and with the distance between x_1 and x_2. Below, we focus on mean-preserving spreads with zero distance between x_1 and x_2 (i.e., from x to $x + \delta$ and $x - \delta$). An inverse transformation to this mean-preserving spread with zero distance is a mean-preserving contraction, where probability masses of the same size are shifted "inward" to a single point, from $x + \delta$ to x and from $x - \delta$ to x.

Like Theorem 6, Theorem 7 is presented for simple lotteries to simplify the exposition. The focus is on the effect of mean-preserving transformations, a second-degree effect. Thus, if and only if conditions are stated in terms of utility functions belonging to $U_2^R(\varepsilon)$, which is more general than $\underline{U}_2(\varepsilon_1, \varepsilon_2)$. More general comparisons involving continuous distributions can be done via the integral condition from Theorem 1.

Theorem 7. *Define distribution G as $p(x_+ - \delta_+) = p(x_+ + \delta_+) = 0.25$ and $p(x_-) = 0.5$, and distribution F as $p(x_+) = 0.5$, $p(x_- - \delta_-) = p(x_- + \delta_-) = 0.25$ with $\delta_- > 0$, $\delta_+ > 0$, $\delta_+^2 \geqslant (1/\varepsilon_2 - 1)\delta_-^2$, and all lottery outcomes contained in $[a, b]$. Then, a decision maker with utility function u prefers F to G for all x_-, x_+, δ_-, and δ_+ satisfying the conditions above if and only if $u \in U_2^R(\varepsilon_2)$.*

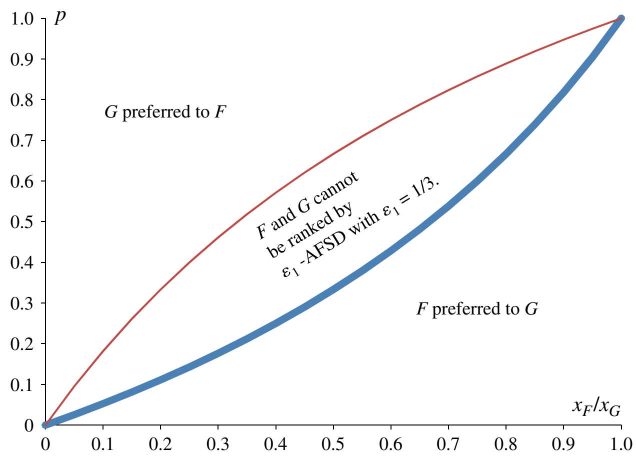
Combining the above conclusions about $(\varepsilon_1, 0)$-GASSD and $(0, \varepsilon_2)$-GASSD, $(\varepsilon_1, \varepsilon_2)$-GASSD is consistent with combinations of first- and second-degree improvements and deteriorations, where $\varepsilon_1(\varepsilon_2)$ corresponds to how much first- (second-) degree improvement should exceed first- (second-) degree deterioration.

The same logic extends to GAnSD for $n > 2$. For example, third-degree SD is about preferences for a decrease in downside risk, and $(0, 0, \varepsilon_3)$-GA3SD allows us to rank a combination of a decrease and increase in downside risk if a decrease is much stronger than an increase. The required strength to allow the ranking is smaller as ε_3 gets larger. Further results regarding GAnSD for $n > 2$ are given in §5, where connections with the concept of combining good with bad are discussed, because it is more convenient to discuss higher-order preferences using those connections.

4. Implementation

Implementation issues include whether it is feasible to investigate dominance by applying the dominance conditions in decision making, and how the resulting dominance information can be presented to the decision maker in an understandable and useful format. In this section, we demonstrate the application of GASD dominance conditions, giving analytical and numerical results, and we show how they might be presented to a decision maker. We consider an example with a format that is consistent with the situation discussed in §1. Suppose that F and G are cdfs corresponding to distributions of X that give x_F with probability p_F (respectively, x_G with probability p_G) and zero otherwise, with $0 < x_F < x_G$. If $p_F \leqslant p_G$, then G dominates F by FSD. If $p_F > p_G$, then F and G cannot be ranked by FSD, and we will explore their ranking by ASD.

Figure 1. (Color online) Regions of values of $(x_F/x_G, p)$ for which F and G can and cannot be ranked by AFSD with $\varepsilon_1 = 1/3$.



For any utility function u with $u(0) = 0$ and $u(x_G) = 1$, $F \succ (\prec) G$ if and only if $u(x_F) > (<) p$, where $p = p_G/p_F$, $0 < p < 1$. Therefore, comparing F and G is equivalent to comparing a sure gain of x_F and a risky gain of x_G with probability p, which amounts to comparing $u(x_F)$ and p. If u, defined on $[0, x_G]$, belongs to a particular class of utilities that limits the values of $u(x_F)$ [below we consider $U_1(\varepsilon_1)$, $\underline{U}_2(\varepsilon_1, 0)$, $\underline{U}_2(0, \varepsilon_2)$, and $\underline{U}_2(\varepsilon_1, \varepsilon_2)$], we can apply the corresponding ASD criterion.

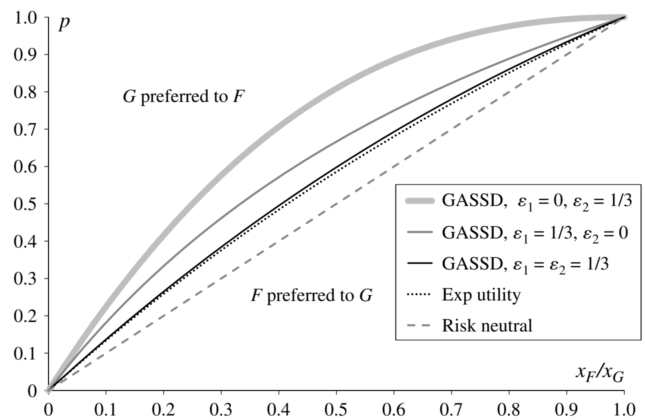
ε_1-AFSD. For ε_1-AFSD, F dominates G if and only if $p \leqslant c\varepsilon_1/(c\varepsilon_1 + (1-c)(1-\varepsilon_1))$, where $c = x_F/x_G$, $0 < c < 1$. Similarly, G dominates F if and only if $p \geqslant c(1-\varepsilon_1)/(c(1-\varepsilon_1) + (1-c)\varepsilon_1)$. Thus, F and G cannot be ranked by ε_1-AFSD if $c\varepsilon_1/(c\varepsilon_1 + (1-c)(1-\varepsilon_1)) \leqslant p \leqslant c(1-\varepsilon_1)/(c(1-\varepsilon_1) + (1-c)\varepsilon_1)$.

Figure 1 displays the ranges of $(x_F/x_G, p)$ for which F and G can and cannot be ranked by AFSD with $\varepsilon_1 = 1/3$. If ε_1 decreases, the inner range (where F and G cannot be ranked by AFSD) gets wider. As ε_1 increases, the inner range gets narrower, and it disappears at $\varepsilon_1 = 1/2$, where AFSD is equivalent to ranking the alternatives based only on the expected value. The example mentioned in §1 corresponds to $x_F/x_G = 0.25$.

$(\varepsilon_1, 0)$-GASSD. If preferences satisfy $(\varepsilon_1, 0)$-GASSD (i.e., $u \in \underline{U}_2(\varepsilon_1, 0)$), then $c \leqslant u(x_F) \leqslant c(1-\varepsilon_1)/(c(1-\varepsilon_1) + (1-c)\varepsilon_1)$. Therefore, G dominates F by $(\varepsilon_1, 0)$-GASSD if and only if $(1-p)x_F\varepsilon_1 \geqslant p(x_G - x_F)(1-\varepsilon_1)$, or $p \geqslant c(1-\varepsilon_1)/(c(1-\varepsilon_1) + (1-c)\varepsilon_1)$, and F dominates G by $(\varepsilon_1, 0)$-GASSD if and only if $p \leqslant x_F/x_G$. In comparison with ε_1-AFSD, which is shown in Figure 1, the upper bound for p to be in the region where F and G cannot be ranked remains the same but the lower bound changes to $p = x_F/x_G$.

$(0, \varepsilon_2)$-GASSD. If preferences satisfy $(0, \varepsilon_2)$-GASSD, then G dominates F if and only if $p \geqslant (c/(2a_2))(2a_2 - c + \sqrt{(2a_2 - c)^2 + 4a_2(1-a_2)})$, where $a_2 = \varepsilon_2/(1-\varepsilon_2)$.

Figure 2. Regions of values of $(x_F/x_G, p)$ for which F and G can and cannot be ranked by $(0, 1/3)$-GASSD, $(1/3, 0)$-GASSD, $(1/3, 1/3)$-GASSD, and exponential utility with risk aversion corresponding to $\varepsilon = 1/3$.



Note. In the region between each curve and the risk-neutral line, F and G cannot be ranked.

Therefore, $u \in \underline{U}_2(0, \varepsilon_2)$ if and only if $c \leqslant u(x_F) \leqslant (c/(2a_2))(2a_2 - c + \sqrt{(2a_2 - c)^2 + 4a_2(1-a_2)})$. In particular, for $\varepsilon_1 = 0$, $\varepsilon_2 = 1/3$, and $x_F/x_G = 0.25$, G dominates F by $(0, \varepsilon_2)$-GASSD if $p \geqslant 0.5$. As shown in Figure 2, $(0, 1/3)$-GASSD provides a higher bound on p and a larger region where F and G cannot be ranked than is provided by GASSD with $(1/3, 0)$-GASSD, where G dominates F by $(1/3, 0)$-GASSD if $p \geqslant 0.4$.

$(\varepsilon_1, \varepsilon_2)$-GASSD. For $(\varepsilon_1, \varepsilon_2)$-GASSD, the results presented in Figure 2 for $\varepsilon_1 = \varepsilon_2 = 1/3$ were generated numerically from the conditions in Theorem 2. (See Figure B.1 in Appendix B for an example involving $(\varepsilon_1, \varepsilon_2)$-GASSD with $\varepsilon_1 \neq \varepsilon_2$.) In comparison with $(\varepsilon_1, 0)$-GASSD and $(0, \varepsilon_2)$-GASSD, $(\varepsilon_1, \varepsilon_2)$-GASSD gives a smaller region where F and G cannot be ranked. If $\varepsilon_1 = \varepsilon_2 = 1/3$ and $x_F/x_G = 0.25$, for example, G dominates F by $(\varepsilon_1, \varepsilon_2)$-GASSD if $p \geqslant 0.326$. This means that having ε_1 *and* ε_2 equal $1/3$ is less rigid than having only one of them equal $1/3$ and the other equal zero, which agrees with intuition.

Finally, we consider preferences that satisfy exponential utility. This is a necessary condition for G to dominate F via ε-almost infinite-degree SD, which corresponds to $u \in \underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = \varepsilon$ for $n = 1, 2, \ldots$. If $u(x) = -\exp(-\rho x)$, $x \in [a, b]$ (in our example, $x \in [0, x_G]$), then $u^{(k)}(b)/u^{(k)}(a) = \exp(-\rho(b-a))$, and ε-almost infinite-degree SD implies that $\exp(-\rho(b-a)) > \varepsilon/(1-\varepsilon)$ if and only if $\rho < (1/x_G)\ln((1-\varepsilon)/\varepsilon)$. Therefore, $\rho^* = (1/x_G)\ln((1-\varepsilon)/\varepsilon)$ is a boundary case and $E[-\exp(-\rho^* x_G)] \geqslant E[-\exp(-\rho^* x_F)]$ if and only if $p \geqslant (1 - (\varepsilon/(1-\varepsilon))^{x_F/x_G})/(1 - \varepsilon/(1-\varepsilon))$. Thus, if utility is CARA with $u^{(1)}(x_G)/u^{(1)}(0) > \varepsilon/(1-\varepsilon)$, $u(0) = 0$, and $u(x_G) = 1$, then $x_F/x_G \leqslant u(x_F) \leqslant (1 - (\varepsilon/(1-\varepsilon))^{x_F/x_G})/(1 - \varepsilon/(1-\varepsilon))$.

In addition to considering $(1/3, 0)$-GASSD and $(0, 1/3)$-GASSD, Figure 2 shows the regions of $(x_F/x_G, p)$ for which F and G can and cannot be ranked via $(1/3, 1/3)$-GASSD and via exponential utility with risk aversion $\rho < (1/x_G)\ln((1-\varepsilon)/\varepsilon)$ for $\varepsilon = 1/3$. Going from $(0, 1/3)$-GASSD to $(1/3, 0)$-GASSD to $(1/3, 1/3)$-GASSD and then to GASSD for exponential utility with $\varepsilon = 1/3$ reduces the region of pairs $(x_F/x_G, p)$ that cannot be ranked by SD. For the example from §1 with $x_F/x_G = 0.25$, GASSD cannot rank the two projects if $0.25 \leqslant p \leqslant 0.50$ when $\varepsilon_1 = 0$, $\varepsilon_2 = 1/3$; if $0.25 \leqslant p \leqslant 0.40$ when $\varepsilon_1 = 1/3$, $\varepsilon_2 = 0$; if $0.25 \leqslant p \leqslant 0.326$ when $\varepsilon_1 = \varepsilon_2 = 1/3$; and if $0.25 \leqslant p \leqslant 0.318$ for exponential utility with $\varepsilon = 1/3$.

Figure 2 and other cases we have investigated suggest that exponential utility with risk aversion corresponding to ε is a reasonably good proxy for $(\varepsilon, \varepsilon)$-GASSD. That needs further study but could be helpful, because we can infer ε on the basis of some certainty equivalent assessments for gambles.

We have focused on dominance conditions in terms of bounds on p for a given ε. Alternatively, we can give the dominance conditions in terms of bounds on ε for a given p. A decision maker might find it easier to assess p than to assess ε. For AFSD, for example, instead of having to assess a specific value for ε_1 that is consistent with her preferences, she can think about whether it is reasonable that ε_1 is on one side or the other of a boundary value. This is similar in form to FSD, for which the boundary value is zero. In that sense, ASD can be thought of as simply shifting the boundary value from its value for SD. This way of thinking about ASD is consistent in spirit with the notion of sensitivity analysis in decision analysis.

Rewriting the expressions for ε_1-AFSD in terms of bounds on ε_1 instead of bounds on p and letting $d = p(1-c)/(p(1-c)+(1-p)c)$, F dominates G if and only if $\varepsilon_1 \geqslant d$ and G dominates F if and only if $\varepsilon_1 \geqslant 1-d$. Note that d is increasing with p, and $d = 0.5$ if and only if $p = x_F/x_G$. These results are summarized as follows and shown graphically in Figure 3:

If $p < x_F/x_G$, then F dominates G for $\varepsilon_1 \geqslant d$, and G does not dominate F for any $\varepsilon_1 \leqslant 0.5$.

If $p > x_F/x_G$, then G dominates F for $\varepsilon_1 \geqslant 1-d$, and F does not dominate G for any $\varepsilon_1 \leqslant 0.5$.

If $p = x_F/x_G$, then F and G are equivalent for $\varepsilon_1 = 0.5$ and cannot be ranked for any $\varepsilon_1 < 0.5$.

For $(\varepsilon_1, 0)$-GASSD, Figure 3 remains the same except that F is preferred to G for any ε_1 if $p \leqslant x_F/x_G$:

If $p \leqslant x_F/x_G$, then F dominates G by SSD.

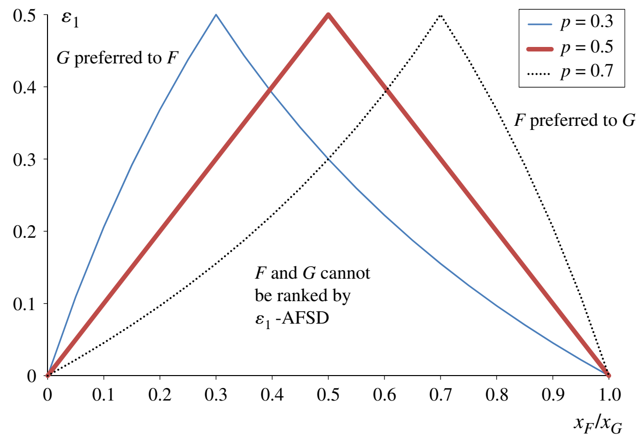
If $p > x_F/x_G$, then G dominates F for $\varepsilon_1 \geqslant 1-d$ and F does not dominate G for any $\varepsilon_1 \leqslant 0.5$.

For $(0, \varepsilon_2)$-GASSD in terms of bounds on ε_2, we find that:

If $p \leqslant x_F/x_G$, then F dominates G by SSD.

If $p > x_F/x_G$, then G dominates F by $(0, \varepsilon_2)$-GASSD if and only if $\varepsilon_2 \geqslant c^2(1-p)/((p-c)^2 + c^2(1-p))$.

Figure 3. (Color online) Regions of values of $(x_F/x_G, \varepsilon_1)$ for which F and G can and cannot be ranked by AFSD for $p = 0.3$, $p = 0.5$, and $p = 0.7$.



Considering GASSD, condition (2) can be stated as follows:

- F dominates G by $(\varepsilon_1, 0)$-GASSD if and only if $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$ and

$$\varepsilon_1 \geqslant \frac{\max_{x\in[a,b]}\{F^{(2)}(x)-G^{(2)}(x)\}}{2\max_{x\in[a,b]}\{F^{(2)}(x)-G^{(2)}(x)\}-[F^{(2)}(b)-G^{(2)}(b)]}. \quad (4)$$

Similarly, condition (3) can be stated as follows:

- F dominates G by $(0, \varepsilon_2)$-GASSD if and only if $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$, $F^{(3)}(b) - G^{(3)}(b) \leqslant 0$, and

$$\varepsilon_2 \geqslant \left(\int_{S_2(F,G)} [F^{(2)}(x) - G^{(2)}(x)]\,dx\right) \cdot \left(2\int_{S_2(F,G)} [F^{(2)}(x) - G^{(2)}(x)]\,dx - \int_a^b [F^{(2)}(x) - G^{(2)}(x)]\,dx\right)^{-1}. \quad (5)$$

Note that even for $\varepsilon_2 = 0.5$, it is possible that G and F cannot be ranked by $(0, \varepsilon_2)$-GASSD, because $(0, 0.5)$-GASSD is equivalent to considering utility functions of the form $u(x) = x - \omega(b-x)^2$ for all $\omega > 0$. Then G and F cannot be ranked by $(0, 0.5)$-GASSD if $F^{(2)}(b) - G^{(2)}(b) < 0$ and $F^{(3)}(b) - G^{(3)}(b) = \int_a^b [F^{(2)}(x) - G^{(2)}(x)]\,dx > 0$.

As noted in §2, the example given here yields analytical results, but in many situations, it may be necessary to find results numerically, especially for $(\varepsilon_1, \varepsilon_2)$-GASSD with $\varepsilon_1 \neq \varepsilon_2$ or higher-degree GASD. Appendix B restates Theorem 2 for numerical implementation and provides an example of its application.

5. Connections with a Preference for Combining Good with Bad

In SD or ASD, the dominance can be expressed either in terms of distributions (as we have done above), or equivalently, in terms of the random variables associated with those distributions. To discuss connections of ASD with a preference for combining good with bad, it is convenient to express dominance in terms of random variables (e.g., X dominates Y).

As shown in Eeckhoudt et al. (2009), nth-degree SD is consistent with a preference for combining good with bad (risk apportionment). If X_i dominates Y_i in the sense of ith-degree SD, $i = m, n$, then $\langle X_m + Y_n, X_n + Y_m\rangle$ dominates $\langle X_m + X_n, Y_m + Y_n\rangle$ via $(n+m)$th-degree SD, where $\langle X, Y\rangle$ denotes a 50-50 lottery of receiving either X or Y. Similarly, if Y_i has more ith-degree risk than X_i, $i = m, n$, then $\langle X_m + X_n, Y_m + Y_n\rangle$ has more $(n + m)$th-degree risk than $\langle X_m + Y_n, X_n + Y_m\rangle$. In this section, we show that GASD and almost nth-degree risk, as defined in §2, are also consistent with a preference for combining good with bad. All random variables are assumed to be independent.

Theorem 8. *Let Y_i have more ε_i-AiR than X_i, $i = n, m$. Then $\langle X_m + X_n, Y_m + Y_n\rangle$ has more ε_{m+n}-almost $(m+n)$th-degree risk than $\langle X_m + Y_n, X_n + Y_m\rangle$, where $\varepsilon_{m+n} = \varepsilon_n \cdot (1 - \varepsilon_m) + \varepsilon_m(1 - \varepsilon_n)$.*

Theorem 8 extends the Eeckhoudt et al. (2009) result about combining mth- and nth-degree risks to almost mth- and nth-degree risks, and provides a way to construct AnR risk by using risks of lower degrees. Note that in Theorem 8, $\varepsilon_{m+n} \geqslant \max\{\varepsilon_m, \varepsilon_n\}$. Also, ε_{m+n} approaches ε_n if ε_m approaches zero and approaches 0.5 if ε_m approaches 0.5.

Theorem 9. *Let X_m dominate Y_m by $(\varepsilon_1, \dots, \varepsilon_m)$-GAmSD, and let Y_n have more ε_n-AnR than X_n. Then $\langle X_m + Y_n, X_n + Y_m\rangle$ dominates $\langle X_m + X_n, Y_m + Y_n\rangle$ by $(0, \dots, \varepsilon^*_{n+1}, \dots, \varepsilon^*_{n+m})$-GA$(m + n)$SD with $\varepsilon^*_{n+i} = \varepsilon_n(1 - \varepsilon_i) + \varepsilon_i(1 - \varepsilon_n)$, $i = 1, \dots, m$.*

Theorem 9 involves combining GAmSD and AnR. By noticing that AFR is equivalent to AFSD, we get the following corollary, showing the recursive property of combining GAmSD and AFSD to obtain GA$(m+1)$SD.

Corollary 1. *Let X_m dominate Y_m by $(\varepsilon_1, \dots, \varepsilon_m)$-GAmSD, and let X_1 dominate Y_1 by ε_1-AFSD. Then $\langle X_m + Y_1, X_1 + Y_m\rangle$ dominates $\langle X_m + X_1, Y_m + Y_1\rangle$ by $(0, \varepsilon^*_2, \dots, \varepsilon^*_{m+1})$-GA$(m + 1)$SD with $\varepsilon^*_{i+1} = \varepsilon_1(1 - \varepsilon_i) + \varepsilon_i(1 - \varepsilon_1)$, $i = 1, \dots, m$.*

As mentioned in §2, obtaining integral conditions for GASD of higher degrees is difficult, and thus it is hard to extend Corollary 1 to combining two pairs of random variables, where each pair is ordered by GASD of degree higher than 1. Theorem 10 addresses the case where one pair is ordered by nth-degree SD.

Theorem 10. *Let X_m dominate Y_m by $(\varepsilon_1, \dots, \varepsilon_m)$-GAmSD, and let X_n dominate X_n by nth-degree SD. Then $\langle X_m + Y_n, X_n + Y_m\rangle$ dominates $\langle X_m + X_n, Y_m + Y_n\rangle$ by $(0, \varepsilon^*_2, \dots, \varepsilon^*_{n+m})$-GA$(m + n)$SD with $\varepsilon^*_k = \max_{\max(k-m,1) \leqslant i \leqslant \min(k-1,n)}(\varepsilon_{k-i})$, $k = 2, \dots, m+n$.*

Theorems 8–10 and Corollary 1 show that AnR and GASD dominance satisfy a preference for combining good with bad, where good and bad are also defined in terms of ASD. These theorems are consistent with similar results for SD and can be used to construct distributions ordered by GASD of higher degrees. More importantly, as illustrated below, they allow the extension of results from various applications that are based on a preference for combining good with bad to distributions that are ordered via ASD. Thus, our concept of GASD substantially enlarges the class of distributions to which these results apply.

5.1. Precautionary Savings

Following Eeckhoudt et al. (2009), consider a simple two-period model of consumption and saving. In a two-period setting with time-separable preferences and no discounting, an individual has random income of X at date $t = 0$ and Y at date $t = 1$. A decision about the amount s to save has to be made before learning the realization of X. If $s < 0$, the consumer is borrowing money, and we assume an interest rate of zero.

Suppose X dominates Y via GAnSD. For $s \geqslant 0$, it follows from Theorem 10 (or Corollary 1) that the 50-50 lottery $\langle X - s, Y + s\rangle$ dominates $\langle X + s, Y - s\rangle$ by GA$(n + 1)$SD. Reinterpreting lottery $\langle A, B\rangle$ as the sequential consumption of A and B at times $t = 0$ and $t = 1$ implies positive saving whenever preferences satisfy GA$(n + 1)$SD.

Note that although it can be difficult to compare distributions in terms of GASD of higher degrees (e.g., Theorem 5), the results regarding a preference for combining good with bad for higher-degree GASD can still be useful. It is not difficult to check whether preferences satisfy GASD by checking whether the decision maker's utility function belongs to a particular class (e.g., $\underline{U}_3(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ in the case of Theorem 5). We can check whether preferences satisfy the higher-degree GASD and then use lower orders to compare the distributions.

5.2. A Corporation with Two Headquarters

Following Eeckhoudt et al. (2009), consider an example that involves a risk-neutral corporation with taxable profit X in country A and taxable profit Y in country B. We assume that the tax schedule is identical in both countries, and that the marginal tax rate is increasing, but at a decreasing rate. This is a realistic assumption, because the marginal rate is typically bounded. Denote the tax owed on profit π by $t(\pi)$. Then after-tax profit is $u(\pi) = \pi - t(\pi)$, and if $t(\cdot)$ is differentiable, the assumptions about t imply that $u^{(2)}(\pi) < 0$ and $u^{(3)}(\pi) > 0$, and for $t^{(1)}(\pi) < 1$, we also have $u^{(1)}(\pi) > 0$.

Suppose that the corporation must decide whether to allocate a new project with pretax distribution of profit Z to country A or B. As Eeckhoudt et al. (2009) show, this project should be allocated to country A if X dominates Y via SSD and $Z > 0$ almost surely. This result can be extended to ASD if we assume that the tax rate is such that $u^{(2)}$ and $u^{(3)}$ are strictly positive, so that $u \in \underline{U}_3(0, \varepsilon_2, \varepsilon_3)$ for some $\varepsilon_i > 0$, $i = 2, 3$. Then, by Corollary 1, the project should be allocated to country A if X dominates Y by (δ_1, δ_2)-GASSD and Z dominates 0 by γ_1-AFSD, provided that $\varepsilon_{i+1} > \delta_i(1-\gamma_1) + \gamma_1(1-\delta_i)$, $i = 1, 2$. Note that if Z is a binary lottery, then the example in §3 provides conditions for Z to be preferred to a sure outcome under AFSD.

5.3. Precautionary Effort

Eeckhoudt et al. (2012) show that the concept of combining good with bad provides conditions for precautionary effort to increase with the deterioration of background risk, as long as lotteries in good and bad states can be ordered by SD. This can be extended by our results from earlier in this section if the lotteries can be ordered by ASD and the deterioration of background risk is also in the sense of ASD.

Denote effort by e, disutility from effort by $v(e)$, the good lottery by X, the bad lottery by Y, and the probability of facing a good lottery by $p(e)$. We assume that disutility from effort $v(e)$ is increasing and convex and that $p(e)$ is increasing and concave. Denote the utility function over lotteries by u, and background risk by Z. Then Equation (9) from Eeckhoudt et al. (2012) can be written as $\max_e p(e)Eu(X+Z) + (1-p(e))Eu(Y+Z) - v(e)$.

The first-order condition (FOC) is

$$\frac{v^{(1)}(e)}{p^{(1)}(e)} = Eu(X+Z) - Eu(Y+Z), \tag{6}$$

and the second-order condition is satisfied by concavity of $p(e)$ and convexity of $v(e)$.

Now, suppose that the background risk changes from Z to W. Since the left-hand side of (6) is increasing in e, precautionary effort will increase if and only if $Eu(X+W) - Eu(Y+W) > Eu(X+Z) - Eu(Y+Z)$. That is, it will increase if and only if $\langle X+W, Y+Z\rangle$ is preferred to $\langle X+Z, Y+W\rangle$ by a decision maker with utility function u.

As noticed in Eeckhoudt et al. (2012), precautionary effort increases if background risk deteriorates in the sense of nth-degree SD, the good lottery X dominates the bad lottery Y in the sense of mth-degree SD, and preferences satisfy $(n+m)$th-degree SD. Results from §4 provide the following extensions if preferences satisfy GASD:

- X dominates Y by GAmSD, and W has more almost nth-degree risk than Z (Theorem 9). In particular, Z dominates W by AFSD (Corollary 1).
- X dominates Y by GAmSD, and Z dominates W by nth-degree SD (Theorem 10).

6. Almost Convex Stochastic Dominance

Almost stochastic dominance, as defined in §2, extends the notion of concave stochastic dominance, which corresponds to risk-averse preferences. Recently, risk-prone preferences have also received attention (Crainich et al. 2013, Ebert 2013), and a multivariate version of convex stochastic dominance (corresponding to risk-prone preferences) is discussed in Denuit et al. (2013). In this section, we discuss the notion of generalized almost convex stochastic dominance.

The class of utility functions related to generalized almost convex stochastic dominance is

$$\bar{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \left\{ u \mid u^{(k)} > 0 \text{ and } \sup\{u^{(k)}(x)\} \leqslant \inf\{u^{(k)}(x)\}\left(\frac{1}{\varepsilon_k} - 1\right), \; k = 1, 2, \ldots, n \right\}.$$

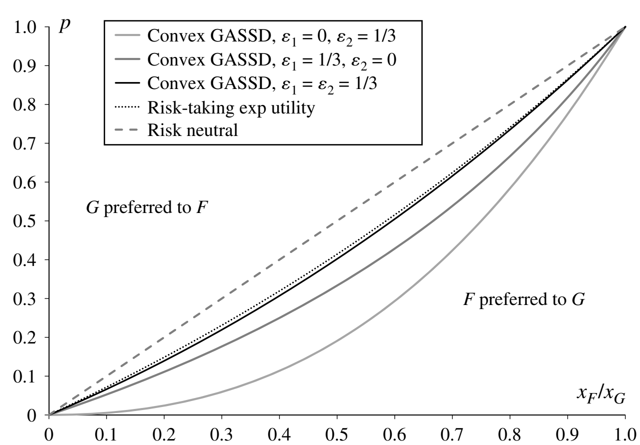
Definition 9. F dominates G by $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$-generalized almost nth-degree convex SD if $E_F(u) \geqslant E_G(u)$ for all $u \in \bar{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$.

For random variables X and Y with support in $[a, b]$, X dominates Y by $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$-generalized almost nth-degree convex SD if and only if $b + a - Y$ dominates $b + a - X$ by $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$-GA$n$SD. This follows from $u(x) \in \bar{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \Leftrightarrow -u(b+a-x) \in \underline{U}_n(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ for u defined on $[a, b]$.

In the context of the example in §4, to compare F and G via generalized almost convex stochastic dominance, we compare $x_G - x_F$ for sure to a risky gain of x_G with probability $1-p$ and zero otherwise. Figure 4 replicates Figure 2 for the convex case.

As §5 shows, GASD is consistent with a preference for combining good with bad. Similarly, generalized almost

Figure 4. Regions of values of $(x_F/x_G, p)$ for which F and G can and cannot be ranked by $(0, 1/3)$-convex GASSD, $(1/3, 0)$-convex GASSD, $(1/3, 1/3)$-convex GASSD, and risk-taking exponential utility corresponding to $\varepsilon = 1/3$.



Note. In the region between each curve and the risk-neutral line, F and G cannot be ranked.

convex SD is consistent with a preference for combining good with good and bad with bad, and the results in §5 directly extend to the convex case. The proof is similar to the proof of Theorem 2.2.12 in Denuit et al. (2013).

7. Concluding Remarks

Our development of GASD builds on the early ASD work with the same motivation: relaxing the rigid SD rules to represent what most decision makers would view as "reasonable" preferences and therefore reflect the choices they would make, and to allow for individual preferences that differ somewhat from the SD rules. This is done by allowing some violation of the SD rules, controlling the extent of that violation with constraints on the derivatives of the utility function.

The contributions of GASD developed here include the following:

- GASD provides integral conditions and utility classes that are consistent with each other.
- The hierarchy property of SD is satisfied by GASD: if F dominates G by nth-degree GASD, then it dominates G by kth-degree GASD for all $k > n$.
- GASD relations can be characterized in terms of allowing some relaxation of the conditions on probability shifts for SD relations, which have a behavioral interpretation and are more intuitive than integral conditions.
- For implementation, decision rules for GASD can be expressed in terms of ranges of ε_i-values, in the spirit of sensitivity analysis, and computational methods to generate numerical results for GASD are illustrated.
- GASD, like SD, satisfies a preference for combining good with bad, and its risk-prone counterpart, convex GASD, satisfies the opposite preference for combining good with good and bad with bad.
- GASD is not limited to first- and second-degree dominance, but covers nth-degree dominance for $n = 1, 2, \ldots,$ allowing the consideration of higher-order risk effects.
- Just as SSD is consistent with second-degree risk, GASD is consistent with the notion of almost nth-degree risk.

ASD has not caught on much in practice, as noted in §1. We expect that GASD, with its integral conditions matched correctly to appropriate utility classes and with more intuitive implementation procedures, can be very useful in decision analysis and empirical studies. Moreover, GASD is applicable in theoretical analysis of applied situations where a preference for combining good with bad can be assumed, or when a preference for combining good with good and bad with bad is more appropriate. Decision makers may anchor on obvious benchmarks like risk aversion, and GASD allows some flexibility via relaxation of those benchmarks, with SD as a boundary case with the GASD epsilons equal to zero.

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Appendix A. Proofs

Proof of Theorem 2. First, the "if" part.

$$E_F(u)-E_G(u)=u^{(1)}(b)[G^{(2)}(b)-F^{(2)}(b)]+\int_a^b[-u^{(2)}(x)][G^{(2)}(x)-F^{(2)}(x)]\,dx.$$

Without loss of generality (WLOG), set $u^{(1)}(b)=1$. From $u^{(1)}(a)\leqslant u^{(1)}(b)(1/\varepsilon_1-1)$, it follows that

$$\int_a^b[-u^{(2)}(x)]\,dx\leqslant\frac{1}{\varepsilon_1}-2.$$

Let $h(x)=-u^{(2)}(x)(\varepsilon_1/(1-2\varepsilon_1))$. Thus, $\int_a^b h(x)\,dx\leqslant 1$ and $\sup h(x)\leqslant\inf\{h(x)\}(1/\varepsilon_2-1)$. Then,

$$E_F(u)-E_G(u)=\frac{1-2\varepsilon_1}{\varepsilon_1}\left\{\frac{\varepsilon_1}{1-2\varepsilon_1}[G^{(2)}(b)-F^{(2)}(b)]+\int_a^b h(x)[G^{(2)}(x)-F^{(2)}(x)]\,dx\right\},$$

and $\quad E_F(u)\geqslant E_G(u)\;\Leftrightarrow\;\int_a^b h(x)[F^{(2)}(x)-G^{(2)}(x)]\,dx\leqslant\frac{\varepsilon_1}{1-2\varepsilon_1}[G^{(2)}(b)-F^{(2)}(b)].$

If $\int_a^b h(x)[F^{(2)}(t)-G^{(2)}(x)]\,dx\leqslant 0$, then the above condition holds. If $\int_a^b h(x)[F^{(2)}(x)-G^{(2)}(x)]\,dx>0$ and $\int_a^b h(x)\,dx<1$, then

$$\int_a^b h(x)[F^{(2)}(x)-G^{(2)}(x)]\,dx<\int_a^b h^*(x)[G^{(2)}(x)-F^{(2)}(x)]\,dx,$$

where $h^*(x)=h(x)/\int_a^b h(x)\,dx\leqslant 1$, so that $\int_a^b h^*(x)\,dx=1$. Therefore, we need to show that

$$\int_a^b h^*(x)[F^{(2)}(x)-G^{(2)}(x)]\,dx\leqslant\frac{\varepsilon_1}{1-2\varepsilon_1}[G^{(2)}(b)-F^{(2)}(b)]$$

for all $h^*(x)\geqslant 0$, $\int_a^b h^*(x)\,dx=1$, and $\sup h^*(x)\leqslant\inf\{h^*(x)\}\cdot(1/\varepsilon_2-1)$.

In turn,

$$\int_a^b h^*(x)[F^{(2)}(x)-G^{(2)}(x)]\,dx\leqslant\max_C\left\{\frac{1}{(1-2\varepsilon_2)|C|+\varepsilon_2(b-a)}\cdot\begin{bmatrix}(1-2\varepsilon_2)\int_C(F^{(2)}(x)-G^{(2)}(x))\,dx\\+\varepsilon_2\int_a^b(F^{(2)}(x)-G^{(2)}(x))\,dx\end{bmatrix}\right\}.$$

The right-hand side corresponds to

$$h^*(x)=\begin{cases}\dfrac{1-\varepsilon_2}{(1-2\varepsilon_2)|C|+\varepsilon_2(b-a)} & \text{if } x\in C,\\[2ex] \dfrac{\varepsilon_2}{(1-2\varepsilon_2)|C|+\varepsilon_2(b-a)} & \text{if } x\notin C,\end{cases}$$

and the left-hand side reaches its maximum if $h^*(x)$ takes only two values: maximal when $G^{(2)}(x)-F^{(2)}(x)$ is above some threshold, and minimal when $G^{(2)}(x)-F^{(2)}(x)$ is below some threshold.

Next, the "only if" part. For Condition (1), take $u(x)=x$, which belongs to the closure of $\underline{U}_2(\varepsilon_1,\varepsilon_2)$. For Condition (2), fix F and G such that $E_F(u)\geqslant E_G(u)$ for all $u\in\underline{U}_2(\varepsilon_1,\varepsilon_2)$. If the left-hand side of Condition (2) is nonpositive, then Condition (2) holds because Condition (1) holds. If the left-hand side of Condition (2) is positive, then define set C^* as

$$C^*=\arg\max_C\left\{\frac{1}{(1-2\varepsilon_2)|C|+\varepsilon_2(b-a)}\cdot\left[\begin{array}{l}(1-2\varepsilon_2)\int_C(F^{(2)}(x)-G^{(2)}(x))\,dx\\ +\varepsilon_2\int_a^b(F^{(2)}(x)-G^{(2)}(x))\,dx\end{array}\right]\right\},$$

and consider $u(x)$ such that $u^{(1)}(b)=1$, $u^{(1)}(a)=1/\varepsilon_1-1$,

$$-u^{(2)}(x)=\frac{1-2\varepsilon_1}{\varepsilon_1}\frac{1}{(1-2\varepsilon_2)|C^*|+\varepsilon_2(b-a)}\cdot\begin{cases}1-\varepsilon_2 & \text{if } x\in C^*,\\ \varepsilon_2 & \text{if } x\notin C^*.\end{cases}$$

Observe that $u\in\underline{U}_2(\varepsilon_1,\varepsilon_2)$. Then,

$$\begin{aligned}0&\leqslant E_F(u)-E_G(u)=u^{(1)}(b)[G^{(2)}(b)-F^{(2)}(b)]\\&\quad+\int_a^b[-u^{(2)}(x)][G^{(2)}(x)-F^{(2)}(x)]\,dx\\&=G^{(2)}(b)-F^{(2)}(b)+\frac{1-2\varepsilon_1}{\varepsilon_1}\frac{1}{(1-2\varepsilon_2)|C^*|+\varepsilon_2(b-a)}\\&\quad\cdot\left[\begin{array}{l}(1-2\varepsilon_2)\int_{C^*}(F^{(2)}(x)-G^{(2)}(x))\,dx\\ +\varepsilon_2\int_a^b(F^{(2)}(x)-G^{(2)}(x))\,dx\end{array}\right].\end{aligned}$$

By the definition of C^*, this is equivalent to Condition (2). □

Proof of Theorem 5. First, the "if" part.

$$\begin{aligned}&E_F(u)-E_G(u)\\&=u^{(1)}(b)[G^{(2)}(b)-F^{(2)}(b)]+[-u^{(2)}(b)][G^{(3)}(b)-F^{(3)}(b)]\\&\quad+\int_a^b u^{(3)}(x)[G^{(3)}(x)-F^{(3)}(x)]\,dx\\&=[-u^{(2)}(b)]\left\{\begin{array}{l}\dfrac{u^{(1)}(b)}{-u^{(2)}(b)}[G^{(2)}(b)-F^{(2)}(b)]\\ +[G^{(3)}(b)-F^{(3)}(b)]+\dfrac{1}{-u^{(2)}(b)}\\ \cdot\int_a^b u^{(3)}(x)[G^{(3)}(x)-F^{(3)}(x)]\,dx\end{array}\right\}.\end{aligned}\tag{A1}$$

WLOG, set $-u^{(2)}(b)=1$. Let $h(x)=u^{(3)}(x)(\varepsilon_2/(1-2\varepsilon_2))$. Then we have

$$-u^{(2)}(x)=-u^{(2)}(b)+\int_x^b u^{(3)}(t)\,dt=1+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_x^b h(t)\,dt \quad\text{and}$$

$$u^{(1)}(a)-u^{(1)}(b)=(b-a)+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b\left(\int_x^b h(t)\,dt\right)dx.$$

Since $u^{(1)}(b)\leqslant u^{(1)}(a)(1/\varepsilon_1-1)$, or $u^{(1)}(a)-u^{(1)}(b)\leqslant u^{(1)}(b)\cdot(1/\varepsilon_1-2)$, we have

$$u^{(1)}(b)\geqslant\frac{\varepsilon_1}{1-2\varepsilon_1}\left((b-a)+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b\left(\int_x^b h(t)\,dt\right)dx\right).$$

Therefore,

$$\begin{aligned}&E_F(u)-E_G(u)\\&\quad\geqslant\frac{\varepsilon_1}{1-2\varepsilon_1}\left((b-a)+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b\left(\int_x^b h(t)\,dt\right)dx\right)\\&\qquad\cdot[G^{(2)}(b)-F^{(2)}(b)]+[G^{(3)}(b)-F^{(3)}(b)]\\&\qquad+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b h(x)[G^{(3)}(x)-F^{(3)}(x)]\,dx,\end{aligned}$$

which is positive if the conditions of Theorem 5 hold.

Now, the "only if" part. For Condition (1), take $u(x)=x$. For Condition (2), take $u(x)=-((b-a)/(1/\varepsilon_1-2))(b-x)+\frac{1}{2}(b-x)^2$. For (3), set $-u^{(2)}(b)=1$, $u^{(1)}(b)=u^{(1)}(a)(1/\varepsilon_1-1)$, and let $u^{(3)}(x)=((1-2\varepsilon_2)/\varepsilon_2)h(x)$. Note that

$$-u^{(2)}(x)=-u^{(2)}(b)+\int_x^b u^{(3)}(t)\,dt=1+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_x^b h(t)\,dt \quad\text{and}$$

$$u^{(1)}(b)=\frac{\varepsilon_1}{1-2\varepsilon_1}\left((b-a)+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b\left(\int_x^b h(t)\,dt\right)dx\right).$$

By Equation (A1), if $E_F(u)-E_G(u)\geqslant 0$, then

$$\begin{aligned}&\frac{u^{(1)}(b)}{-u^{(2)}(b)}[G^{(2)}(b)-F^{(2)}(b)]+[G^{(3)}(b)-F^{(3)}(b)]\\&\quad+\frac{1}{-u^{(2)}(b)}\int_a^b u^{(3)}(x)[G^{(3)}(x)-F^{(3)}(x)]\,dx\geqslant 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{\varepsilon_1}{1-2\varepsilon_1}\left((b-a)+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b\left(\int_x^b h(t)\,dt\right)dx\right)\\&\quad\cdot[G^{(2)}(b)-F^{(2)}(b)]+[G^{(3)}(b)-F^{(3)}(b)]\\&\quad+\frac{1-2\varepsilon_2}{\varepsilon_2}\int_a^b h(x)[G^{(3)}(x)-F^{(3)}(x)]\,dx\geqslant 0,\end{aligned}$$

or

$$\begin{aligned}&\int_a^b h(x)[F^{(3)}(t)-G^{(3)}(x)]\,dx\\&\qquad+\frac{\varepsilon_1}{1-2\varepsilon_1}\int_a^b\left(\int_x^b h(t)\,dt\right)dx[F^{(2)}(b)-G^{(2)}(b)]\\&\quad\leqslant\frac{\varepsilon_2}{1-2\varepsilon_2}\left(\frac{\varepsilon_1}{1-2\varepsilon_1}(b-a)[G^{(2)}(b)-F^{(2)}(b)]\right.\\&\qquad\left.+[G^{(3)}(b)-F^{(3)}(b)]\right).\quad\square\end{aligned}$$

Proof of Theorem 6.

$$E_F(u)-E_G(u)=0.5(u(x_++\delta_+)-u(x_+)+u(x_--\delta_-)-u(x_-))=0.5(\delta_+u^{(1)}(x_1)-\delta_-u^{(1)}(x_2))$$

for some $x_1\in[x_+,x_++\delta_+]$ and $x_2\in[x_--\delta_-,x_-]$. If $u\in U_1(\varepsilon_1)$, $u^{(1)}(x_2)\leqslant u^{(1)}(x_1)(1/\varepsilon_1-1)$, and $\delta_+u^{(1)}(x_1)-\delta_-u^{(1)}(x_2)\geqslant u^{(1)}(x_1)(\delta_+-(1/\varepsilon_1-1)\delta_-)>0$.

Now, suppose $u\notin U_1(\varepsilon_1)$. Then there exist $x_1\in(a,b)$ and $x_2\in(a,b)$ such that $u^{(1)}(x_2)>u^{(1)}(x_1)(1/\varepsilon_1-1)$. For small enough δ_- and $\delta_+=(1/\varepsilon_1-1)\delta_-$, there exist x_+ and x_- (in neighborhoods of x_1 and x_2 correspondingly) such that $u(x_++\delta_+)-u(x_+)=\delta_+u^{(1)}(x_1)$ and $u(x_--\delta_-)-u(x_-)=-\delta_-u^{(1)}(x_2)$. Then $E_F(u)-E_G(u)<0$, a contradiction. □

Proof of Theorem 7. Observe that $0.5(u(x+\delta)+u(x-\delta))-u(x)=0.25\delta^2(u^{(2)}(x_1)+u^{(2)}(x_2))$ for some $x_1\in[x-\delta,x]$ and $x_2\in[x,x+\delta]$. Then,

$$E_F(u)-E_G(u)=0.125(u^{(2)}(x_1)\delta_-^2+u^{(2)}(x_2)\delta_-^2-u^{(2)}(x_3)\delta_+^2-u^{(2)}(x_4)\delta_+^2)$$

for some $x_1\in[x_--\delta_-,x_-]$, $x_2\in[x_-,x_-+\delta_-]$, $x_3\in[x_+-\delta_+,x_+]$, and $x_4\in[x_+,x_++\delta_+]$. If $u\in U_2^R(\varepsilon_2)$, $-u^{(2)}(x_3)\leqslant -u^{(2)}(x_1)(1/\varepsilon_2-1)$ and $-u^{(2)}(x_4)\leqslant -u^{(2)}(x_2)(1/\varepsilon_2-1)$. Then $E_F(u)\geqslant E_G(u)$.

Now, suppose $u\notin U_2^R(\varepsilon_2)$. Then, there exist $x_+\in(a,b)$ and $x_-\in(a,b)$ such that $-u^{(2)}(x')>-u^{(2)}(x'')(1/\varepsilon_2-1)$ for all x' in the neighbourhood of x_+ and all x'' in the neighborhood of x_-. Then, for small enough δ_- and $\delta_+=\sqrt{1/\varepsilon_2-1}\delta_-$, $E_F(u)-E_G(u)<0$. □

The proofs of Theorems 8 and 9 involve Lemma 1.

Lemma 1. *Let Y have more ε_n-AnR than X, and let $u\in U_{n+m}(\varepsilon^*)$ with $\varepsilon^*=\varepsilon_n(1-\varepsilon_m)+\varepsilon_m(1-\varepsilon_n)$. Then $v(z)=E[u(Y+z)]-E[u(X+z)]\in U_m(\varepsilon_m)$.*

Proof of Lemma 1. We need to show that

$$\sup\{(-1)^{m+1}v^{(m)}(z)\}\leqslant\inf\{(-1)^{m+1}v^{(m)}(z)\}(1/\varepsilon_m-1).$$

Let $[a,b]$ contain X and Y, and let F and G be cdfs of X and Y, respectively. Then,

$$E[u^{(m)}(Y+z)]-E[u^{(m)}(X+z)]=\sum_{k=1}^{n-1}(-1)^ku^{(m+k)}(b+z)(G^{(k+1)}(b)-F^{(k+1)}(b))+(-1)^n\int_a^b u^{(m+n)}(x+z)(G^{(n)}(x)-F^{(n)}(x))\,dx.$$

Since X and Y are ordered by ε_n-AnR, $F^{(k)}(b)=G^{(k)}(b)$, $k=1,\dots,n$, and

$$(-1)^{m+1}v^{(m)}(z)=(-1)^{m+1}(E[u^{(m)}(Y+z)]-E[u^{(m)}(X+z)])=(-1)^{n+m+1}\int_a^b u^{(m+n)}(x+z)(G^{(n)}(x)-F^{(n)}(x))\,dx.$$

Denote $A=\int_{F^{(n)}>G^{(n)}}(F^{(n)}(x)-G^{(n)}(x))\,dx$ and $B=\int_{G^{(n)}>F^{(n)}}(G^{(n)}(x)-F^{(n)}(x))\,dx$. Since Y has more ε_n-AnR than X, $A\leqslant B(\varepsilon_n/(1-\varepsilon_n))$. If $B=0$, then $A=0$, and Lemma 1 holds because $v^{(m)}(z)\equiv 0$.

Consider $B>0$, and let $\bar\theta_{n+m}=\sup\{(-1)^{n+m+1}u^{(n+m)}(x)\}$ and $\underline{\theta}_{n+m}=\inf\{(-1)^{n+m+1}u^{(n+m)}(x)\}$. Then $B\underline{\theta}_{n+m}-A\bar\theta_{n+m}\leqslant(-1)^{m+1}v^{(m)}(z)\leqslant B\bar\theta_{n+m}-A\underline{\theta}_{n+m}$, and

$$\frac{\inf\{(-1)^{m+1}v^{(m)}(z)\}}{\sup\{(-1)^{m+1}v^{(m)}(z)\}}\geqslant\frac{B\underline{\theta}_{n+m}-A\bar\theta_{n+m}}{B\bar\theta_{n+m}-A\underline{\theta}_{n+m}}.$$

The right-hand side is decreasing in A, and reaches a minimum at $A=B(\varepsilon_n/(1-\varepsilon_n))$ (since $A\leqslant B(\varepsilon_n/(1-\varepsilon_n))$). Therefore,

$$\frac{\inf\{(-1)^{m+1}v^{(m)}(z)\}}{\sup\{(-1)^{m+1}v^{(m)}(z)\}}\geqslant\frac{\underline{\theta}_{n+m}-\bar\theta_{n+m}(\varepsilon_n/(1-\varepsilon_n))}{\bar\theta_{n+m}-\underline{\theta}_{n+m}(\varepsilon_n/(1-\varepsilon_n))}.$$

Using $\underline{\theta}_{n+m}/\bar\theta_{n+m}=\varepsilon^*/(1-\varepsilon^*)$ with $\varepsilon^*=\varepsilon_n(1-\varepsilon_m)+\varepsilon_m(1-\varepsilon_n)$,

$$\frac{\underline{\theta}_{n+m}-\bar\theta_{n+m}(\varepsilon_n/(1-\varepsilon_n))}{\bar\theta_{n+m}-\underline{\theta}_{n+m}(\varepsilon_n/(1-\varepsilon_n))}=\frac{(\varepsilon^*/(1-\varepsilon^*)-\varepsilon_n/(1-\varepsilon_n))}{1-(\varepsilon^*/(1-\varepsilon^*))(\varepsilon_n/(1-\varepsilon_n))}=\frac{\varepsilon^*-\varepsilon_n}{1-\varepsilon^*-\varepsilon_n}=\frac{\varepsilon_m(1-2\varepsilon_n)}{1-\varepsilon_n(1-\varepsilon_m)-\varepsilon_m(1-\varepsilon_n)-\varepsilon_n}=\frac{\varepsilon_m}{1-\varepsilon_m}.\quad\square$$

Proof of Theorem 8. Consider $u\in U_{n+m}(\varepsilon^*)$ and denote $v(z)=E[u(Y_n+z)]-E[u(X_n+z)]$. By Lemma 1, $v(z)\in U_m(\varepsilon_m)$. We need to show that $E[u(X_m+X_n)]+E[u(Y_m+Y_n)]\leqslant E[u(X_m+Y_n)]+E[u(Y_m+X_n)]$, which is equivalent to $E[v(X_m)]\geqslant E[v(Y_m)]$, which holds by $v(z)\in U_m(\varepsilon_m)$. □

Proof of Theorem 9. Consider $u\in\underline{U}_{n+m}(0,\dots,0,\varepsilon^*_{n+1},\dots,\varepsilon^*_{n+m})$ and denote $v(z)=E[u(Y_n+z)]-E[u(X_n+z)]$. We need to show that $E[u(X_m+X_n)]+E[u(Y_m+Y_n)]\leqslant E[u(X_m+Y_n)]+E[u(Y_m+X_n)]$, which is equivalent to $E[v(X_m)]\geqslant E[v(Y_m)]$. In turn, $E[v(X_m)]\geqslant E[v(Y_m)]$ holds if $v(z)\in\underline{U}_m(\varepsilon_1,\dots,\varepsilon_m)$. By Lemma 1, $v(z)\in U_k(\varepsilon_k)$ for $k=1,\dots,m$, which is equivalent to $v(z)\in\underline{U}_m(\varepsilon_1,\dots,\varepsilon_m)$. □

Proof of Theorem 10. Consider $u\in\underline{U}_n(\varepsilon_1^*,\varepsilon_2^*,\dots,\varepsilon^*_{n+m})$ and let $v(z)=E[u(Y_m+z)]-E[u(X_m+z)]$. Then

$$0.5E[u(X_m+Y_n)]+0.5E[u(X_n+Y_m)]\geqslant 0.5E[u(X_m+X_n)]+0.5E[u(Y_n+Y_m)]$$

is equivalent to

$$E[u(X_n+Y_m)]-E[u(X_m+X_n)]\geqslant E[u(Y_n+Y_m)]-E[u(X_m+Y_n)]\;\Leftrightarrow\; E[v(X_n)]\geqslant E[v(Y_n)].$$

It remains to show that $v\in\underline{U}_n(0,\dots,0)$. For any $k=1,\dots,n$, $(-1)^ku^{(k)}(x)\in\underline{U}_{m+n-k}$. In turn, $\underline{U}_{m+n-k}(\varepsilon^*_{k+1},\dots,\varepsilon^*_{n+m})\subset\underline{U}_m(\varepsilon_1,\dots,\varepsilon_m)$ because $n+m-k\geqslant m$ and $\varepsilon^*_{k+i}\geqslant\varepsilon_i$ for $i=1,\dots,m$ because $\varepsilon^*_j=\max_{\max(j-m,1)\leqslant k\leqslant\min(j-1,n)},(\varepsilon_{j-k})$, $j=2,\dots,m+n$. Therefore, $(-1)^kv^{(k)}(z)\geqslant 0$, so $v\in\underline{U}_n(0,\dots,0)$. □

Appendix B. Restatement of GASSD for Implementation

In this appendix, Theorem 11 restates Theorem 2 in a form more convenient for numerical implementation, and we provide an example of the application of Theorem 11.

Theorem 11. *Define* $C(\gamma) = \{x\colon F^{(2)}(x) - G^{(2)}(x) > \gamma,\ x \in [a, b]\}$, $A(\gamma) = \int_{C(\gamma)} (F^{(2)}(x) - G^{(2)}(x))\,dx$, $B(\gamma) = |C| = \int_{C(\gamma)} dx$, *and* $D = \int_a^b (F^{(2)}(x) - G^{(2)}(x))\,dx$. *Assume WLOG that* $F^{(2)}(b) - G^{(2)}(b) \leqslant 0$, *so that* $E_F(X) \geqslant E_G(X)$.

(a) *If* $A(0) = 0$, *then F dominates G by SSD, which is equivalent to* $(0, 0)$*-GASSD.*

(b) *If* $A(0) > 0$ *and* $(1 - 2\varepsilon_2)A(0) + \varepsilon_2 D \leqslant 0$, *then F dominates G by* $(0, \varepsilon_2)$*-GASSD.*

(c) *If* $A(0) > 0$ *and* $(1 - 2\varepsilon_2)A(0) + \varepsilon_2 D > 0$, *then F dominates G by* $(\varepsilon_1, \varepsilon_2)$*-GASSD if and only if* $\gamma^* \leqslant (\varepsilon_1/(1 - 2\varepsilon_1)) \cdot [G^{(2)}(b) - F^{(2)}(b)]$, *or* $\varepsilon_1 \geqslant \gamma^*/(2\gamma^* + (G^{(2)}(b) - F^{(2)}(b)))$, *where* γ^* *is the solution of the equation* $(1 - 2\varepsilon_2)(\gamma B(\gamma) - A(\gamma)) + \varepsilon_2 \gamma(b - a) = \varepsilon_2 D$.

Proof of Theorem 11. First, (a) follows from SSD, since $A(0) = \int_{S_2(F, G)} (F^{(2)}(x) - G^{(2)}(x))\,dx = 0$. For (b) and (c), note that γ^* decreases in ε_2 with $\gamma^*(\varepsilon_2 = 0) = \gamma_M = \max_{x \in [a, b]}\{F^{(2)}(x) - G^{(2)}(x)\}$ and γ^* reaching 0 at $\varepsilon_2 = A(0)/(2A(0) - D)$. Also, $A(\gamma)$ and $B(\gamma)$ are decreasing with γ, $A'(\gamma) = \gamma B'(\gamma)$, and $A(\gamma_M) = B(\gamma_M) = 0$.

The integral condition in Theorem 2 can be restated as $\max_\gamma\{w(\gamma)\} \leqslant (\varepsilon_1/(1 - 2\varepsilon_1))[G^{(2)}(b) - F^{(2)}(b)]$, where $w(\gamma) = ((1 - 2\varepsilon_2)A(\gamma) + \varepsilon_2 D)/((1 - 2\varepsilon_2)B(\gamma) + \varepsilon_2(b - a))$ and $w'(\gamma) = ((1 - 2\varepsilon_2)/[(1 - 2\varepsilon_2)B(\gamma) + \varepsilon_2(b - a)]^2)(A'(\gamma)[(1 - 2\varepsilon_2)B(\gamma) + \varepsilon_2(b - a)] - B'(\gamma)[(1 - 2\varepsilon_2)A(\gamma) + \varepsilon_2 D])$. Since $A'(\gamma) = \gamma B'(\gamma)$ and $B'(\gamma) \leqslant 0$, $w'(\gamma)$ has the same sign as $-[\gamma(1 - 2\varepsilon_2)B(\gamma) + \gamma\varepsilon_2(b - a) - (1 - 2\varepsilon_2)A(\gamma) - \varepsilon_2 D]$, the derivative of which simplifies to $-[(1 - 2\varepsilon_2)B(\gamma) + \varepsilon_2(b - a)] < 0$. Therefore, $w'(\gamma)$ changes sign at most once, and if it does, it is first positive and then negative.

The FOC for maximizing $w(\gamma)$ (and determining γ^*) is $(1 - 2\varepsilon_2)(\gamma B(\gamma) - A(\gamma)) + \varepsilon_2 \gamma(b - a) = \varepsilon_2 D$. When $\gamma = 0$, the LHS is $-A(0)(1 - 2\varepsilon_2)$. Then, if $-A(0)(1 - 2\varepsilon_2) > \varepsilon_2 D$, $w(\gamma)$ is decreasing at $\gamma = 0$, and F dominates G by $(0, \varepsilon_2)$-GASSD by Equation (3), which proves (b).

When $\gamma = \gamma_M$, the LHS is $\varepsilon_2 \gamma_M(b - a)$, which is greater than $\varepsilon_2 D$. Therefore, $w(\gamma)$ is decreasing as $\gamma \to \gamma_M$. Then, for $-A(0)(1 - 2\varepsilon_2) \leqslant \varepsilon_2 D$, $w(\gamma)$ is nondecreasing at $\gamma = 0$. Therefore, $0 \leqslant \gamma^* < \gamma_M$ and $\max_\gamma\{w(\gamma)\} = w(\gamma^*)$. Solving $w(\gamma^*) \leqslant (\varepsilon_1/(1 - 2\varepsilon_1))[G^{(2)}(b) - F^{(2)}(b)]$ with respect to ε_1 yields (c). □

As an illustration, we apply Theorem 11 to the following example:

Let $a = 0$, $b = 1$, and consider

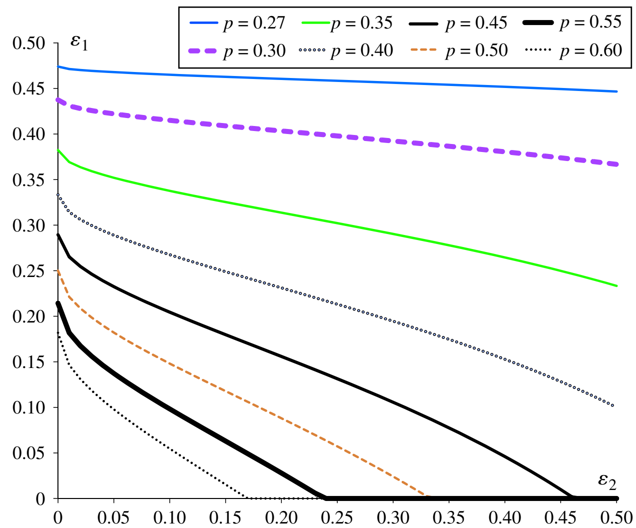
$$F^{(1)}(x) = \begin{cases} 1 - p, & x < 1, \\ 1, & x = 1, \end{cases} \quad \text{and} \quad G^{(1)}(x) = \begin{cases} 0, & x < c, \\ 1, & x > c. \end{cases}$$

(WLOG, this is equivalent to $a = 0$, $b = x_F$, and $c = x_G/x_F$.) In words, F corresponds to a lottery paying 0 with probability $1 - p$ and 1 with probability p, and G corresponds to receiving c for sure. Then $F^{(2)}(x) = (1 - p)x$ and

$$G^{(2)}(x) = \begin{cases} 0, & x < c, \\ (x - c), & x > c. \end{cases}$$

If $p \leqslant c$, then G dominates F via SSD. Suppose that $0 < c < p < 1$. Then $C(\gamma) = \{x\colon \gamma/(1 - p) < x < (c - \gamma)/p\}$ for $0 \leqslant \gamma \leqslant (1 - p)c$ and $A(\gamma) = \int_{\gamma/(1-p)}^{c} (1 - p)x\,dx + \int_c^{(c-\gamma)/p} (c - px)\,dx = (1/(2p(1 - p))) \cdot ((1 - p)^2 c^2 - \gamma^2)$.

Figure B.1. (Color online) Regions of values of $(\varepsilon_1, \varepsilon_2)$ for which F and G can and cannot be ranked by $(\varepsilon_1, \varepsilon_2)$-GASSD for different values of p with $c = 0.25$, as described in Appendix B.



Note. F dominates G if $(\varepsilon_1, \varepsilon_2)$ is located northeast from the corresponding curve.

Denote $\gamma_M = (1 - p)c$. Then $A(\gamma) = (1/(2p(1 - p)))(\gamma_M^2 - \gamma^2)$ and $B(\gamma) = (\gamma_M - \gamma)/p(1 - p)$ with $A'(\gamma) = \gamma B'(\gamma)$ and $A(\gamma_M) = B(\gamma_M) = 0$. $D = \int_0^c (1 - p)x\,dx + \int_c^1 (c - px)\,dx = \frac{1}{2}(1 - p - (1 - c)^2)$.

Now, F dominates G by $(0, \varepsilon_2)$-GASSD if $(1 - 2\varepsilon_2)A(0) + \varepsilon_2 D \leqslant 0 \Leftrightarrow \varepsilon_2 \geqslant (1 - p)c^2/(2(1 - p)c^2 - p(1 - p - (1 - c)^2))$.

For $\varepsilon_2 < (1 - p)c^2/(2(1 - p)c^2 - p(1 - p - (1 - c)^2))$, the equation for γ^* is $(1 - 2\varepsilon_2)(\gamma B(\gamma) - A(\gamma)) + \varepsilon_2 \gamma(b - a) = \varepsilon_2 D$, which simplifies to $K(\gamma - \gamma_M)^2 - (\gamma - \gamma_M) + (D - \gamma_M) = 0$, where $K = (1/\varepsilon_2 - 2)(1/(2p(1 - p)))$. The solution is $\gamma^* = \gamma_M + (1 - \sqrt{1 - 4K(D - \gamma_M)})/2K$, taking the negative root so that $\gamma - \gamma_M < 0$, and F dominates G by $(\varepsilon_1, \varepsilon_2)$-GASSD if $\varepsilon_1 \geqslant 1/(2 + (p - c)/\gamma^*)$. Note that if $\varepsilon_2 \to 0$, then $K \to \infty$ and $\gamma^* \to \gamma_M$.

Figure B.1 illustrates GASSD dominance when $c = 0.25$, showing ε_1 as a function of ε_2 for different values of p. For a given p, F dominates G via $(\varepsilon_1, \varepsilon_2)$-GASSD if and only if $(\varepsilon_1, \varepsilon_2)$ is located northeast of the corresponding curve. As p increases, the $(\varepsilon_1, \varepsilon_2)$-boundary moves toward the origin, which means that smaller ε_1 and ε_2 can yield GASSD dominance. Note that although in this example we are able to get analytical expressions to demonstrate the various steps in the process, these steps could be done numerically to obtain results like Figure B.1.

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