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Search before Trade-offs are Known

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Abstract

Search, broadly defined, is a critical managerial activity. Our contribution is a model of search for multiattribute alternatives. Most of the search literature considers univariate alternatives, and can be applied to a multiattribute setting provided the trade-offs to be used at the final selection stage were known at the search stage. However, uncertainty about trade-offs is likely to occur, especially in settings (e.g., vendor selection, new product development, innovation tournaments) that involve parallel search. We show that incorporating uncertainty about trade-offs into a model changes its search strategy recommendations. Failing to account for such uncertainty, which is likely in practice, leads to suboptimal search and potentially large losses. For parallel search and a multivariate elliptical (e.g., normal) distribution of the alternatives, the solution is equivalent to univariate search with appropriately adjusted standard deviation. We prove that the optimal number of draws increases if uncertainty about tradeoffs increases.

Keywords: Simultaneous Search; Parallel Search; Unknown Trade-offs; Multiattribute Search
1. Introduction

After searching for five months and considering more than a hundred CEO contenders, Microsoft directors chose internal candidate Satya Nadella. The search process was exhausting; the Board discussed not only potential CEO candidates but also the pros and cons of an outsider versus an insider, future governance of the $300 billion company, and Bill Gates assuming a new Board role as Technology Advisor—which in turn affected the CEO appointment. In the language of decision analysis, Microsoft’s Board of Directors searched for multiattribute alternatives and the trade-offs among attributes were not precisely known at the start.

The extensive search (i.e., considering more than 100 alternatives) conducted by Microsoft is consistent with insight from the search literature — bigger difference of payoffs from different alternatives justifies the higher search effort and cost. We focus on other aspects of this example—that the search is for multiattribute alternatives and that trade-offs among different attributes (e.g., relative importance of the ability to lead change versus an understanding of Microsoft’s complex business) are not entirely known at the beginning yet are clarified later.

Search, broadly defined, is ubiquitous. Research and development activities, new product development, idea generation, innovation tournaments, and finding a supplier or business partner are just some examples. In practically all situations, the alternatives are characterized by multiple attributes. Within expected utility framework, the value of each multiattribute alternative is given by its utility, which depends on trade-offs among the attributes. This implies that search for multiattribute alternatives boils down to univariate search, subject to one caveat: the trade-offs to be used at the final selection stage are known in advance.

The search process can be viewed as consisting of two stages: the search stage, during which alternatives are explored; and the selection stage, at which one of the explored alternatives is chosen. At the selection stage, the discovered alternatives are ranked by explicit comparison of their utilities or, more

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simply, via ordinal ranking. In either case, this ranking is based on (implicit or explicit) trade-offs among the attributes. If the search process takes time, then it is reasonable to assume that—during that time—new information about the trade-offs (independent of the search process itself) will come to light; therefore, the trade-offs to be used at the selection stage might not be known at the search stage. As noticed by March (1978):

We expect change in our preferences. As we contemplate making choices that have consequences in the future, we know that our attitudes about possible outcomes will change in ways that are substantial but not entirely predictable. The subjective probability distribution over possible future preferences (like the subjective probability distribution over possible future consequences) increases its variance as the horizon is stretched. As a result, we have a tendency to want to take actions now that maintain future options for acting when future preferences are clearer.

Uncertain trade-off might occur even in the case of a single-attribute utility function. Bell (1988) discusses the setting where mean–variance trade-off is affected by the resolution of a side bet. That setting would match ours if the side bet is resolved between the search and selection stages, and we focus on how this anticipated resolution affects the search strategy.

Our methodological contribution is a model of search for multiattribute alternatives. In §4 we consider parallel search when the trade-offs between attributes are unknown at the search stage. If the alternatives are drawn from a multivariate elliptical distribution, then the solution to multivariate search is equivalent to univariate one with appropriately adjusted standard deviation, and it is optimal to search more as the uncertainty about trade-offs increases. The family of elliptical distributions is a large one and includes the multivariate normal, $t$, Cauchy, logistic, and other distributions. As a multivariate elliptical (e.g., normal) distribution is likely to occur in practice, our results provide managerial guidelines for search decisions, as illustrated in Example 3. In §5 we consider parallel search with outside (already available) alternatives and the implications for sequential search.

A review of the literature is in §2, where we give special attention to settings where parallel search is used (e.g., vendor selection) and to the multiattribute nature of the alternatives in such settings.
Parallel search is used when exploring an alternative takes so much time that sequential search is precluded. Then the search and selection decisions are separated by substantial time, during which some information about trade-offs is likely to arrive. It is for these cases that our model is most relevant. The univariate parallel search is reviewed in §3.

2. Background and Literature Review

Search models have been extensively studied and applied to a variety of settings (for a recent review, see Rogerson et al. 2005). In sequential search, the decision is whether to explore one more alternative or to stop (e.g., Lippman and McCall 1976); in parallel search (Nelson 1961, Stigler 1961), the decision concerns the number of alternatives to explore, after which the alternative with the highest payoff is selected. A more general model combines these two decisions, so that in each period the searcher decides the number of alternatives to explore (Morgan 1983, Morgan and Manning 1985).

The search literature focuses almost exclusively on univariate alternatives. A few papers (e.g., Lim et al. 2006; see also the references therein) consider sequential search for multiattribute options along with different costs for discovering a new option and for learning the attributes of a particular option. In that research, the focus is on balancing search over options and search within options. In spirit, this is similar to the case of univariate options, whose uncertain values can be learned at extra cost (Lippman and McCardle 1991).

2.1. When Parallel Search Is Used

When exploring an alternative takes a lot of time, the decision maker seldom has the luxury of searching for more than one period. In such cases, parallel search is the only feasible option. Parallel search is also favored by the very design of tournaments and procurements.

Indeed, the model of parallel search has been applied in many contexts that include procurement, new product development, and innovation tournaments. In a procurement setting, the time-consuming assessment of suppliers and the demand for transparency both argue for a parallel search approach (Costantino et al. 2012, Heijboer and Telgen 2002). With new product development, parallel search is
used to reduce the time required to develop and test new solutions with the aim of benefitting from swifter responses to market opportunities (Loch et al. 2001). Dahan and Mendelson (2001) and Srinivasan et al. (1997) apply a parallel search model to new product development. In innovation tournaments, both the tournament design and the low marginal sampling cost favor a parallel search process (Terwiesch and Ulrich 2009, Terwiesch and Xu 2008). Kornish and Ulrich (2011) show empirically that redundancy (one potential weakness of parallel search for new ideas) is quite small. The various streams of the literature have developed special terms for the optimal number of alternatives to explore, such as “economic tender quantity” (de Boer et al. 2000) and “optimal batch size” for companies developing new products (Loch et al. 2001).

2.2. Multiattribute Alternatives and Uncertain Trade-offs

Most of the time, alternatives are characterized by multiple attributes. In the context of vendor selection, Ho et al. (2010) conclude: “The traditional single criterion approach based on lowest cost bidding is no longer supportive and robust enough in contemporary supply management.” An alternative criterion for evaluating bids is the “most economically advantageous tender” (MEAT; see Costantino et al. 2012). A survey of procurement practitioners concludes that incorporating multiple criteria (in addition to price) into the decision support tool used to determine the optimal number of bidders would increase considerably the number of situations to which the model is applicable (Heijboer and de Boer 2001). Different attributes are typically aggregated using a weighted sum (Lorentziadis 2010), which corresponds to additive utility in the decision analysis literature. The value of an alternative with attributes $y_1, \ldots, y_M$ is given by $k_1 y_1 + \cdots + k_M y_M$; here $k_1, \ldots, k_M$ are the attribute weights, often called “importance weights” (Lorentziadis, 2010). Because weight $k_i$ is also a trade-off between money (search cost) and attribute $i$, $i = 1, \ldots, M$, throughout the paper we will refer to these weights as trade-offs.

If parallel search is used, then there is a significant time delay between the search stage (when the number of alternatives to explore is chosen) and the selection stage (when the best of the explored alternatives is selected). Trade-offs $k_1, \ldots, k_M$ are numbers to be used at the selection stage, and given the significant time that elapses between search and selection decisions, they are likely to be unknown at the
search stage. Lorentziadis (2010) discusses just such a case in the context of supplier selection.

3. Univariate Model of Parallel Search

In the classical model of parallel search (Nelson 1961, Stigler 1961), the decision variable is the number of independent identical draws of random variable \(X\). The searcher knows the distribution of \(X\). Each draw costs \(c > 0\). After collecting \(n \geq 1\) draws \(X_1, \ldots, X_n\), the decision maker chooses the best of them. Thus, the payoff equals \(\max_{i=1,\ldots,n}(X_i)\), the highest of these \(n\) draws, minus the search cost \(nc\). For \(n\) draws, the expected payoff is given by\(^2\) \[\pi(c, X, n) = -nc + E[\max_{i=1,\ldots,n}(X_i)].\]

**Theorem 1.** Let \(E[|X|]\) exist. Then the expected payoff \(\pi(c, X, n) = -nc + E[\max_{i=1,\ldots,n}(X_i)]\) is concave in \(n\) and \(\lim_{n \to \infty} (\pi(c, X, n + 1) - \pi(c, X, n)) = -c\). So for \(c > 0\), the expected payoff is maximized at \(n^*(c, X) = \arg\max_n (\pi(c, X, n)) = \min\{n: \pi(c, X, n + 1) - \pi(c, X, n) < 0\}\).

**Proof.** Denote by \(F_X\) the cumulative distribution function (cdf) of \(X\). Then the cdf of \(\max_{i=1,\ldots,n}(X_i)\) is \(F_X^n\). Since \(E[|X|]\) exists, it follows that \(E\left[\max_{i=1,\ldots,n}(X_i)\right]\) also exists and that, by David and Nagaraja (2003, p. 34, footnote 1, and equation (3.1.10')), 

\[
E\left[\max_{i=1,\ldots,n}(X_i)\right] = \int_0^\infty (1 - F_X^n(x) - F_X^n(-x)) \, dx.
\]

Therefore, \(\pi(c, X, n + 1) - \pi(c, X, n) = \int_0^\infty \left(F_X^n(x)(1 - F_X(x)) + F_X^n(-x)(1 - F_X(-x))\right) \, dx - c\).

This expression is decreasing in \(n\) and approaches \(-c\) as \(n\) goes to infinity. \(\Box\)

For a random variable \(Z\), denote 

\[
\omega(n, Z) = E[\max_{i=1,\ldots,n}(Z_i)], \\
n^*(c, Z) = \min(n: \omega(n + 1, Z) - \omega(n, Z) < c).
\]

Note that if the shape of the distribution is preserved, then the optimal number of draws is adjusted in a simple manner: for \(X = \mu + \sigma Z\) with \(\sigma > 0\), we have \(\omega(n, X) = \mu + \sigma \omega(n, Z)\) and \(n^*(c, X) = n^*(c/\sigma, Z)\).

Let \(X\) be normal with mean \(\mu\) and standard deviation \(\sigma\). Then \(\pi(c, X, n) = -nc + \mu + \sigma \omega(n, Z_N)\)

\(^2\)This expected payoff is shown to be concave in Nelson (1961) for bounded \(X\) and in Benhabib and Bull (1983), de Boer et al. (2000), and Srinivasan et al. (1997) for \(X\) bounded from below. For completeness, we formally state and prove Theorem 1, because we could find no proof in the literature for the case where \(X\) is unbounded from below and from above.
and \( n^*(c, X) = n^*(c/\sigma, Z_N) \), where \( Z_N \) is standard normal. Silver (1987, Table 1) presents the corresponding values and uses them to estimate the number of bids. An interesting observation is that \( n^*(c, Z_N) \approx 0.5/c \), which implies a total search cost of \( cn^*(c/\sigma, Z_N) \approx 0.5 \sigma \). So if the values of the alternatives are normally distributed with standard deviation \( \sigma \), then one should spend about \( 0.5\sigma \) on search. As the search cost \( c \) decreases, the total optimal search cost also decreases albeit quite slowly. For example, if \( c = 0.002 \), then \( n^*(0.002, Z_N) = 169 \) and the total optimal search cost is 0.338.

4. Parallel Search before Trade-offs Are Known

We now extend the parallel search model from §3 to multiattribute alternatives with uncertain trade-offs. Each draw yields a multiattribute alternative \( Y \), and the uncertain trade-offs are denoted by \( K \). (We use boldface capital letters to indicate random vectors.) The vector \( Y \) captures uncertainty among attribute values of alternatives that might be available because of search, and the vector \( K \) captures uncertainty about relative merits of different components of \( Y \). Components of \( Y \) can be dependent and components of \( K \) can be dependent, but components of \( Y \) are independent of components of \( K \). For a particular \( k \), the payoff from alternative \( y \) is given by the valuation function \( v(y, k) \). If \( v(y, k) = k_1 y_1 + \cdots + k_M y_M \), then each \( k_i \) is a trade-off between attribute \( y_i \) and money (search cost); we will focus on this case and refer to \( k \) as trade-offs. However, some of our theoretical results (in particular, Theorem 2 and Theorem 4) are applicable to arbitrary \( v(y, k) \); then \( k \) corresponds to uncertain parameters of the valuation function.

As in §3, the decision maker collects \( n \) independent identical draws from a multivariate distribution that is known to this decision maker, and each draw costs \( c \). If alternative \( y \) is selected then the payoff is \(-nc + v(y, k)\), so the payoff depends both on the selected alternative \( y \) and on the realized trade-offs \( k \). A risk-neutral decision maker maximizes the expected payoff.

The decision maker faces two decisions: choosing the number of draws \( n \) at the search stage and, after collecting \( n \) alternatives, choosing one of them at the selection stage. The second decision is fairly straightforward. The first decision — about the number of draws — is more challenging and depends on
when the uncertainty about $K$ is resolved.

There are three different scenarios that correspond to the different times at which $K$ may become known (Figure 1). First, suppose $K$ is known at the search stage—that is, before the decision about the number of draws. Then, after observing $n$ draws with realizations $y_1, \ldots, y_n$, the decision maker will choose the alternative that maximizes $v(y_i, k)$. By Theorem 1, the optimal number of draws is then $n^*(c, X)$ with $X = v(Y, k)$.

**Figure 1.** Resolving uncertainty about trade-offs: Three scenarios.

1) **Trade-offs are known**

   - **Search Stage:** Choose the number of draws.
   - **Selection Stage:** Choose one Alternative.

   Uncertainty about $K$ is resolved before the search stage.

2) **Trade-offs remain unknown**

   - **Search Stage**
   - **Selection Stage**

   Uncertainty about $K$ is resolved only after the selection stage.

3) **Trade-offs become known**

   - **Search Stage**
   - **Selection Stage**

   Uncertainty about $K$ is resolved between the search stage and the selection stage.

Second, suppose $K$ is not known at the search stage and remains unknown at the selection stage (i.e., when the decision maker is choosing one of the $n$ alternatives). In this scenario, after observing $n$
draws, the decision maker will choose the alternative that maximizes \( E_K[v(y_i, K)] \) and—again by Theorem 1—the optimal number of draws is \( n^*(c, X) \) with \( X = E_K[v(Y, K)] \). In sum: if trade-offs either are known before the search stage or remain unknown at the selection stage, then multiattribute search is equivalent to the classical univariate search problem in which each alternative is characterized by a single attribute (payoff). To determine the optimal number of draws in such cases, it is enough to estimate the univariate distribution of \( v(Y, k) \) in the first scenario and of \( E_K[v(Y, K)] \) in the second.

The focus of our paper is on the third scenario, where trade-offs become known before selecting one of the discovered alternatives but only after the decision about the number of draws has been made. (We refer to this scenario as “trade-offs become known”.) After \( n \) alternatives are drawn, the choice among \( y_1, \ldots, y_n \) might depend on the realized value of \( k \). In turn, the optimal number of draws is affected by knowing that the uncertainty about \( K \) will be resolved before the selection stage. The expected payoff for \( n \) draws is given by

\[
\pi(c, Y, K, n) = -nc + E[\max_{i=1,\ldots,n} v(Y_i, K)],
\]

where the expectation is over \( K \) and \( Y_1, \ldots, Y_n \).

**Theorem 2.** Let \( E[|v(Y, K)|] \) exist. Then the expected payoff \( \pi(c, Y, K, n) \) given by (2) is concave in \( n \) and \( \lim_{n \to \infty} (\pi(c, Y, K, n + 1) - \pi(c, Y, K, n)) = -c \). So for \( c > 0 \), the expected payoff is maximized at

\[
n^*(c, Y, K) = \arg\max_n \left\{ \pi(c, Y, K, n) : \pi(c, Y, K, n + 1) - \pi(c, Y, K, n) < 0 \right\}.
\]

**Proof.** The expected payoff (2) can be written as

\[
\pi(c, Y, K, n) = E_K \left[ -nc + E_{Y_1, \ldots, Y_n} \left[ \max_{i=1,\ldots,n} v(Y_i, K) \right] \right] = E_K \left[ \pi(c, v(Y, K), n) \right].
\]

It is the expectation (with respect to \( K \)) of a function that, by Theorem 1, is concave in \( n \) and has marginal change approaching \(-c\) as \( n \) goes to infinity. Therefore, the optimal number of draws \( n^*(c, Y, K) \) is the smallest \( n \) for which the change in the expected payoff becomes negative. \( \square \)

By Theorem 2, the expected payoff when trade-offs become known is a well-behaved function of \( n \), and maximizing it numerically (given the distributions of \( Y \) and \( K \)) is feasible in each particular case. Because it is beneficial to know trade-offs before choosing one of the alternatives, the expected payoff (2)
is not less than \( \pi(c, E_K]\{v(Y, K), n\}) = -nc + E_{Y_1, \ldots, Y_n} \max_{i=1, \ldots, n} E_K v(Y_i, K) \), which is the expected payoff when trade-offs remain unknown (and the two payoffs are equal only if the optimal alternative is, almost surely, the same for any \( k \)).

It is tempting to conjecture that if the uncertainty about \( K \) is resolved before the selection stage, the optimal number of draws should be no smaller than if \( K \) remains unknown—as it is more beneficial to create a wider choice of alternatives in the former case. In §4.1 we show that this is indeed the case if \( v(y, k) \) is linear in \( y \) and if \( Y \) has a multivariate elliptical distribution. In §4.2 we present counterexamples that underscore the difficulty of generalizing this result to nonelliptical distributions.

4.1. Search from a Multivariate Elliptical Distribution

Let the random vector \( Y \) have \( M \)-variate elliptical distribution with mean vector \( \mu_Y = (\mu_1, \ldots, \mu_M)^T \) and covariance matrix \( \Sigma \). (Superscript \( T \) denotes transposition.) The family of elliptical distributions is large and includes multivariate normal, \( t \), Cauchy, logistic and others (Fang et al. 1990, Table 3.1). An important property of an elliptically distributed random vector is that, up to normalization, all its components—and any linear combination of its components—have the same marginal univariate distribution (Owen and Rabinovitch 1983, Definition (b)). Denote by \( Z_Y \) the corresponding standardized random variable, which can also be defined as \( Z_Y = (Y_i - \mu_i)/\sqrt{\Sigma_{ii}} \) for any \( i = 1, \ldots, M \). For example, if \( Y \) is \( M \)-variate normal then \( Z_Y \) is standard normal.

In this section we assume also that \( v(y, k) = \sum_{j=1}^M y_j k_j = k^T y \). As before, \( K \) is independent of \( Y \) and we assume that \( E(\{K_j\}) \) exists for \( j = 1, \ldots, M \). Let \( \mu_K = E(K) \). Theorem 3 summarizes the solution for optimal parallel search in this setting and shows that it is equivalent to univariate search with appropriately adjusted standard deviation.

**Theorem 3.** Let \( Y \) be multivariate elliptical with mean vector \( \mu_Y \) and covariance matrix \( \Sigma \), and let \( v(y, k) = k^T y \). Then the expected payoff (2) is \( \pi(c, Y, K, n) = -nc + \mu_K^T \mu_Y + E[\sigma(K)]\omega(n, Z_Y) \). Here \( \sigma(K) = \sqrt{k^T \Sigma k}, Z_Y = (Y_i - \mu_i)/\sqrt{\Sigma_{11}} \) and \( \omega(n, Z_Y) \), as defined in (1), is the expected value of the nth-order statistic of \( Z_Y \). For \( c > 0 \), the optimal number of draws is \( n^* = (c/E[\sigma(K)], Z_Y) \) with \( n^* \) defined in (1).
Furthermore, \( \sigma(k) \) is convex and so \( E[\sigma(K)] \geq \sigma(\mu_K) \); hence the optimal number of draws if trade-offs become known is never less than the optimal number of draws if trade-offs remain unknown (or are known to equal \( \mu_K \)).

**Proof.** For \( y \) multivariate elliptical and \( v(y, k) = k^T y \), equation (2) becomes

\[
\pi(c, Y, K, n) = -nc + E_K \left[ E_{Y_1, \ldots, Y_n} \max_{i=1, \ldots, n} \left( K^T Y_i \right) \right].
\]

For given \( k \), the random variable \( k^T Y \) has mean \( k^T \mu_Y \) and standard deviation \( \sigma(k) = \sqrt{k^T \Sigma k} \), and its shape corresponds to \( Z_Y \) — that is,

\[
k^T Y = k^T \mu_Y + \sigma(k) Z_Y.
\]

Then, by (1), we have

\[
E_{Y_1, \ldots, Y_n} \left[ \max_{i=1, \ldots, n} \left( k^T Y_i \right) \right] = k^T \mu_Y + \sigma(k) \omega(n, Z_Y);
\]

taking the expectation over \( K \) yields

\[
\pi(c, Y, K, n) = -nc + \mu_k^T \mu_Y + E[\sigma(K)] \omega(n, Z_Y).
\]

In words, multiattribute search from a multivariate elliptical distribution is equivalent to search from a \( Z_Y \)-shaped univariate distribution with mean \( \mu_k^T \mu_Y \) and standard deviation \( E[\sigma(K)] \). If trade-offs remain unknown, then the expected payoff from \( n \) draws is equal to

\[
-nc + \mu_k^T \mu_Y + \sigma(\mu_k) \omega(n, Z_Y).
\]

It remains to prove that \( \sigma(k) \) is convex; that is, for any \( k_1, k_2 \) and \( \gamma \in (0,1) \),

\[
\sqrt{(\gamma k_1 + (1-\gamma)k_2)^T \Sigma (\gamma k_1 + (1-\gamma)k_2)} \leq \gamma \sqrt{k_1^T \Sigma k_1 + (1-\gamma)k_2^T \Sigma k_2}.
\]

Put \( X_1 = k_1^T Y \) and \( X_2 = (1-\gamma)k_2^T Y \). Then the inequality’s left-hand side is the standard deviation of \( X_1 + X_2 \) and its right-hand side is the sum of the standard deviations of \( X_1 \) and \( X_2 \) (since \( \Sigma = E(YY^T) - \mu_Y \mu_Y^T \)). The inequality is strict unless \( X_1 \) and \( X_2 \) are perfectly positively correlated. \( \square \)

Theorem 3 exploits two facts. First, the standard deviation \( \sigma(k) \) of a random variable \( v(Y, k) = k^T Y \) is convex in \( k \), which is true for any distribution of \( Y \). Second, \( k^T Y \) is a linear transformation of \( Z_Y \) because \( Y \) is elliptical. Then

\[
\pi(c, k^T Y, n+1) - \pi(c, k^T Y, n) = \sigma(k)(\omega(n+1, Z_Y) - \omega(n, Z_Y))
\]

is also convex in \( k \). For nonelliptical distribution of \( Y \), the shape of \( k^T Y \) depends on \( k \). Because of that, as shown in §4.2, it is difficult to extend Theorem 3 to nonelliptical distributions.

By Theorem 3, one should search more if the distribution of trade-offs becomes riskier (i.e., if multivariate zero-mean noise is added to \( K \)). According to Corollary 1 below, for \( M = 2 \) one should also search more if correlation between trade-offs \( K_1 \) and \( K_2 \) decreases in the sense of correlation–decreasing transformations (Epstein and Tanny 1980, Richard 1975).
**Corollary 1.** For $M = 2$, let $K > 0$ be obtained from $K_0 > 0$ via a sequence of correlation-increasing transformations. Then $n^* (c/E[\sigma(K_0)], Z_Y) \geq n^* (c/E[\sigma(K)], Z_Y)$. This result does not extend to $M > 2$.

**Proof.** For $M = 2$, we need to prove that $E[\sigma(K_0)] \geq E[\sigma(K)]$. This holds because

$$\frac{\partial^2}{\partial k_1 \partial k_2} \sigma(k) = \frac{\partial^2}{\partial k_1 \partial k_2} \frac{\Sigma_{11} k_1^2 + 2 \Sigma_{12} k_1 k_2 + k_2^2 \Sigma_{22} - \Sigma_{12}^2}{(\sigma(k))^3} \leq 0.$$ 

To show that this result does not extend to $M > 2$, consider the case $M = 3$ with $\Sigma_{13} = \Sigma_{23} = 0$. Then

$$\frac{\partial^2}{\partial k_1 \partial k_2} \sigma(k) = -\frac{k_1 k_2 (\Sigma_{11} \Sigma_{22} - \Sigma_{12}^2)}{(\sigma(k))^3} + \frac{k_2^2 \Sigma_{12} \Sigma_{23}}{(\sigma(k))^3},$$

which is negative for $\Sigma_{12} \leq 0$ but is positive for $\Sigma_{12} > 0$ and large enough $k_3$. □

To gain more intuition about Theorem 3, it is useful to consider an equivalent setting: one where the alternatives are drawn from a spherical distribution (a special case of a symmetric elliptical distribution in which all components are uncorrelated). The $M \times M$ covariance matrix $\Sigma$ can be written as $\Sigma = A^T A$, where $A$ is a $p \times M$ matrix with $\text{rank}(\Sigma) = p$. Then there exists a $p$-variate spherically distributed $Y'$ with $E(Y'_i) = 0$ and $\text{Var}(Y'_i) = 1$, $i = 1, \ldots, p$, such that $Y = \mu_Y + A^T Y'$ (Fang et al. 1990, Definition 2.2). For example, if $Y$ is multivariate normal then $Y'$ consists of independent standard normal variables. Note that $Z_Y$ (defined in Theorem 3) has the same distribution as $Y'_i$, $i = 1, \ldots, p$, and therefore $\omega(n, Z_Y) = \omega(n, Y'_i)$. Let $K' = AK$. Then $\sigma(k) = \sqrt{k^T \Sigma k} = \sqrt{k'^T k'}$. In words, the parallel search setting in which alternatives are drawn from an $M$-variate elliptical distribution with covariance matrix $\Sigma = A^T A$ is equivalent to a setting where alternatives are drawn from a $p$-variate spherical distribution ($p \leq M$) and where the distribution of trade-offs is adjusted as $K' = AK$. Then $k'$ is a vector in a $p$-dimensional space and $\sigma(k') = \sqrt{k'^T k'}$ is its length, which is convex because the length of the sum is less than the sum of the lengths. Uncertainty about the adjusted trade-off vector $K'$ can then be decomposed into uncertainty about length (i.e., the value of $\sigma(k')$) and uncertainty about direction (i.e., the unit vector $k'/\sigma(k')$).

Returning to the original trade-offs $k$, we can say that uncertainty about length is uncertainty about $\sigma(k) = \sqrt{k^T \Sigma k}$ and uncertainty about direction is uncertainty about $Ak/\sigma(k)$. Decomposing uncertainty about trade-offs into uncertainty about length and uncertainty about
direction is useful for considering the value of information — that is, how much the decision maker would benefit if trade-offs were resolved earlier. As Corollary 2 will show, it is beneficial to know trade-offs before the selection stage (compared with the trade-offs remaining unknown) if there is uncertainty about direction and it is beneficial to know trade-offs before the search stage (compared with the trade-offs becoming known) if there is uncertainty about length.

**COROLLARY 2.** Consider the setting in Theorem 3, and let $\Sigma = A^T A$.

(i) Suppose there is no uncertainty about direction; that is, suppose there exists a unit vector $s$ such that $A K / \sigma(K)$ equals $s$ almost surely. Then the expected payoff and the optimal number of draws are the same in the “trade-offs remain unknown” and “trade-offs become known” scenarios.

(ii) Suppose there is no uncertainty about length; that is, there exists a value $b$ such that $\sigma(K)$ equals $b$ almost surely. Then the expected payoff and the optimal number of draws are the same in the “trade-offs become known” and “trade-offs are known” scenarios.

**Proof.** (i) If there is no uncertainty about direction, then $A K = s \sigma(K)$ and so $\sigma(\mu_K) = E[\sigma(K)]$. By Theorem 3, the payoffs in the “trade-offs remain unknown” and “trade-offs become known” scenarios are identical.

(ii) If trade-offs are known, then the optimal number of draws for a particular realization of $k$ is given by $n^*(c/\sigma(k), Z_Y) = n^*(c/b, Z_Y)$. Also note that $E[\sigma(K)] = b$. So the decision at the search stage is the same in both scenarios. The decision at the selection stage is also the same given that, in each scenario, the trade-offs are known by the selection stage. □

The intuition behind Corollary 2 is that, if there is no uncertainty about direction, then the same discovered alternative remains optimal for all realizations of $K$ and so the action at the selection stage is independent of $k$ — in which case the information about $K$ has no value. Uncertainty about length corresponds to the uncertainty about search cost, which in turn affects the optimal number of draws given by $n^*(c/\sigma(k), Z_Y)$. If $\sigma(K)$ is the same for all realizations of $K$, then information about $K$ does not change the decision at the search stage.

For example, consider $M = 2$; let $\Sigma = A^T A$ be an identity matrix (and then $A$ is also an identity.
matrix), let $K^T$ be either $(0.5, 0.5)$ or $(1.5, 1.5)$ with equal probabilities, and let $K_0^T$ be either $(1.5, 0.5)$ or $(0.5, 1.5)$ with equal probabilities. Then there is no uncertainty about direction if uncertain trade-offs are given by $K$ and no uncertainty about length if uncertain trade-offs are given by $K_0$. According to Corollary 2, it is worthwhile to resolve trade-offs before the selection stage (i.e., to be in the “trade-offs become known” rather than the “trade-offs remain unknown” scenario) if trade-offs are given by $K_0$ but not if they are given by $K$. Conversely, it is worthwhile to resolve trade-offs before the search stage (i.e., to be in the “trade-offs are known” rather than the “trade-offs become known” scenario) if trade-offs are given by $K$ but not if they are given by $K_0$. Note also that, in line with Corollary 1, the optimal number of draws when trade-offs become known is greater for $K_0$ than for $K$.

4.2. Examples Where It Is Optimal to Search Less If Trade-offs Become Known

Here we present two examples that illustrate the difficulty of generalize §4.1’s results to nonelliptical distributions. Example 1 uses a discrete three–point distribution of $Y$ and demonstrates that the resolution of uncertainty about $K$ before the selection stage may either decrease or increase the optimal number of draws as compared with the “trade-offs remain unknown” scenario. Example 2 demonstrates the same with $Y$ being independent uniform.

**Example 1.** Let $Y^T = (Y_1, Y_2) = \begin{cases} (1, 0) & \text{with probability } p, \\ (20, -10) & \text{with probability } (1 - p)/2, \\ (-20, 10) & \text{with probability } (1 - p)/2; \end{cases}$

$K^T = (K_1, K_2) = \begin{cases} (1, 1) & \text{with probability } 0.5, \\ (1, 3) & \text{with probability } 0.5; \end{cases}$

$v(y, k) = y_1 k_1 + y_2 k_2; \ c = 0.01.$

Set $p = 0.05$. If $K$ remains unknown then the optimal number of draws is 31, but if $K$ becomes known then the optimal number of draws is 10.

Set $p = 0.5$. Now if $K$ remains unknown then the optimal number of draws is 5, but if $K$ becomes known then the optimal number of draws is 18. (See the Appendix for details.) The intuition behind this result is that, if $K$ remains unknown, then the alternative $(1, 0)$ is preferable because it gives the highest expected payoff. If $K$ becomes known then one wants to get two alternatives, $(20, -10)$ and $(-20, 10)$, that
yield the highest respective payoffs for \( k_2 = 1 \) and \( k_2 = 3 \). If \( p \) is small (e.g., \( p = 0.05 \)) then, when \( K \) remains unknown, many draws are required (at a small cost of \( c = 0.01 \) each) to discover one alternative of \((1, 0)\). However, if \( K \) becomes known then even a relatively small number of draws would practically guarantee that there will be one \((20, -10)\) and one \((-20, 10)\) alternative. If \( p \) is not small (e.g., \( p = 0.5 \)) then the opposite holds: there is no need to take many draws when \( K \) remains unknown, since the chances of getting \((1, 0)\) are quite high; but one must search more to discover one \((20, -10)\) and one \((-20, 10)\) alternative. □

Example 1 shows that (a) the resolution of uncertainty about \( K \) at the selection stage does affect the optimal number of draws at the search stage and (b) this number can be either greater or less than the optimal number of draws when \( K \) remains unknown. However, that example involves a relatively special distribution of \( Y \). In our next example \( Y \) is independent uniform.

**Example 2.** Consider \( M = 2 \), and let \( Y_1 \) and \( Y_2 \) be independent and uniformly distributed on \([0, 1]\). Let \( K^T = (K_1, K_2) = \{(1, 0) \text{ with probability } 0.5, (0, 1) \text{ with probability } 0.5 \}; \) and \( v(y, k) = y_1k_1 + y_2k_2 \).

If trade-offs become known then

\[
E\left[\max_{i=1,\ldots,n} v(Y_i, K)\right] = 0.5E\left(\max_{i=1,\ldots,n} Y_{1i}\right) + 0.5E\left(\max_{i=1,\ldots,n} Y_{2i}\right) = \omega(n, Z_U),
\]

where \( Z_U \) is uniformly distributed on \([0, 1]\). Therefore, search with trade-offs becoming known is equivalent to univariate search from a uniform distribution on \([0, 1]\). The optimal number of draws is \( n^* (c, Z_U) \), as defined in (1), with \( \omega(n, Z_U) = n/(n + 1) \).

If trade-offs remain unknown, then the distribution of \( Z_T = E_K v(Y, K) = 0.5(Y_1 + Y_2) \) is symmetric triangular on \([0, 1]\). Then the optimal number of draws is \( n^* (c, Z_T) \) with

\[
\omega(n, Z_T) = \int_0^{0.5} n2^{1-n}x^n \, dx + \int_{0.5}^1 nx(2-x)(4x-x^2-2)^{n-1}2^{1-n} \, dx.
\]

If it were always optimal to search more in the “trade-offs become known” scenario, then \( n^* (c, Z_U) \geq n^* (c, Z_T) \) for all \( c \). By (1), this is equivalent to

\[
\omega(n + 1, Z_U) - \omega(n, Z_U) \geq \omega(n + 1, Z_T) - \omega(n, Z_T) \text{ for all } n.
\]

Calculation reveals that this inequality holds for \( n \leq 7 \) but not for \( n > 7 \). As a result, \( n^* (c, Z_U) \geq n^* (c, Z_T) \) for \( c \geq 0.012 \) and
\( n^*(c, Z_U) \leq n^*(c, Z_T) \) for \( c < 0.012 \), with strict inequalities for some \( c \).

In this example, even though the standard deviation of \( \nu(Y, k) = k_1 Y_1 + k_2 Y_2 \) is convex in \( k \), the shape of this distribution depends on \( k \). It is uniform for \( k_1 = 1 \) and \( k_2 = 0 \) but is triangular for \( k_1 = 0.5 \) and \( k_2 = 0.5 \) (which corresponds to the case where trade-offs remain unknown). □

In Example 2, \( Y \) is uniform over a square. We remark that if \( Y \) were uniform over a circle (or an ellipse) then it would be spherically distributed and so, by Theorem 3, the optimal number of draws would be greater when trade-offs become known — because in this case the shape of the distribution of \( k^T Y \) would not depend on \( k \).

### 4.3. Illustration: Bivariate Normal Distribution of Alternatives

Let \( Y = (Y_1, Y_2)^T \) be bivariate normal with respective standard deviations \( \sigma_1, \sigma_2 \) and correlation \( \rho \). We consider the case of one uncertain trade-off \( K^T = (1, K_2) \) and assume without loss of generality that \( E(K_2) = 1 \). Then \( \nu(y, k) = y_1 + k_2 y_2 \).

In a setting with \( K_1 \equiv 1 \), the first attribute is in the same units as search cost. In the case of monetary search cost, \( y_1 \) can be price, cost, or profit. The second attribute is nonmonetary — for example, capacity, reliability, quality, or supplier sustainability. Then \( k_2 \) is the weight of the second attribute, or the trade-off between \( y_2 \) and \( y_1 \) (e.g., between quality and price), and it might be not known at the search stage.

Although alternatives are typically characterized by multiple attributes (§2 provides the references for multiple attributes in the vendor selection context), for illustration purposes we focus here on the \( M = 2 \) case with a single uncertain trade-off. Example 3 (to follow) describes a realistic setting with two attributes.

If that trade-off remains unknown at the selection stage, then the optimal number of draws is

\[
\begin{align*}
n^*(c/\sigma(\mu_K), Z_N) \quad & \text{with} \quad \sigma(\mu_K) = \sqrt{\sigma_1^2 + 2\sigma_1\sigma_2\rho + \sigma_2^2} \\
\end{align*}
\]

If the trade-off becomes known by the selection stage, then — by Theorem 3 — the optimal number of draws is \( n^*(c/E[\sigma(K)], Z_N) \) with

\[
\begin{align*}
E[\sigma(K)] = E\left(\sqrt{\sigma_1^2 + 2\sigma_1\sigma_2\rho K_2 + \sigma_2^2 K_2^2}\right).
\end{align*}
\]
Figure 2. Optimal number of draws (vertical axis) as a function of correlation $\rho$ for $K_2 = 1.0 \pm 0.8$ (with equal probability); $c = 0.05$ and $\sigma_1 = \sigma_2 = 1$.

Figure 2 plots the optimal number of draws in these two scenarios as a function of $\rho$, where $K_2 = 1 \pm 0.8$ (with equal probability), $c = 0.05$, and $\sigma_1 = \sigma_2 = 1$. Confirming Theorem 3, the graph shows that one should search more if the trade-off becomes known. This effect is stronger for negative correlations (e.g., for $\rho = -0.9$, the optimal number of draws is 5 when the trade-off remains unknown but is 9 when the trade-off becomes known) and vanishes for positive correlations. This dynamic is consistent with Corollary 2; uncertainty about direction decreases with correlation and goes to zero at $\rho = 1$, in which case $\mathbb{E}[\sigma(K)] = \sigma(K)$.

Figure 3. Difference between expected payoff with the optimal number of draws and expected payoff with a single draw (vertical axis) as a function of correlation $\rho$, for three different information scenarios; $K_2 = 1.0 \pm 0.8$ (with equal probability), $c = 0.05$, and $\sigma_1 = \sigma_2 = 1$. 
Figure 3 plots the difference between the expected payoff with the optimal number of draws and the expected payoff with a single draw for the three different scenarios (of when the trade-off becomes known) described in Figure 1. (This difference does not depend on $\mu_Y$.) As expected, the earlier the trade-off is known, the better. We can also see that, at $\rho = -1$, the expected payoffs are the same for the “trade-off is known” and “trade-off becomes known” scenarios; in other words, there is no benefit in resolving uncertainty about the trade-off before the search stage as opposed to resolving it before the selection stage. This result is consistent with Corollary 2 because $\sigma(K) \equiv 0.8$ at $\rho = -1$, eliminating all uncertainty about length. At $\rho = 1$ the situation is different. Now the payoffs are the same for the “trade-off becomes known” and “trade-off remains unknown” scenarios, because in this case there is no uncertainty about direction. We now present a realistic situation where the foregoing analysis could prove useful.

**EXAMPLE 3.** A corporation needs a new floor space and must decide how many vendors it should invite to bid on the construction tender. According to observed practice, the corporation will evaluate the bids in terms of two attributes: the price to construct the building and the speed, in number of months, required for completion (Palaneeswaran and Kumaraswamy 2000).

Meanwhile, the corporation will rent temporary space until the permanent building is constructed. The monthly rental cost determines the price–speed trade-off that will be used for selecting the winner of the tender. At the time of deciding how many bidders to invite, the rental cost of the temporary space is unknown. Yet the tender process for construction lasts several months (Elfving et al. 2005), so the corporation will learn the rental cost before selecting a construction vendor.

The corporation may follow one of two different approaches when deciding how many vendors should be asked to bid. If the corporation accounts for the resolution of rental cost uncertainty, then its decision making follows the “trade-off becomes known” scenario and the decision is based on the probability distribution of the rental cost; if the corporation does not account for the resolution of rental cost uncertainty (or entirely ignores this uncertainty), then its decision making follows the “trade-off remains unknown” scenario and the decision is based on the expected rental cost. (The “trade-off is
known” scenario would occur if the corporation had an option to invite the bidders after the rental cost becomes known.

For illustration purposes, assume that the cost $c$ of including each vendor is $50,000, that the monthly rental cost $k_2$ is either $0.2$ or $1.8$ million (with equal probability), and that the attribute values $(Y_1, Y_2)$ for price and speed are jointly normally distributed with $\sigma_1 = \$1$ million and $\sigma_2 = 1$ month. These numbers correspond to Figures 2 and 3, where the latter’s vertical axis is denominated in millions of dollars.

Figure 2 shows that the corporation can safely ignore the uncertainty about the price–speed trade-off if it expects $\rho$ to be positive. That would be the case if different vendors use different technologies, some fitting better to the corporation’s requirements and some fitting worse. Then both price and speed are either good or bad. However, we would rather expect a negative correlation, as when a vendor offering rapid construction demands a higher price for speediness. This scenario corresponds to negative $\rho$ in Figure 2, in which case the corporation should approach more vendors than would be recommended by a model that uses only the expected rental cost (and thus ignores uncertainty about that cost at the search stage). As mentioned previously, for $\rho = -0.9$ the optimal number of vendors is $5$ if the trade-off (rental cost) remains unknown and $9$ if the trade-off becomes known.

Figure 3 plots the benefit of soliciting bids from the optimal number of vendors relative to approaching just a single vendor. When $\rho = -0.9$, the payoff in the “trade-off becomes known” scenario is $0.63$ million higher than that in the “trade-off remains unknown” scenario. In this case, failing to account for resolution of the rental cost uncertainty leads not only to underinvesting in search but also to underestimating the benefits of search. The latter is costly because it may lead the corporation to forgo the tender altogether. □

For illustrative purposes we have considered a bivariate normal distribution. It follows from §4.1 that all results would remain qualitatively the same for any other bivariate elliptical distribution.

5. Search with Outside Alternatives and Sequential Search
So far we have assumed that the decision maker must make at least one draw (i.e., \( n \geq 1 \)), that there are no previously discovered (and still available) alternatives, and that there is no option to abandon. We now consider the case where outside alternatives are present. This setting also allows us to discuss when one should stop searching and the implications for sequential search.

Consider the setting of Theorem 2, and assume that the decision maker has already discovered \( m \geq 1 \) alternatives \( y_1^*, \ldots, y_m^* \). We call them “outside alternatives”. (An option to abandon would be an alternative that yields zero payoff regardless of the value of \( K \).) Set \( \Theta = \{y_1^*, \ldots, y_m^*\} \). Then, for \( n \geq 0 \) draws, the expected payoff is given by

\[
\pi(c, Y, K, n|\Theta) = -nc + E[\max_{i=1,\ldots,n,j=1,\ldots,m}\{v(Y_i, K), v(y_j^*, K)\}].
\] (3)

We can adapt the proof of Theorem 2 to show that the expected payoff (3) is concave in \( n \). However, computing this payoff becomes more tedious because the optimal \( n \) could depend on all available alternatives \( y_1^*, \ldots, y_m^* \). That situation arises whenever, for each \( j \) (\( j = 1, \ldots, m \)), there is a positive probability that \( \max\{v(y_1^*, K), \ldots, v(y_m^*, K)\} = v(y_j^*, K) \) — in other words, if none of the alternatives \( y_1^*, \ldots, y_m^* \) is dominated by others. However, if trade-offs remain unknown at the time of selection, then the decision maker needs to retain only one of the \( m \) alternatives: the one that maximizes \( E_K[v(y_j^*, K)] \).

Example 1 showed that, in the absence of outside alternatives, the optimal number of draws when trade-offs become known at the search stage could be either greater or less than when trade-offs remain unknown. This conclusion holds in the more general setting in which some outside alternatives are available. Also, if \( n \) is large compared to \( m \) and if outside alternatives are viewed as being drawn from the same distribution \( Y \), then it is unlikely that the best alternative would be one of the alternatives discovered earlier. In such a case, ignoring outside alternatives will probably yield a good approximation (so that, numerically, (2) and (3) would be close to each other).

We now consider sequential search before trade-offs are known. The setting is the same as in Figure 1 except for change to the search stage: instead of choosing the number of draws (the only decision
variable in parallel search), the decision maker performs a sequential search. There might be multiple periods, and in each period there might be multiple draws. Taking zero draws is equivalent to stopping the search; for that stopping decision, the decision maker must compare the expected payoff with no additional draws \((n = 0)\) and the expected payoff with at least one additional draw \((n \geq 1)\). By the concavity of \((3)\) in \(n\), one should stop searching if \(\pi(c, Y, K, 0|\theta) > \pi(c, Y, K, 1|\theta)\); that is, one need only compare the expected payoffs with \(n = 0\) and \(n = 1\). We now compare the stopping decisions in two scenarios— one where trade-offs become known and one where they remain unknown— for the same set of outside alternatives \(y_1^*, ..., y_m^*\).

**Theorem 4.** Suppose that exactly one outside alternative is available (i.e., \(m = 1\)). If it is optimal to continue searching when trade-offs remain unknown, then it is optimal to continue searching when trade-offs become known.

**Proof.** When trade-offs become known, one should continue searching if

\[
E[v(y_1^*, K)] < -c + E[\max\{v(Y, K), v(y_1^*, K)\}].
\]

When trade-offs remain unknown, one should continue searching if

\[
E[v(y_1^*, K)] < -c + E_Y[\max\{E_K[v(Y, K)], E_K[v(y_1^*, K)]\}].
\]

Theorem 4 now follows from the inequality

\[
E_Y[\max\{E_K[v(Y, K)], E_K[v(y_1^*, K)]\}] \leq E_{Y,K}[\max\{v(Y, K), v(y_1^*, K)\}].
\]

This theorem exploits the fact that if one stops searching (i.e., \(n = 0\)) then, for \(m = 1\), the expected payoff is the same in both scenarios; however, if one more alternative is drawn then the expected payoff is greater when trade-offs become known. For \(m \geq 2\), the expected payoff with \(n = 0\) is also greater in the scenario where trade-offs become known. As a result, Theorem 4 does not extend to \(m \geq 2\); it could be that, for some choice of \(y_1^*, ..., y_m^*\), the searcher will stop if trade-offs become known at the selection stage but will continue searching if those trade-offs remain unknown.

For example, consider the stopping decision in the setting of Example 1 with \(p = 0.5\), 
\(m = 2, y_1^* = (20, -10)^T\), and \(y_2^* = (-20, 10)^T\). The searcher will stop in the scenario where trade-offs become known at the selection stage but will continue searching in the scenario where trade-offs remain unknown. The reason is that when trade-offs remain unknown, the searcher will stop if and only if the
alternative (1, 0) is available. In the scenario where trade-offs become known, the searcher will stop if and only if the alternatives (20, -10) and (-20, 10) are both available.

Therefore, if trade-offs become known at the selection stage, then the stopping decision is different from the one in a univariate search. One important difference is that search with recall is never the same as search without recall. In a standard sequential search model with independent payoffs drawn from the same distribution, the stopping rule is characterized by a reservation price. Then the infinite-horizon settings with recall and with no recall are equivalent, because the decision maker never needs to keep an alternative that is below the reservation price and stops searching if the discovered alternative is above the reservation price. The same holds for a multivariate search where trade-offs remain unknown. But if trade-offs become known then the searcher would like to keep multiple discovered alternatives — all that are not dominated. Therefore, settings with and without recall are different. (And search without recall is again equivalent to search where trade-offs remain unknown.)

Another difference concerns draws from different distributions. Weitzman (1979) considers a setting in which the searcher has several closed boxes (i.e., different univariate distributions) to explore. Each box is characterized by a reservation price, and the search strategy is to open the box with the highest reservation price. If trade-offs become known then the search strategy is not that simple and elegant. Each box does not have a single reservation value because the attractiveness of opening a particular box depends on the combination of alternatives discovered previously. For example, suppose the searcher can take any number of draws (sequentially, with a fixed cost per draw) from two different distributions. In a univariate setting, one would never use both distributions: all draws will be taken from the distribution with the highest reservation price. If trade-offs become known then the searcher might switch between the distributions depending on previously discovered alternatives.

6. Summary and Discussion

Almost all existing models of search consider single-attribute alternatives, even though real-world alternatives are usually characterized by multiple attributes. The payoff from a multiattribute alternative is
determined by the trade-offs among—or weights attached to—its attributes. A decision maker explores
the alternatives at the search stage and then, at the selection stage, chooses one of them.

If the trade-offs determining final payoffs are known at the search stage, then a search for
multiattribute alternatives is equivalent to a univariate search in which each alternative is characterized by
its payoff. If trade-offs are not known at the search stage and remain unknown at the selection stage, then
the multiattribute setting is again equivalent to a univariate search; in this scenario, each alternative is
characterized by its expected payoff and we can assume that each uncertain trade-off is equal to its
expected value. In both cases, trade-offs that will be used at the selection stage are known at the search
stage. However, when the uncertainty about trade-offs is resolved after the search stage yet before the
selection stage, the multiattribute search setting differs from a univariate one.

We consider parallel search in §4. If trade-offs become known then, in general, the optimal
number of draws (i.e., the optimal decision at the search stage) could be either greater or less than the
optimal number of draws if trade-offs remain unknown (Examples 1 and 2). In a setting where
alternatives are drawn from a multivariate elliptical distribution, the optimal number of draws when the
trade-offs become known cannot be less than the optimal number of draws when trade-offs remain
unknown (Theorem 3). The family of elliptical distributions, which is a large one, includes the
multivariate normal distribution. The emerging prescription is that one should search more if some
information about trade-offs is likely to arrive before the selection stage. Example 3 presents and
discusses a realistic application.

If the uncertainty about trade-offs is resolved by the selection stage, then losses from nonoptimal
search—as when one wrongly assumes that trade-offs remain unknown at the selection stage (which is
equivalent to ignoring the uncertainty about trade-offs)—can be substantial. As illustrated in §4.2 for a
two-attribute example, these losses are higher when attributes are negatively correlated (which could
reasonably be expected in many situations) and also when trade-offs are negatively correlated
(Corollary 1).
Naturally, it is better to resolve uncertainty about trade-offs sooner rather than later; thus the expected payoff in the scenario where trade-offs remain unknown is smaller than the one where trade-offs become known, which in turn is smaller than the payoff where trade-offs are known at the outset. However, these benefits depend on the nature of uncertainty (Corollary 2). Uncertainty about trade-offs can be decomposed into uncertainty about direction and uncertainty about length of the adjusted trade-off vector. Resolving uncertainty about trade-offs at the selection stage matters to the extent that it might change the best alternative, which is captured by uncertainty about direction (an uncertainty that, as Theorem 3 shows, in turn affects the search stage decision). Resolving uncertainty about trade-offs at the search stage matters to the extent that it might change the optimal number of draws, which is captured by uncertainty about length.

In §5 we extend our setting to the case in which some alternatives (e.g., an option to abandon) are present before search starts, and we also consider the decision of when to stop searching. An important difference here from univariate search (and thus from the setting where trade-offs remain unknown) is that the decision maker must retain multiple alternatives because, depending on information about trade-offs at the selection stage, different alternatives might turn out to be optimal. When only one outside alternative is available, it is better to search more if trade-offs become known at the selection stage (Theorem 4). We would think that the case of a single outside alternative does occur often in practice. After all, a decision maker who perceives the alternatives to be univariate has no reason to retain multiple alternatives—not even if some had been discovered previously. Another difference from univariate search is that a searcher who can draw from different distributions might prefer switching among them in the “trade-offs become known” scenario, in response to previously discovered alternatives. The reason is that, unlike the case of univariate search, here each distribution cannot be characterized by a single reservation price.

Our conclusion is that both parallel (§4) and sequential (§5) search with multiattribute alternatives—and with trade-offs becoming known at the selection stage—are quite different from univariate search. We expect that this conclusion and our model will be relevant also in richer settings. In
Morgan and Manning (1985), the searcher decides how many alternatives to draw in each period. Chade and Smith (2006) consider the portfolio choice problem in which the decision maker determines how many draws to take simultaneously and also which distributions to explore (as in the literature on directed search, where participants in the labor market identify the jobs for which they should apply). Smith and Ulu (2012) consider technology adoption with uncertain costs and qualities. In all these settings it would be reasonable to view each alternative as having multiple attributes, and it would also be reasonable to expect that some additional information about trade-offs will become available over time (i.e., before the selection decision is made).

Our approach and results could be useful in modeling innovation tournaments and delegated search (i.e., search by agents on behalf of the principal). Chao and Erat (2012) discuss a setting in which the contestants are engaged in search activity but the principal will (with some probability) evaluate the quality of their solution as zero. Such a setting can be viewed as an extreme case of uncertainty about trade-offs, and more detailed modeling of this uncertainty could well yield extra insights. Chao et al. (2013) consider a stage–gate process whereby the agent searches and the principal is responsible for deciding whether (and when) the project should be stopped. A natural extension of that model would be to consider multiattribute projects, for which one could reasonably assume that the agent has some uncertainty about the trade-offs to be used by the principal. The model could also incorporate some gradual learning about the trade-offs.

It might be often overlooked that uncertainty about trade-offs will be reduced at the selection stage. This oversight could be due to the commonly observed underappreciation of uncertainty: a decision maker who does not perceive trade-offs to be uncertain at the search stage will, of course, be unaware that this uncertainty may be reduced at the selection stage. Furthermore, decision makers often omit important objectives (Bond et al. 2008) and make typical mistakes when assessing trade-offs (Keeney 2002); similar mistakes are likely to occur when assessing uncertainty about those trade-offs. Accounting both for uncertain trade-offs and for multiattribute alternatives places more of a cognitive burden on the decision maker, who must estimate not only the joint multivariate distribution of alternatives’ attributes (as
opposed to the univariate distribution of alternatives’ payoffs) but also the reduction in trade-off uncertainty at the selection stage. Even so, the resulting solution need not be overly complicated—for instance, it is fairly tractable in the case of a multivariate elliptical distribution of attributes. More importantly, such analysis leads to a better search decision (i.e., one with a higher expected payoff) and also enables us to assess the value of information about uncertain trade-offs.

Appendix: DERIVATION OF EXAMPLE 1

Let \( y_1 = (1, 0)^T, y_2 = (20, -10)^T, \) and \( y_3 = (-20, 10)^T. \)

First consider the case where \( K \) remains unknown. Then \( E_K[v(y_1, K)] = 1, E_K[v(y_2, K)] = 0, \)
\( E_K[v(y_3, K)] = 0, X = E_K[v(Y, K)] = \begin{cases} 1 \text{ with probability } p, \\ 0 \text{ with probability } (1 - p), \end{cases} \)
and the expected payoff after \( n \) draws is given by \( \pi(c, X, n) = -nc + 1 - (1 - p)^n. \) In accordance with Theorem 1, this value is maximized at \( n^*(c, X) = \max(1, [\ln(c/p)/\ln(1 - p)]); \) here \([\cdot]\) signifies rounding down to the nearest integer.

Now consider the case where \( K \) becomes known before choosing one of the \( n \) alternatives.

For both possible realizations of \( K, \) we have
\[
\max_{i=1,\ldots,n} v(Y_i, K) = \begin{cases} -10 \text{ with probability } ((1 - p)/2)^n, \\ 1 \text{ with probability } (p + (1 - p)/2)^n - ((1 - p)/2)^n, \\ 10 \text{ with probability } 1 - (p + (1 - p)/2)^n. \end{cases}
\]

The expected payoff (2) is
\[
\pi(c, Y, K, n) = -nc + 10 \left( 1 - \left( \frac{1 + p}{2} \right)^n \right) + \left( \frac{1 + p}{2} \right)^n - \left( \frac{1 - p}{2} \right)^n - 10 \left( \frac{1 - p}{2} \right)^n.
\]

Then, for any \( c \) and \( p, \) we can compute the optimal number of draws \( n^*(c, Y, K). \) □

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