Pricing and Revenue Management:
The Value of Coordination

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Pricing and Revenue Management: The Value of Coordination

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The integration of systems for pricing and revenue management must trade off potential revenue gains against significant practical and technical challenges. This dilemma motivates us to investigate the value of coordinating decisions on prices and capacity allocation in a stylized setting. We propose two pairs of sequential processes for making static decisions—on pricing and revenue management—that differ in their degree of integration (hierarchical versus coordinated) and their pricing inputs (deterministic versus stochastic). For a large class of stochastic, price-dependent demand models, these four processes admit tractable solutions satisfying intuitive sensitivity properties. We assess the relative performance of hierarchical and coordinated approaches, revealing the benefits of coordination and the advantages of modeling demand uncertainty in pricing. We use industry data to establish that coordination can yield significantly more revenue than a traditional hierarchical process that first sets prices using a deterministic model and then optimizes booking limit decisions. Yet we also find that most benefits of a fully coordinated process can be obtained using a hierarchical process in which prices are adjusted to reflect demand uncertainty. We conclude that stochastic pricing (i.e., capturing demand risk in pricing decisions) can mitigate the effects of poorly coordinated pricing and revenue management functions.

1. Introduction

Revenue management is common in capacity-constrained service industries—including airlines, hotels, car rentals, event ticketing, and TV advertising—where demand is responsive to price changes. However, revenue management models and practice have traditionally focused on capacity allocation decisions while treating price and demand as exogenous. This focus is partly explained by rigid organizational structures that separate the functions of marketing (including pricing) and operations (revenue management) and also by the technical and operational difficulties inherent in implementing an integrated price–availability decision support system. Indeed, “departmental differences in personnel, expertise and decision-support systems make it difficult to coordinate ... pricing and yield management decisions” (Jacobs et al. 2000). As a result, a sequential decision process is common in many industries (Talluri and van Ryzin, 2004, chap. 10).
Over the past decade, the importance of coordinating decisions on tactical pricing and revenue management has been widely acknowledged in the revenue management literature (McGill and van Ryzin 1999) and by practitioners (Garrow et al. 2006). In a wide-ranging review, Fleischmann et al. (2004) observe that “pricing decisions have a direct effect on operations and vice versa. Yet, the systematic integration of operational and marketing functions remains in an emerging stage, both in academia and in business practice.” In fact, a recent survey finds that only 11% of 479 companies practicing revenue management in Europe and North America manage both price and capacity allocation decisions, even as an overwhelming consensus points to price management as having the highest potential for revenue management (Kolisch and Zatta 2010).

The need to learn more about the value of integrating pricing and revenue management motivates two broad types of research questions. First, from a modeling perspective, what are the technical challenges entailed by incorporating pricing decisions into a revenue management framework? In particular, what types of demand specifications lead to tractable problems, how should we model price-sensitive demand uncertainty, and in what stage of the process is it actually important to do so? Second, from the practical perspective of assessing benefits, when is it important to integrate pricing and availability decisions, and what is the financial impact of doing so—for example, as compared with a traditional sequential approach? In particular, given the practical limitations of coordination, are there simpler alternatives that can achieve comparable revenues?

Our research addresses these issues by studying four sequential processes that combine pricing with subsequent revenue management decisions and differ along two dimensions: the extent of coordination between price and allocation decisions and the firm’s approach to pricing. These processes, which are modeled as two-stage stochastic programs, build price sensitivity and optimization into a stylized framework of static, two-fare-class revenue management (Belobaba 1987, Littlewood 1972). This standard building block model of revenue management theory and practice optimizes the allocation of limited capacity between two customer segments, where higher-paying customers arrive later in the horizon and where prices and demand are exogenously fixed.

The design of our study is simple but does capture the key elements of pricing and revenue management while allowing us to assess the value of coordinating these decisions under demand uncertainty. We focus on static two-fare-class pricing; preliminary analysis suggests that our main insights do extend to multiple classes. Static pricing is frequently observed in practice, for advertising, administrative and competitive reasons (Talluri and van Ryzin 2004, p. 334) and it is theoretically supported by consumer behavior considerations (e.g. Besanko and Winston 1990, Nasiry and Popescu 2011). Static models with few prices and independent demand can serve as good sources

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1 Although not necessarily in a coordinated manner.
of approximation for more realistic problems (Gallego and van Ryzin 1994, Bitran and Caldentey 2003) and may even outperform choice-based approaches (Zhang and Lu 2010). This paper makes the following main contributions, as intimated by the questions raised at the outset.

First, unlike a fully coordinated system, all the sequential policies studied here are proved to be tractable for a broad class of stochastic price-dependent demand models that capture increasing elasticity in the firm’s lost sales rate (LSR). Examples include attraction models and additive-multiplicative specifications (e.g., with linear and isoelastic price dependence) with increasing failure rate (IFR). These regularity conditions on stochastic demand naturally extend the deterministic regular demand concept (Gallego and van Ryzin 1994, Ziya et al. 2004), as well as single-product newsvendor model assumptions (Kocabıyıkoğlu and Popescu 2011), and they allow for sensitivity results characterizing the interaction of price and capacity decisions. For example, we show that, in a hierarchical environment, an increase in the high-end price should be met with a lower protection level—in contrast with implications of the standard revenue management model, which does not capture the price response. If LSR elasticity is increasing in price and quantity, then firms with expanding capacity should set lower high-end prices but higher protection levels because they will see lower revenue rates—for example, lower revenue per available seat (RAS) for airlines and lower revenue per available room (REVPAR) for hotels.

Second, we quantify the value of coordinating the decisions on pricing and capacity allocation. Using numerical simulations on data from the car rental industry, we find that the revenue gains from full coordination can be large (typically 1%–10%) relative to a hierarchical process that sets prices based on deterministic demand models—a common norm in practice (Valkov 2005, Westermann and Lancaster 2011), according to our exchanges with industry experts. The value of coordination is higher when demand is greater (relative to capacity) or more uncertain and also when segments are more differentiated. We find it interesting that the coordination gap can be nearly closed, however, by using a (tractable) hierarchical process that adjusts prices to reflect demand uncertainty and subsequently optimizes booking limits. We conclude that using stochastic pricing (i.e., capturing demand risk in the price-setting stage of a hierarchical process) can effectively compensate for the lack of coordination in a static revenue management framework.2

These insights have important practical consequences for capacitated firms when one considers the substantial organizational and implementation challenges posed by the integration of pricing and revenue management (Jacobs et al. 2000), especially when compared to the modest (data and estimation) requirements of using stochastic pricing in an extant uncoordinated environment. Moreover, the financial consequences can be significant because small positive changes in revenue

2 Throughout the paper, “coordination” refers to the full or partial integration of pricing and allocation decisions.
translate into spectacular profit gains for revenue management industries grappling with high fixed costs and extremely thin margins. For example, a 1% increase in revenue would have allowed the car rental company whose data was used in our numerical experiments—which in 2009 posted net profit margins of -1% on revenues of $5 billion (US)—to break even that year.

2. Relation to the Literature

Our work contributes to the vast literature on revenue management, for which the most comprehensive references to date are the books by Talluri and van Ryzin (2004) and Phillips (2005). McGill and van Ryzin (1999) review the earlier revenue management literature, Elmaghraby and Keskinocak (2003) focus on dynamic pricing. There is a growing body of work (reviewed by Bitran and Caldentey 2003) in the revenue management literature that addresses the problem of joint pricing and allocation. Several papers in this area use deterministic demand models to capture complex multiproduct, multiresource, or dynamic environments (e.g., Cote et al. 2003, Kachani and Perakis 2006, Kuyumcu and Popescu 2006). Ziya et al. (2004) analyze demand conditions that ensure regularity in deterministic models.

In contrast, we focus on stochastic demand models: we provide corresponding regularity conditions and assess the value of capturing price-sensitive demand uncertainty, relative to the value of coordination. Toward this end, we focus on a static, two-fare-class capacity allocation model (Belobaba 1987, Littlewood 1972) and extend it to manage and coordinate pricing decisions. A first step in this direction is due to Weatherford (1997), who evaluates numerically the revenue benefits—as a function of the requisite computational effort—from integrating allocation decisions and pricing in a static, single-resource environment with normally distributed additive-linear demand.

A few revenue management papers study joint pricing and allocation problems with aggregate demand uncertainty; they all use additive and/or multiplicative demand forms, which are special cases of our model. Bertsimas and de Boer (2005) provide regularity conditions for a static, partitioned allocation model and additive-multiplicative demand (similar to our model in Section 4.1) and then use that model to devise a heuristic for a multiperiod price–capacity allocation problem. In the context of nonprofit applications, de Vericourt and Lobo (2009) jointly optimize prices and allocations in a dynamic setting under a multiplicative demand model; their single-stage regularity condition is a special case of our LSR elasticity conditions. In a dynamic setting with competition, Mookherjee and Friesz (2008) assume increasing price elasticity in a multiplicative demand model with increasing generalized failure rate (IGFR) risk. These papers all rely on static regularity conditions to characterize more complex dynamic problems. Our results extend the static regularity conditions in these papers to more general demand models.

Several other approaches have been used for modeling price-sensitive demand uncertainty in revenue management. Multiperiod problems characterize price-sensitive stochastic demand as a
Markov arrival process, which is typically described as being Poisson distributed with known price and time-dependent intensity (Feng and Xiao 2006, Gallego and van Ryzin 1994, Maglaras and Meissner 2006). Uncertainty about the arrival rate has been addressed in Bayesian learning frameworks (Aviv and Pazgal 2005) or by using robustness methods (Adida and Perakis 2010). Our modeling choice favors instead the simplest framework that allows us to explore the interplay of coordination and uncertainty about (price-sensitive) demand in a revenue management context.

Finally, our work is also related to a vast operations literature on coordinating pricing and inventory decisions, as reviewed by Chan et al. (2004) and Fleischmann et al. (2004). An important distinction is that models in this stream focus on storable goods rather than services. Our model can be viewed as a multi-product extension of static newsvendor pricing models (for reviews, see Petruzzi and Dada 1999, Yano and Gilbert 2003). Most of this literature characterizes price-sensitive demand uncertainty in terms of additive and/or multiplicative models. Our general demand model and approach are based on Kocabıyıkoğlu and Popescu (2011), who use the concept of increasing LSR elasticity to provide general regularity conditions for the newsvendor pricing problem. Our analytical results in the first part of this paper show that similar demand regularity conditions are sufficient for several sequential pricing and revenue management problems. However, our primary concern differs from the concerns of this literature in that we aim to assess the value of coordinating pricing and capacity allocation decisions relative to a status quo hierarchical business process.

3. Hierarchical and Coordinated Revenue Management Models

In the standard revenue management model (Belobaba 1987, Littlewood 1972), a monopolistic firm optimizes the allocation of a fixed quantity of a flexible resource between two market segments with uncertain demands; the high-price segment arrives after the low-price segment, and prices are predetermined. In reality, firms have the ability to control prices, which in turn affect demand. In particular, the demand in major application areas of revenue management, such as airline travel and car rental, is sensitive to price changes (Talluri and van Ryzin 2004, chap. 7). To capture price response, we model demand as a general stochastic function of price, \( D(p) \) (detailed in Section 4) and extend the standard revenue management problem to optimize segment prices (Section 3.1).

To assess the value of coordinating pricing and allocation decisions, we introduce pricing models (Section 3.2) that provide input to hierarchical decision processes (Section 3.3).

3.1. Price Sensitive Revenue Management

Let \( p \) and \( \bar{p} \) denote the average contribution margins of the high- and low-fare classes (respectively); the corresponding random demands at these prices are \( D(p) \) and \( \bar{D}(\bar{p}) \), which are assumed to be independent. Throughout this paper, the parameters pertaining to the low-fare class are denoted by a bar (overline). Table 9 in the Appendix summarizes our notation.
The standard revenue management model allows for nested allocations of the firm’s capacity $K$, which means that all capacity that is not sold to the low-fare class is made available for sale to the high-fare class. Given a protection level $x$ for the high-fare class, sales to the low-price segment are constrained by the booking limit $K - x$ and by low-fare demand $\bar{D}(\bar{p})$, so they amount to $\min\{\bar{D}(\bar{p}), K - x\}$. Thus, the inventory available for sale to the high-fare class is ex-ante uncertain and amounts to $\max\{x, K - \bar{D}(\bar{p})\}$; in particular, it exceeds the protection level $x$ if the low-fare demand falls short of the booking limit—that is, if $\bar{D}(\bar{p}) \leq K - x$. Since low-fare demand is realized before high-fare demand and the two are independent, it follows that expected sales to the high-fare class (conditional on the low-fare demand realization $\bar{D}(\bar{p}) = \bar{D}$) can be calculated as $E_D\left[\min\{D(p), \max\{x, K - \bar{D}(\bar{p})\}\}\right]$. Taking sequential expectations, the firm’s expected revenue from the two nested fare classes may be written as follows:

$$R(\bar{p}, p, x) = \bar{p}E\left[\min\{\bar{D}(\bar{p}), K - x\}\right] + pE\left[\min\{D(p), \max\{x, K - \bar{D}(\bar{p})\}\}\right]. \tag{1}$$

The classical revenue management model optimizes the protection level $x$, given fixed prices $\bar{p}$ and $p$. To reflect the hierarchical nature of this process, we refer to this model as (H):

$$\text{(H)} \quad \mathcal{V}[\text{H}] = \max_x R(\bar{p}, p, x); \tag{2}$$

the operator $\mathcal{V}$ denotes the optimal policy value. In contrast, a fully coordinated pricing and revenue management model, (F), simultaneously optimizes the prices $p, \bar{p}$ and protection level $x$:

$$\text{(F)} \quad \mathcal{V}[\text{F}] = \max_{\bar{p}, p, x} R(\bar{p}, p, x). \tag{3}$$

To mitigate the difficulty of solving model (F), we study a partially coordinated model (C), which jointly optimizes the price and allocation for the high-end market given a low price $\bar{p}$:

$$\text{(C)} \quad \mathcal{V}[\text{C}] = \max_{p, x} R(\bar{p}, p, x). \tag{4}$$

Our focus on optimizing the high-end price in model (C) reflects industry practice: low-end prices are often constrained by competitive, historical, or social considerations, whereas the firm enjoys more pricing power and flexibility in the high-end market (see Section 5 for further discussion). Moreover, in a well-segmented market, the timing of arrivals allows decisions regarding the high-end segment to be made later in the horizon—that is, after the low-end price $\bar{p}$ has been set.

As broadly discussed in the Introduction, our goal is to assess the value of coordinating decisions on pricing and capacity allocation. We therefore benchmark the fully coordinated, but generally intractable model (F) against tractable hierarchical and partially coordinated policies. These policies employ pricing models to provide segment prices, which are then used as input into revenue management model (H) or (C). In Section 3.2 we describe two pricing models commonly used in the literature and that determine the first stage of the sequential policies studied in this paper.
3.2. Pricing Models

Depending on the industry, several pricing approaches (A) are conceivable and used in practice; these include price fixing, value-based pricing, cost-plus methods, and matching the competition (Phillips 2005). Our discussions with industry experts indicate that—unlike revenue management models—the price optimization models used in practice are typically deterministic, based on average demand curves for each product and market (e.g., Valkov 2005). Hence we use a deterministic model (D) as a benchmark for making pricing decisions and then compare it to a stochastic pricing model (S) in order to assess the value of stochastic pricing.

The deterministic pricing model (Bitran and Caldentey 2003, Gallego and van Ryzin 1994) is a certainty-equivalent (or fluid) benchmark that replaces random demands with their means \( E[D(p)] \) and \( E[\tilde{D}(\tilde{p})] \) while solving for optimal “deterministic” prices \( p^D, \tilde{p}^D \) subject to capacity constraints:

\[
(D) \quad V[D] = \max_{p,\tilde{p}} p E[D(p)] + \tilde{p} E[\tilde{D}(\tilde{p})] \quad \text{s.t.} \quad E[D(p)] + E[\tilde{D}(\tilde{p})] \leq K. \tag{5}
\]

The stochastic pricing model (Belobaba 1987, Bertsimas and de Boer 2005) jointly optimizes prices \( p, \tilde{p} \) and allocation by partitioning capacity into separate blocks of size \( x \) and \( K-x \) that can be sold only to the respective market segments. The optimal segment prices \( p^S, \tilde{p}^S \) solve

\[
(S) \quad V[S] = \max_{p,\tilde{p},x} p E\{D(p)\} + \tilde{p} E\{\tilde{D}(\tilde{p})\} \quad \text{min}\{D(p),x\} + \tilde{p} E\{\min\{\tilde{D}(\tilde{p}),K-x\}\}. \tag{6}
\]

This stochastic (so-called partitioned allocation) model (S) has also been used to approximate more complex, nested, or multiperiod revenue management models by Belobaba (1987) and Bertsimas and de Boer (2005). In contrast with those papers, which use model (S) as a benchmark for making allocation decisions \( x \), we will use model (S) to make pricing decisions. Unlike model (D), model (S) captures demand uncertainty in pricing decisions—in particular, the variance of both demand classes affects \( (\tilde{p}^S, p^S) \) but does not affect \( (\tilde{p}^D, p^D) \). Absent demand risk, the two models and corresponding prices coincide, so we can say that (S) adjusts deterministic prices set by (D) to account for (price-sensitive) demand uncertainty; we refer to \( p^S \) and \( \tilde{p}^S \) as risk-adjusted prices.

3.3. Sequential Pricing and Revenue Management Policies

We are now ready to introduce two pairs of sequential policies, which combine a (deterministic or stochastic) pricing model (D) or (S) with a hierarchical or partially coordinated revenue management approach based on (H) or (C), respectively. A generic sequential policy is a two-stage stochastic program for making pricing and capacity allocation decisions, as follows.

\textit{Stage 1:} Solve a generic pricing model (A), such as \( A \in \{D,S\} \), to obtain prices \( \tilde{p}^A, p^A \).

\textit{Stage 2:} Use these prices \( \tilde{p}^A, p^A \) as input for model (H) or (C) to yield the following possibilities.

- Model (HA): Solve (H) \( \max_x R(\tilde{p}^A, p^A, x) \) for the optimal allocation \( x \).

- Model (CA): Solve (C) \( \max_{p,x} R(p^A, \tilde{p}^A, x) \) for the high-end price \( p \) and allocation \( x \).


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<th><strong>DETERMINISTIC PRICING</strong></th>
<th><strong>STOCHASTIC PRICING</strong></th>
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<td><strong>COORDINATED</strong></td>
<td>(CD)</td>
<td>(CS)</td>
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<td></td>
<td>$V[CD] = \max_{x,p} R(\tilde{p}^D, p, x)$</td>
<td>$V[CS] = \max_{x,p} R(\tilde{p}^S, p, x)$</td>
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<tr>
<td><strong>HIERARCHICAL</strong></td>
<td>(HD)</td>
<td>(HS)</td>
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<td>$V[HD] = \max_{x} R(\tilde{p}^D, p^D, x)$</td>
<td>$V[HS] = \max_{x} R(\tilde{p}^S, p^S, x)$</td>
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<td></td>
<td>${p^D, \tilde{p}^D} = \arg \max(D)$</td>
<td>${p^S, \tilde{p}^S} = \arg \max(S)$</td>
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In hierarchical (H)-processes, pricing decisions are oblivious to subsequent allocation decisions. By contrast, in coordinated (C)-processes the price and allocation decisions for the high-end market are integrated—in other words, high-end prices are set by anticipating optimal booking limits.

Table 1 formalizes the four models of interest: (HD) and (HS) (resp., (CD) and (CS)) are the hierarchical (resp., coordinated) models with, respectively, deterministic and stochastic pricing. The upper and lower benchmarks are, respectively, the fully coordinated model (F) and the hierarchical model (HD) itself—a stylized reflection of business practice, where prices are set deterministically and subsequently used as input for optimizing decisions on booking limits. The other three models improve on (HD) along two dimensions: coordination (CD), capturing demand stochasticity in pricing decisions (HS), or both (CS); see Table 1.

Two main questions emerge from the design of these models, questions that reflect the broader issues raised in the Introduction. First, when are these problems tractable from an optimization standpoint? In particular, what regularity conditions need to be imposed on the demand model under price-sensitive uncertainty? Second, which of these policies performs better under what conditions, and how closely do they approximate the fully coordinated model (F)? In particular, what is the value of coordination, and how much of this value can be achieved with a hierarchical process? We address the first (tractability) question by deriving analytical results in Sections 4 and 5. We provide preliminary analytical answers to the second (performance) question in Proposition 1 and develop them numerically in Section 6.2. Let $cv^D$ and $\tilde{cv}^D$ denote the coefficients of variation of demand at deterministic prices, $D(p^D)$ and $\tilde{D}(\tilde{p}^D)$, respectively. Proofs are in the Appendix.

**Proposition 1.** For $A \in \{D, S\}$, $V[D] \geq V[F] \geq V[CA] \geq V[HA] \geq \left(1 - \frac{1}{2} \max\{cv^D, \tilde{cv}^D\}\right) V[D]$. Moreover, $V[HS] \geq V[S]$.

This result formalizes the idea that coordination improves performance whereas demand uncertainty does not. A general upper bound (e.g. Bitran and Caldentey 2003, Prop. 6) limits the value...
of coordination and shows that all models and revenues coincide as demand variability goes away.

4. Demand Model and Results for Hierarchical Processes

In this section we obtain regularity conditions for the hierarchical pricing and revenue management models (HD) and (HS) presented in Section 3 and then characterize sensitivity properties for the corresponding price and allocation decisions.

Throughout the paper, we use the following general price-sensitive stochastic demand model with distribution $F(p,v) = \mathbb{P}[D(p) \leq v]$, survival function $L(p,v) = 1 - F(p,v)$, and density $f(p,v)$:

$$D(p) = d(p,Z).$$

(7)

Here $d$ is a deterministic demand function and $Z$ is a random variable with finite mean, price-independent density function $\phi$, and distribution $\Phi$. The random variable $Z$ captures the demand risk; in empirical estimation, this can be random noise or an independent variable in a regression model. Conceptually, $Z$ can be any sales driver that is uncertain and not perfectly controlled by the firm; examples include market size, personal disposable income of the target market, brand awareness, and a reference price (see, e.g., Hanssens et al. 2001).

The riskless demand function $d(p,z)$ is decreasing in price $p$, strictly increasing in $z$, and twice differentiable in $p$ and $z$. We use the terms increasing/decreasing and positive/negative in their weak sense throughout. We assume that the pathwise (riskless) unconstrained revenue $\pi(p,z) = pd(p,z)$ is strictly concave in $p$ for any realization $z$ of $Z$ (i.e., $2d_p(p,z) + pd_{pp}(p,z) < 0$). This assumption is not necessary for all our results, but it does simplify the analysis. In particular, it ensures that the deterministic pricing model (D) has a unique solution.

For technical convenience, variables are restricted to positive compact intervals, e.g. $x \in X = [0,K]$ and $p \in P$. We also set $p_{\min} = \min P = \arg \max \{d(p,\Phi^{-1}(1-\bar{p}/p)) | p \geq \bar{p}\}$; this assumption is needed for regularity of coordinated (but not hierarchical) models and seems to be practically unrestrictive (see Section 4.3 and the Appendix). Our results apply to any subinterval of $P$ and $X$.

4.1. Structural Results for Pricing Models

Next we present structural properties for the pricing models introduced in Section 3.2. We first briefly review the well-known micro-economic results for the deterministic pricing model (D) as a base for comparison with the stochastic model (S).

Regularity conditions for (D) follow by concavity of $\pi(p,z)$ and $\bar{\pi} (\bar{p}, \bar{z})$ in $p$ and $\bar{p}$, respectively. They are also ensured if expected demands $\mu(p) = \mathbb{E}[D(p)]$ and $\bar{\mu}(\bar{p}) = \mathbb{E}[\bar{D}(\bar{p})]$ have increasing elasticity: $E(p) = -p \frac{\mu'(p)}{\mu(p)}$ and $\bar{E}(\bar{p}) = -\bar{p} \frac{\bar{\mu}'(\bar{p})}{\bar{\mu}(\bar{p})}$, respectively. Optimal prices $(\bar{p}^D, p^D)$ satisfy:

$^3$ Partial derivatives are denoted by corresponding subscripts. Low-fare demand $\bar{D}(\bar{p}) = \tilde{d}(\bar{p},\bar{Z})$ is defined similarly.
\[ \mu(p)(1 - E(p)) = \bar{\mu}(\bar{p})(1 - \bar{E}(\bar{p})) = \lambda \geq 0, \lambda(\mu(p) + \bar{\mu}(\bar{p}) - K) = 0 \text{ and } \mu(p) + \bar{\mu}(\bar{p}) \leq K. \quad (8) \]

In particular, expected demand at the optimal prices is elastic: \( E(p^D) \geq 1, \bar{E}(\bar{p}^D) \geq 1. \)

Regularity conditions for the stochastic pricing model (S) can be expressed in terms of a different concept of elasticity, one that captures the underlying demand uncertainty. Thus, the lost sales rate (LSR) elasticity corresponding to \( D(p) \) is defined as \( \mathcal{E}(p, x) = -\frac{pL(p, x)}{L(p, x)} = \frac{pF_p(p, x)}{1 - F_p(p, x)} \) (Kocabıyıköglu and Popescu 2011). This is the price elasticity of the rate of lost sales—that is, the percentage change in the lost sales rate \( L(p, x) \) with respect to the percentage change in price for a given capacity allocation \( x \). The LSR elasticity \( \mathcal{E}(\bar{p}, x) \) for the low-fare class is defined similarly.

The next proposition shows how the structural results for the deterministic model (D) extend to its stochastic counterpart (S) through the concept of LSR elasticity. In particular, the pricing problem (S) is tractable for stochastic demand models with LSR elasticity increasing in \( x \). This condition is equivalent to demand (7) being stochastically decreasing in price in the hazard rate order, a condition satisfied by most demand specifications used in the literature (see Section 4.3).

**Proposition 2.** Assume that \( \mathcal{E}(p, x) \) and \( \bar{\mathcal{E}}(\bar{p}, x) \) are increasing in \( x \) for all \( p \) and \( \bar{p} \).

(a) The stochastic pricing model (S) has a unique optimal solution \( (\bar{p}^S, p^S, x^S) \), that satisfies

\[ \int_0^u L(p, v)(1 - \mathcal{E}(p, v)) \, dv = \int_0^{K-u} \bar{L}(\bar{p}, v)(1 - \bar{\mathcal{E}}(\bar{p}, v)) \, dv = 0, \quad (9) \]

\[ pL(p, u) = \bar{p}L(\bar{p}, K - u). \quad (10) \]

(b) The optimal price for each product under model (S), keeping all other variables constant, is decreasing in its own allocation and is independent of the other product’s price.

Although the three-variable objective of model (S) is not jointly concave in general, Proposition 2 shows that it can be optimized as a concave univariate function along the optimal price paths for each segment as determined by (9). Condition (10) states that capacity should be partitioned so as to balance the marginal expected revenue per inventory unit from each segment. These conditions resemble the deterministic marginal revenue conditions (8).

The increasing LSR elasticity conditions thus extend the elasticity results for the deterministic model (D); in particular, from (9), the lost sales rate at the optimal solution is elastic, \( \mathcal{E}(p^S, x^S) \geq 1 \) and \( \bar{\mathcal{E}}(\bar{p}^S, K - x^S) \geq 1. \) The first part of Proposition 2 extends the single-product newsvendor results in Kocabıyıköglu and Popescu (2011) to the case of two products sharing a limited resource. A multiproduct extension of Proposition 2 follows along the same lines, generalizing the result obtained by Bertsimas and de Boer (2004) under an additive-multiplicative demand model.
4.2. Structural Results for Hierarchical Models

A hierarchical process uses the prices determined by models such as (D) or (S) to make capacity allocation decisions based on the revenue management model (H). We next investigate how these capacity allocation decisions should be made and how they respond to a change in prices. Suppose that, in an uncoordinated environment, the marketing department announces a price cut for the high-end segment. Should the revenue management department respond by increasing or decreasing the allocation for this segment? The answer depends on the underlying price-sensitive demand uncertainty, and it helps also to establish structural properties for coordinated models in Section 5.

For any given prices $p$ and $\bar{p}$, the objective function $R(\bar{p}, p, x)$ in (1) is quasi-concave in $x$, so the optimal protection level $x^*(\bar{p}, p)$ for (H) is uniquely characterized by Littlewood’s (1972) rule:

$$L(p, x) = \mathbb{P}[D(p) \geq x] = \bar{p}/p.$$  \hspace{1cm} (11)

Proposition 3 provides the optimal solution and sensitivity results for hierarchical models (HD) and (HS) based on results from Section 4.1 and existing comparative statics for the newsvendor with pricing problem (Kocabıyıkolu and Popescu 2011, Theorem 1(b)). We focus on sensitivity of $x^*(p) = x^*(p, \bar{p})$ to the high-end price $\bar{p}$; the optimal protection level increases in the low-fare price $\bar{p}$, regardless of price-sensitivity, so we selectively omit functional dependence on $\bar{p}$ from notation.

**Proposition 3.**

(a) Model (HD) admits a unique optimal solution $(p^D, \bar{p}^D, x^{HD} = x^*(p^D, \bar{p}^D))$ that solves (8) and (11). If $\mathcal{E}(p, x)$ and $\tilde{\mathcal{E}}(\bar{p}, x)$ are increasing in $x$, then model (HS) also admits a unique optimal solution $(p^S, \bar{p}^S, x^{HS} = x^*(p^S, \bar{p}^S))$ that solves (9), (10) and (11).

(b) The optimal protection level $x^*(p)$ is decreasing in the high-end price $p$ if and only if $\mathcal{E}^*(p) = \mathcal{E}(p, x^*(p)) \geq 1$. Moreover, the following alternative conditions are sufficient for $x^*(p)$ to be decreasing in $p$: (i) $\mathcal{E}^*(p)$ is increasing in $p$; and (ii) $\mathcal{E}(p, x)$ is increasing in $p$ for all $x$.

Part (a) shows how hierarchical models can be efficiently solved, as a system of equations, if demand is stochastically decreasing in price in hazard rate order. The optimal (HD) and (HS) policies are generally not comparable, even for additive-linear models, as illustrated in Section 6.2.

Part (b) elucidates the relationship between price and protection level for the high-end segment; intuitively, this is determined by two effects that are typically opposed. On the one hand, a price hike increases the marginal return from protecting more capacity for this class, suggesting higher protection levels. On the other hand, increasing (high-end) prices implies a lower rate of lost sales (due to decreased demand) and hence a decrease in protection levels. Whichever effect dominates will determine the direction of change in $x^*(p)$. For example, when demand is not a function of

---

4 The capacity-constrained optimal protection level is $\min\{K, x^*(p, \bar{p})\} \in X$, but this does not affect the results.
price (D(p) ≡ D), price changes have no impact on the rate of lost sales (E ≡ 0) and the protection level increases in p. This effect is reversed, however, when demand is sufficiently price sensitive—specifically, whenever the rate of lost sales is elastic with respect to changes in price (along the optimal allocation path; i.e., when \( E^*(p) \geq 1 \)). The pathwise bound on LSR elasticity fully characterizes this sensitivity result, since it is both necessary and sufficient. Verifying the bound or condition (i) requires inverting the demand distribution to obtain \( x^*(p) \). A sufficient condition which does not require calculating an inverse, is that LSR elasticity be increasing in price.

### 4.3. Implications for Modeling Demand

The results so far have shown that increasing LSR elasticity conditions, which emulate the well-known deterministic elasticity conditions for model (D), guarantee structural properties for both stochastic pricing (S) and revenue management (H) and hence also for (HD) and (HS) models. We briefly argue that these demand conditions are intuitive, easy to verify, and relatively unrestrictive.

It is natural to assume that demand is decreasing in price in a stochastic sense; \( E(p, x) \) increasing in \( x \) is precisely equivalent to \( D(p) \) decreasing in \( p \) with respect to the hazard rate order (Kocabıyıkoğlu and Popescu 2011, Prop. 3). These authors show that a broad class of demand models have increasing LSR elasticity with respect to both \( x \) and \( p \); these include additive-multiplicative and attraction models, in particular (i) additive-linear, logit, and exponential models with IFR risk \( Z \) and (ii) multiplicative isoelastic and power models with IGFR risk \( Z \).

To illustrate the assumptions underlying our results, Table 2 provides expressions for \( E(p, x) \), \( E^*(p) \), and \( p_{\text{min}} \) for the additive-linear and the multiplicative isoelastic demand models frequently used in the literature (see, e.g., Petruzzi and Dada 1999) with uniform \((0,1)\) or mean-1 exponential risk \( Z \); both distributions are IFR. It is easy to verify that \( E(p, x) \) is increasing and that \( E^*(p) \geq 1 \) whenever \( p \geq p_{\text{min}} \). Moreover, if \( b > 1/\bar{b} \) (i.e., if high-fare demand is sufficiently price sensitive), then \( p_{\text{min}} = \bar{p} \) for additive models and so the lower bound \( p_{\text{min}} \) used in Proposition 3b(i,ii) is unrestrictive.

Not all demand models take the form of (7). For example, the Poisson model with price-dependent demand rate \( \lambda(p) \), which is commonly used in revenue management (Feng and Xiao

\[ E(p, x) = \frac{b p}{x + a - b p} \]

\[ E^*(p) = b p \frac{b p}{x + a - b p} \]

\[ p_{\text{min}} = \max \left( \bar{p}, \sqrt{1/p} \right) \]

\[ p_{\text{min}} = \bar{p} \left( 1 + \frac{1}{b} \right) \]

<table>
<thead>
<tr>
<th>( d(p, Z) )</th>
<th>( E(p, x) )</th>
<th>( E^*(p) )</th>
<th>( p_{\text{min}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a - b p + Z )</td>
<td>Uniform Exponential</td>
<td>Uniform Exponential</td>
<td>Uniform Exponential</td>
</tr>
<tr>
<td>( a p - b Z )</td>
<td>Uniform Exponential</td>
<td>Uniform Exponential</td>
<td>Uniform Exponential</td>
</tr>
</tbody>
</table>

---

5 An additive-multiplicative model is \( d(p, Z) = \alpha(p) Z + \beta(p) \). By definition, \( Z \) is IFR if it has increasing failure rate \( \frac{d_z}{1 - F_z} \) and \( Z \) is IGFR if it has increasing generalized failure rate \( \frac{d_z}{1 - F_z} \). The IFR assumption, which implies IGFR, is common in the operations literature and imposes mild restrictions on the demand distribution.
2006, Gallego and van Ryzin 1994), does not fit the $d(p, Z)$ form. However, its normal approximation $D(p) = \lambda(p) + \sqrt{\lambda(p)}Z$, with $Z \sim N(0,1)$ (hence IFR), fits the additive-multiplicative form. For this model, $\mathcal{E}(p, x)$ increases in $x$; it also increases in $p$ if $\lambda(p)$ is concave (or if $p\lambda'(p)$ is decreasing).

We conclude that the hierarchical models (HD) and (HS) can be efficiently solved for a large class of demand specifications. The LSR elasticity conditions leave much freedom for statistical model selection, as we shall illustrate with data in Section 6.1.

5. Structural Results for Coordinated Models

In a centralized environment, pricing and allocation decisions are made jointly by a single unit of the firm. Alternatively, coordination can be achieved if the marketing function makes pricing decisions while considering the subsequent optimal allocation decision to be made by the revenue management system. In this section we investigate the coordinated model (C), which optimizes expected revenue $R(p, x)$ as a function of high-end price $p$ and allocation $x$; we omit for simplicity the functional dependence on the low-end price $\bar{p}$, which is kept fixed in this section. In contrast with the full recourse problem (F), which is generally nonconcave, model (C) is shown to be tractable under similar conditions as the hierarchical policies studied in Section 4.

Practical considerations endorse the relevance of managing the price and allocation decisions for the high-end segment for a given low-end price. In many revenue management settings, such as concerts and sporting events, low-end prices are kept fixed for brand image and for historical, fairness, or social considerations while high-end prices are actively managed. There are also settings—such as airlines, hotels, car rentals, and advertising—in which the low-end market is highly competitive and with little degree of pricing power relative to the high-end segment (Zhang and Kallesen 2008). In fact, the first North American revenue management initiative, the American Airlines “Ultimate Super Saver” program, was purposely designed to conditionally match low-fare competitor People Express in the low-end segment while reserving capacity for higher-margin sales. Major airlines continue to offer low-fare products on a limited basis in order to compete against low-cost carriers such as Southwest, Ryanair, and EasyJet. In the high-end market, however, airline price dispersion is extremely high (up to 700%, according to Donofrio 2002) and competition less severe, suggesting that price is an important profit lever. Motivated by these examples, we focus on jointly optimizing allocation and pricing decisions for the high-end segment in this section.

5.1. Regularity Conditions

The coordinated model (C) is generally not jointly concave in the price and allocation decision for the high-end class. This problem can be viewed, equivalently, as a pricing model with recourse: the high-end price $p$ is determined by anticipating that the protection level is optimally set in response to this price, $x = x^*(p)$, so the problem amounts to optimizing the univariate objective
\(R^*(p) = R(p, x^*(p))\). We show that this univariate objective is concave if the LSR elasticity is increasing in price or, alternatively, if it is larger than 1/2 along the optimal allocation path \(x^*(p)\).

**Proposition 4.** Suppose that one of the following conditions holds: (a) \(\mathcal{E}^*(p) \geq 1/2\) for all \(p\); (b) \(\mathcal{E}^*(p)\) is increasing in \(p\); or (c) \(\mathcal{E}(p, x)\) is increasing in \(p\) for all \(x\). Then the coordinated pricing and revenue management (C) model can be efficiently solved as a concave univariate problem and admits a unique optimal solution \((p^{**}, x^{**})\).

In short, the conditions that guaranteed sensitivity results for hierarchical models (Proposition 3b) ensure regularity of the coordinated model (C). The conditions in Proposition 4 are satisfied by most demand functions of practical interest, as we argued in Section 4.3. This result also shows that regularity conditions in the revenue management context are no stronger than those that coordinate the simpler, price-setting newsvendor problem (Kocabıyıkolu and Popescu 2011, Theorem 2). In some cases, the lower bounds of 1/2 on LSR elasticity are not only sufficient but also necessary for concavity of the revenue function. For example, if \(d\) is linear in \(p\) (i.e., if \(d(p, z) = \delta(z) - p\gamma(z)\)), then it can be shown that \(\mathcal{E}^* \geq 1/2\) is both necessary and sufficient for the concavity of \(R^*(p)\). Therefore, no weaker constant bound can be expected to hold for all demand functions.

**5.2. Extension: Substitution Effects**

The coordinated model described so far assumes that demand for each class depends on its own fare price but not on the fare price of the other class, since the market is perfectly segmented into low- and high-fare customers. Traditionally, airlines have achieved this segmentation by designing product fences (restrictions) such as booking more than 14 days prior to departure or staying over a Saturday night. These restrictions allowed airlines to charge prices up to 7 times higher for the greater flexibility offered to the business segment (Donofrio 2002). However, in other practical settings (e.g., event ticketing), where perfect segmentation is more difficult to achieve, firms offer comparable products and the demand for a product may increase with the price of a substitute.

In this section we show that our results for model (C) extend when decisions on the high-end price \(p\) also affect low-fare demand, \(\bar{D}(p) = \bar{d}(p, \bar{Z})\), where \(\bar{d}(p, z)\) is increasing in \(p\); we omit again the functional dependence on \(\bar{p}\) for notational convenience. The effect of the high-end price \(p\) on both demand classes complicates our original revenue management model as follows:

\[
\max_{p,x} \mathbb{E}_{\bar{D}} \left[ \bar{p} \min \{ \bar{D}(p), K - x \} + p \mathbb{E}_{D} \left[ \min \{ D(p), \max \{ x, K - \bar{D}(p) \} \} \right] \right].
\]  

(12)

**Proposition 5.** Assume that \(\bar{d}_{pp} \leq 0\). Then (12) has a unique price–allocation solution if either of the following conditions holds: (a) \(\mathcal{E}(p, x)\) is increasing in \(p\); or (b) \(\mathcal{E}^*(p)\) is increasing in \(p\).
This result shows that increasing LSR elasticity conditions continue to ensure structural properties even when the segmentation between classes is imperfect. The additional assumption of diminishing marginal impact of substitute high-end prices on low-fare demand holds for additive-linear demand systems $D(p) = Z - b_1 p, \bar{D}(p) = Z + b_2 p$ (e.g., Elmaghraby and Keskinocak 2003), as well as for isoelastic multiplicative models $D(p) = p - b_1 Z, \bar{D}(p) = p + b_2 \bar{Z}$, where $b_i \geq 0$ for $i = 1, 2$. For these models, $E(p, x)$ increases in $p$ if $Z$ is IGFR (see Kocabıyıkoğlu and Popescu 2011, table 2).

### 5.3. Summary and Sensitivity Results

We conclude our analytical investigation by providing sensitivity results that characterize the impact of capacity on joint pricing and allocation decisions as well as on optimal revenues.

In a sequential revenue management process, Littlewood’s rule (11) implies that, for a given price $p$, the optimal protection level is independent of capacity (or equal to it). However, this statement no longer holds when price and allocation decisions are made jointly. Our next result characterizes the effect of capacity on the optimal coordinated price–allocation solution. In particular, it confirms that optimal high-end prices decrease with capacity even as these prices are coordinated with booking limit decisions. We shall further study the effect of capacity on the (marginal) revenues of model (C), $R^{**}(K) = R(p^{**}, x^{**}; K)$ and on the revenue rate per capacity unit $R^{**}(K)/K$.

**Proposition 6.** (a) If $E(p, x)$ is increasing in $p$ and $x$, then $p^{**}(K)$ decreases with capacity $K$ and $x^{**}(K)$ increases with capacity $K$. (b) The optimal revenue $R^{**}(K)$ from the coordinated model (C) is increasing and concave in capacity $K$, whereas the optimal revenue per unit of capacity, $R^{**}(K)/K$, is decreasing in $K$.

In sum, firms that experience a freeing up or expansion of capacity should expect more revenue but lower revenue rates (e.g., lower RAS for airlines and lower REVPAR for hotels). Such firms should therefore set lower prices for the high-end segment, but at the same time increase the protection level, if LSR elasticity is increasing in price and quantity. Our numerical results in Section 6.2 suggest that these sensitivity properties for model (C) mirror those for the fully coordinated model (F), and extend to all the sequential processes described in Table 1.

To conclude, our analytical results suggest that increasing LSR elasticity is a unifying condition that enables us to solve efficiently the four pricing and revenue management models in Table 1 and also to characterize their sensitivity properties.

**Corollary 1.** Sufficient regularity conditions for the processes (HD), (HS), (CD), and (CS) are summarized in Table 3. In particular, these models can all be solved as concave univariate problems for demand models that feature increasing LSR elasticity in $p$ and $x$. 
Table 3  Regularity conditions for hierarchical and coordinated models in terms of LSR elasticity

<table>
<thead>
<tr>
<th></th>
<th>Deterministic Pricing</th>
<th>Stochastic Pricing</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Coordinated</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(CD)</td>
<td>$\mathcal{E}$ increasing in $p$</td>
<td>(CS) $\mathcal{E}$ increasing in $p$ and $x$</td>
</tr>
<tr>
<td><strong>Hierarchical</strong></td>
<td>(HD) —</td>
<td>(HS) $\mathcal{E}$ increasing in $x$</td>
</tr>
</tbody>
</table>

6. Performance Assessment: Numerical Insights

In this section we provide a numerical analysis using industry data in order to evaluate the performance of various hierarchical and coordinated models for pricing and revenue management. We quantify the benefits of coordinating decisions on pricing and allocation and of incorporating demand uncertainty into pricing. We evaluate how these benefits vary with respect to capacity levels and to such demand parameters as location, scale, and variability. We first describe the data set and then use it to fit a stochastic, price-sensitive demand model, as detailed in Section 4.

6.1. Demand Model for Car Rental Bookings

We begin by illustrating the applicability of the general class of regular demand models discussed in Section 4 to an industry data set provided by Avis Europe. The demand for car rentals has complex dynamics driven by regional and socioeconomic factors; it is also highly heterogeneous, especially across national boundaries. Because the Avis data set contains only pricing information, it is not possible to provide an exhaustive analysis of the price–demand relationship for car rentals in the absence of other factors. Instead, our aim here is simply to identify the functional form of a demand model that is appropriate for this data, and use these inputs in the numerical analysis of the performance of different policies in Section 6.2.

The booking data analyzed here consists of all car rentals at major airports in four European countries (France, Germany, Italy, and Spain) between 1 January 2008 and 31 March 2008. The data set contains the length of rental, the car group, and the price (in euros) per rental day quoted at the time of booking. We consider in the analysis only rentals by individual customers, since corporate rentals are less price sensitive and have different demand patterns. Avis categorizes cars into different groups depending on vehicle characteristics, and different price ranges are applicable to each group. Within each country, segmentation is performed by considering the bookings in the high-price and low-price car groups separately. Given our lack of inventory records, we can only assume that the Avis bookings represent the true (i.e., not truncated) demand. Although this assumption is unlikely to affect our insights, any analysis of booking data should start with demand untruncation (Queenan et al. 2007) before determining the specific impact of prices. Our subsequent results emphasize the practical importance of capturing the true demand distribution.
Figure 1  Demand versus price per day for the high-price car class in France and Germany

As an illustration, Figure 1 plots the demand in France and Germany versus price per day for the high-price vehicle group. As expected, price has a negative impact on demand. Analogous plots for Italy and Spain as well for all bookings aggregated across the four countries (not reported here) all show similar patterns. The scatter plots suggest that an additive or multiplicative model would be appropriate for capturing the relationship between price and demand. We shall therefore fit the additive-linear model $D(p) = a - bp + Z$ and the multiplicative demand model $D(p) = ap^{-b}Z$.

In order to assess the price-demand relationship, we use a two-stage analysis. In the first stage we identify and calibrate the impact of price on demand using a least-squares approach, and in the second stage we test the distributional assumptions on the stochastic residuals $Z$. If more detailed data were available (on seasonal factors, customer characteristics, market features, etc.), then these covariates would be included in the analysis under the deterministic component of the demand function in order to isolate the effect of price on demand (see, e.g., Luo and Zhang 2010).

We establish the most appropriate functional form for the relationship between price and demand by comparing the adjusted $R^2$ for the additive and multiplicative demand models; these are reported in Table 4, where the data are stratified by country. The additive specification generally provides a better fit than the multiplicative one, although the difference is marginal in some cases. The parameter estimates vary slightly across countries. The standard errors of the parameters (reported in parentheses) imply that the estimated negative impact of price on demand is highly statistically significant, which is consistent with our expectations and with Figure 1.

As a second step, we determine the best-fitting distribution for the random component $Z$ under the additive demand model by analyzing the residuals from the least-squares fit. We test four distributional assumptions: normal, exponential, gamma, and uniform. The $\chi^2$ goodness-of-fit tests

---

6 A joint fully parametric maximum likelihood estimation for price impact coefficients $a$ and $b$ and for the distributional form of $Z$ is generally not possible because the likelihood is unbounded for some residual distributions, leading to inconsistent estimators (Cheng and Amin 1983). For normal residual distributions $Z$, however, the joint estimation approach converges and is equivalent to the two-stage approach.
Table 4 Parameter estimates and goodness of fit of the additive and the multiplicative demand models for high-price car rentals by country; standard errors in parentheses

<table>
<thead>
<tr>
<th>Country</th>
<th>Additive</th>
<th></th>
<th></th>
<th>Multiplicative</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R^2$</td>
<td>$a$</td>
<td>$b$</td>
<td>$R^2$</td>
<td>log $a$</td>
<td>$b$</td>
</tr>
<tr>
<td>Germany</td>
<td>0.25</td>
<td>11.46</td>
<td>0.09</td>
<td>0.23</td>
<td>13.24</td>
<td>2.73</td>
</tr>
<tr>
<td></td>
<td>(2.44)</td>
<td>(0.03)</td>
<td>(3.74)</td>
<td>(0.82)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>France</td>
<td>0.43</td>
<td>28.74</td>
<td>0.25</td>
<td>0.41</td>
<td>20.89</td>
<td>4.33</td>
</tr>
<tr>
<td></td>
<td>(5.04)</td>
<td>(0.05)</td>
<td>(4.23)</td>
<td>(0.93)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spain</td>
<td>0.23</td>
<td>11.46</td>
<td>0.09</td>
<td>0.23</td>
<td>13.74</td>
<td>2.82</td>
</tr>
<tr>
<td></td>
<td>(2.43)</td>
<td>(0.03)</td>
<td>(3.68)</td>
<td>(0.81)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Italy</td>
<td>0.21</td>
<td>13.39</td>
<td>0.11</td>
<td>0.22</td>
<td>13.58</td>
<td>2.79</td>
</tr>
<tr>
<td></td>
<td>(3.33)</td>
<td>(0.04)</td>
<td>(4.06)</td>
<td>(0.89)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

reject the exponential, uniform, and gamma distributions with respective $p$-values of 0.00001, 0.0016, and 0.0449. The $p$-value of the test for the normal distribution is 0.6533, indicating that it is an appropriate parametric model for the residuals $Z$. Maximum likelihood estimation leads to a mean of 0 and standard deviation $\sigma = 2.45$ for the normal $Z$. A similar analysis of the low-price car group reveals that the average rental price per day ranges between €28.45 in Italy and €33.64 in France; the demand risk $\bar{Z}$ from the additive model has a normal distribution with mean 0 and standard deviation $\bar{\sigma} = 11.43$.

This analysis motivates us to focus the presentation of our numerical experiments on an additive-linear demand model: $D(p) = a - bp + Z$ and $\bar{D}(\bar{p}) = \bar{a} - \bar{b}\bar{p} + \bar{Z}$, where $Z$ and $\bar{Z}$ have independent normal distributions with mean 0 and standard deviations $\sigma$ and $\bar{\sigma}$, respectively. Under this model, LSR elasticity is increasing in both $p$ and $x$, and $\pi(p, z)$ is strictly concave in $p$. For this data set, the lower bound on price $p_{\text{min}}$ introduced in Section 4 is practically unconstraining, as illustrated in the Appendix. According to Corollary 1, all models in Table 1 can be solved efficiently and admit a unique solution. We find the optimal (F) solution via a search algorithm; preliminary analysis suggests that our demand conditions may not be sufficient for (F) to be (pathwise) quasi-concave.

### 6.2. Numerical Results: Benefits Assessment

Proposition 1 provided a partial ordering of the various pricing and revenue management policies introduced in Section 3, and a theoretical bound on the value of coordination. In this section we use numerical experiments to complement those results by quantifying the magnitude of performance gaps between various policies and thereby assessing the benefits of coordination and of modeling price-sensitive demand uncertainty.

---

7 Probability plots of the residuals under the four distributional assumptions (not reported here) are consistent with the results of the goodness-of-fit tests, confirming that the normal distribution is likely to provide the best fit.
Because it is closest to actual business practice, we consider the hierarchical model with deterministic pricing (HD) as a benchmark and shall report the performance of each policy relative to this benchmark. The relative improvement of policy \( A \) over \( B \) is denoted \( \Delta(A, B) = (V[A] - V[B])/V[B] \). In particular, we consider as performance metrics the marginal value of coordination \( \Delta(CD, HD) \) and that of stochastic pricing \( \Delta(HS, HD) \), relative to model (HD), and investigate how these measures are affected by overall demand, including its slope and variability.

For purposes of illustration, we present numerical results for specific demand models and parameters. The default parameters of the additive-linear demand model are set to \( a = 30 \) and \( b = 0.25 \) for the high-fare class and to \( \bar{a} = 80 \) and \( \bar{b} = 2.00 \) for the low-fare class. The corresponding demand risks \( Z \) and \( \bar{Z} \) have independent normal distributions with mean 0 and standard deviations \( \sigma = 2 \) and \( \bar{\sigma} = 12 \), respectively. This choice of model and parameters is motivated by our analysis of the French bookings of Avis car rentals in the previous section; among all countries, this is where the additive demand model has the highest explanatory power. Extensive numerical experiments with a wide range of parameters suggest that the insights illustrated here are robust (see Section 6.3).

### 6.2.1. The Effect of Capacity.

In revenue management, the load in the market is measured ex ante by the demand factor, which is the ratio of expected demand to capacity. But expected demand in our model is a function of selling prices, which are not determined a priori, and so the demand factor is policy specific. The results in this section are obtained by varying the capacity \( K \) via the (unconstrained) demand factor, \( \Lambda = (\mu(p^o) + \bar{\mu}(\bar{p}^o))/K = \frac{1}{2}(a + \bar{a})/K \); this definition is based on expected demand at the optimal unconstrained prices \( p^o = p^D(K = \infty) = a/2b \) and \( \bar{p}^o = \bar{p}^D(K = \infty) = \bar{a}/2\bar{b} \).\(^8\) Revenue management is most relevant when capacity is binding yet ample enough to serve both segments \( (\Lambda \in [1, 4]) \) for the fluid model; we report results for \( \Lambda \in [1, 4] \).

The left panel of Figure 2 shows how the percentage increase in revenues relative to the (HD) policy varies with capacity, as reflected in the demand factor \( \Lambda \). With stochastic pricing, the performance of the hierarchical (HS) heuristic is almost equal to its partially coordinated counterpart (CS, not reported here for brevity), and both are close to the upper bound of (F). In contrast, (CD) shows significantly smaller revenue improvements. Confirming Proposition 6, the absolute revenue per capacity unit (not reported here) decreases with capacity for all policies, as do the high-end prices (right panel of Figure 2). Finally, (HS) systematically sets nearly optimal high-end prices (low-end prices exhibit a similar pattern), which can be significantly higher or lower than those set by (HD) or (CD); this explains the superior performance of (HS) relative to these policies.

---

\(^8\) We emphasize that \( \Lambda \) is different from (and typically much larger than) the actual load that materializes in the market, which depends on the realization of demand and the firm’s pricing policy. In fact, as long as \( \Lambda \geq 1 \), the demand factor at deterministic prices is \( \Lambda(D) = (\mu(\bar{p}^D) + \bar{\mu}(\bar{p}^D))/K = 1 \), i.e. capacity is binding in the fluid model.
Table 5  Relative policy benefits (percentages) for varying demand factors: Marginal value of coordination, \( \Delta_{(CD, HD)} \) and of stochastic pricing, \( \Delta_{(HS, HD)} \); and the optimality gap, \( \Delta_{(F, HS)} \)

<table>
<thead>
<tr>
<th>Demand factor ( \Lambda )</th>
<th>Marginal value</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
<th>3.5</th>
<th>4.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta_{(CD, HD)} )</td>
<td></td>
<td>0.40</td>
<td>0.01</td>
<td>0.30</td>
<td>0.75</td>
<td>1.24</td>
<td>1.77</td>
<td>2.28</td>
</tr>
<tr>
<td>( \Delta_{(HS, HD)} )</td>
<td></td>
<td>2.19</td>
<td>0.22</td>
<td>2.50</td>
<td>5.78</td>
<td>9.47</td>
<td>13.43</td>
<td>17.66</td>
</tr>
<tr>
<td>( \Delta_{(F, HS)} )</td>
<td></td>
<td>0.00</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Relative to the benchmark hierarchical model with deterministic prices (HD), the value of full coordination \( \Delta_{(F, HD)} \) typically exceeds 2% and can reach above 10%; see Table 5. Relative to (HS), however, the value of coordination is much lower (\( \Delta_{(F, HS)} \leq 0.03\% \)), suggesting that most coordination benefits actually stem from adjusting prices (up or down) to reflect demand risk.

6.2.2. Demand Variability. We next investigate the improvement in optimal revenue with respect to overall demand variability, complementing the theoretical bound in Proposition 1. Keeping the same parameters as in the previous section (\( a = 30, b = 0.25, \bar{a} = 80, \bar{b} = 2.00, \sigma = 2, \bar{\sigma} = 12, \) and \( \Lambda = 2 \)), we scale \( \sigma \) and \( \bar{\sigma} \) proportionally by a factor \( \theta \in [0, 1] \) and then plot, in the left panel of Figure 3, the percentage increase in revenues over (HD) as a function of \( \theta \).

Figure 3 (left panel) confirms that all policies converge as demand becomes more predictable (\( \theta \to 0 \)). The relative value of full coordination increases with overall demand variability—in other words, as the system becomes more difficult to control. As before, the (HS) policy outperforms (HD) and (CD) and is close to the fully coordinated upper bound (F). As variability increases, prices set with (HD) and (CD) are increasingly distant from the (F)-optimal ones, which are closely replicated by (HS) (Figure 3, right panel). Low-end prices, not reported here, exhibit similar patterns. This disparity in prices drives the trend in the value of coordination, emphasizing the high cost of ignoring demand uncertainty when deciding on prices.
We also study the revenue impact of unilaterally increasing either high-end or low-end demand variability as measured by the coefficients of variation. We separately vary the values of the standard deviations $\sigma$ and $\bar{\sigma}$ of $Z$ and $\bar{Z}$, so that the coefficients of variation of the base demand $D(p_0)$ and $D(\bar{p}_0)$, $cv = 2\sigma/a$ and $\bar{cv} = 2\bar{\sigma}/\bar{a}$ range between 0.1 and 0.6. This corresponds to a range of $(1.5, 9.0)$ for $\sigma$ and of $(4.0, 24.0)$ for $\bar{\sigma}$. For consistency with the values in the rest of this section, when $\sigma$ varies we fix $\bar{\sigma} = 12$ and when $\bar{\sigma}$ varies we fix $\sigma = 2$.

Figure 4 plots the relative increase in revenues over the benchmark (HD) as a function of variability in the high- and respectively low-end demand, and it confirms our previous insights. The value of full coordination is greater when low-end demand becomes more variable, but it is less when high-end demand becomes more variable; that is, it increases with the differentiation between segments. The marginal benefit of capturing demand uncertainty in pricing is again prevalent and exhibits the same trend (see also Table 6). We emphasize that all measures reported here are relative; the absolute expected revenues from all policies (not reported here) decrease with variability in both demands, and with overall variability ($\theta$), because the value of information increases.

6.2.3. Demand Slope. With an additive model $D(p) = a - bp + Z$, price influences the mean of the demand distribution without affecting the variability. We investigate the impact of changes in (high or low) price sensitivity on the relative performance of these policies. First, we vary the market parameter $b$ while keeping all other model parameters constant ($a = 30$, $\bar{a} = 80$, $\bar{b} = 2$, $\sigma = 2$, $\bar{\sigma} = 12$, $\Lambda = 2$) in order to control for the effects of capacity or demand variability in the market; then we fix $b = 0.25$ and vary the low-end slope $\bar{b}$. We find that the value of coordination increases with the differentiation in price sensitivity between segments. Table 7 summarizes relevant performance metrics as functions of $b$ and $\bar{b}$, and it confirms that the (HS) heuristic—unlike its deterministic counterparts (HD) and (CD)—performs nearly as well as the fully coordinated optimum (F).
6.3. Summary and Robustness Tests

To complement our theoretical bounds in Proposition 1, the following systematic performance ordering of the models in Table 1 emerges from our numerical experiments: $V[HD] \approx V[CD] \leq V[HS] \approx V[F]$. To summarize, our numerical analysis generated the following insights.

1. The value of fully integrating pricing and revenue management is high (typically 1-10 %), relative to a common hierarchical approach with deterministic prices (HD). This value is higher when capacity is tighter, when demand is more uncertain, and/or when markets are more differentiated.

2. Most of this value comes from adjusting prices to reflect demand risk. Indeed, the performance of a fully coordinated (F) system can be nearly replicated by a hierarchical process with stochastic pricing (HS), which captures demand uncertainty when making pricing decisions.

3. The cost of ignoring demand risk when making pricing decisions is significant, and may not be effectively mitigated by improving coordination. The relative advantage of the hierarchical policy

**Table 6** Value of coordination and stochastic pricing versus high- and low-end coefficient of variation

<table>
<thead>
<tr>
<th>$cv$</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
<th>0.60</th>
<th>$cv$</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
<th>0.50</th>
<th>0.60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta(CD, HD)$</td>
<td>0.22</td>
<td>0.20</td>
<td>0.22</td>
<td>0.20</td>
<td>0.28</td>
<td>0.27</td>
<td>0.05</td>
<td>0.16</td>
<td>0.21</td>
<td>0.26</td>
<td>0.29</td>
<td>0.31</td>
<td></td>
</tr>
<tr>
<td>$\Delta(HS, HD)$</td>
<td>2.45</td>
<td>1.85</td>
<td>1.35</td>
<td>0.94</td>
<td>0.61</td>
<td>0.31</td>
<td>0.09</td>
<td>1.01</td>
<td>2.24</td>
<td>3.74</td>
<td>5.53</td>
<td>7.61</td>
<td></td>
</tr>
<tr>
<td>$\Delta(F, HS)$</td>
<td>0.01</td>
<td>0.05</td>
<td>0.10</td>
<td>0.18</td>
<td>0.27</td>
<td>0.39</td>
<td>0.06</td>
<td>0.03</td>
<td>0.02</td>
<td>0.01</td>
<td>0.01</td>
<td>0.01</td>
<td></td>
</tr>
</tbody>
</table>

**Table 7** Value of coordination and stochastic pricing versus high- and low-end demand slope

<table>
<thead>
<tr>
<th>$b$</th>
<th>0.10</th>
<th>0.15</th>
<th>0.20</th>
<th>0.25</th>
<th>0.30</th>
<th>0.35</th>
<th>$b$</th>
<th>0.70</th>
<th>1.00</th>
<th>1.30</th>
<th>1.60</th>
<th>1.90</th>
<th>2.20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta(CD, HD)$</td>
<td>0.06</td>
<td>0.13</td>
<td>0.21</td>
<td>0.30</td>
<td>0.40</td>
<td>0.49</td>
<td>0.89</td>
<td>0.60</td>
<td>0.41</td>
<td>0.30</td>
<td>0.22</td>
<td>0.13</td>
<td></td>
</tr>
<tr>
<td>$\Delta(HS, HD)$</td>
<td>1.38</td>
<td>1.87</td>
<td>2.24</td>
<td>2.50</td>
<td>2.67</td>
<td>2.78</td>
<td>2.72</td>
<td>2.84</td>
<td>2.69</td>
<td>2.50</td>
<td>2.31</td>
<td>2.13</td>
<td></td>
</tr>
<tr>
<td>$\Delta(F, HS)$</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.05</td>
<td>0.05</td>
<td>0.09</td>
<td>0.06</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
<td>0.03</td>
<td></td>
</tr>
</tbody>
</table>
(HS) over the partially coordinated policy (CD) is significant and increases with system variability. These insights emphasize that, when deciding on static prices, it is more important to capture market uncertainty than to improve the coordination with revenue management. Extensive simulations with a wide range of parameters and distribution classes indicate that these results are robust. Finally, we remark that revenue figures can have a strong influence on profitability: given the industry’s notoriously thin margins, a 1% increase in revenue could actually double profits.

An implication of our insights is that estimating the distribution of demand is important, not only for revenue management but also for pricing decisions. In particular, more research is needed to quantify how censoring or underestimating demand variability would bias prices and revenues. To conclude, we assess the sensitivity of optimal revenues to assumptions about the distribution of demand risk. Motivated by our analysis of car rental data, the numerical experiments have so far focused on the case when $Z$ and $\bar{Z}$ have normal distributions. We next study the impact on optimal revenue of incorrectly assuming a certain distribution (i.e., when another distribution fits the data better). How does the potential revenue loss from such misestimation compare with the revenue impact of using different pricing policies with the correct demand distribution? Is the robustness of the demand assumption more or less important than the approach to pricing?

To answer these questions, we focus on three distributions (normal, gamma, and uniform) for the demand risks $Z$ and $\bar{Z}$; for each of these distributions, we generate one data set of 100 demand realizations. Then, for each of the three demand data sets and each pricing policy, we compute the optimal revenues separately under the assumption that the demand risks have normal, gamma, or uniform distributions. Table 8 gives the robustness gaps, which for each case are computed as the percentage difference between the expected revenues under the assumed demand model and the true model for all pricing policies. The robustness gaps are small unless the uniform distribution is incorrectly assumed for the demand risks. Assuming a normal distribution when the true model is not normal has a small negative impact on optimal revenue across all pricing policies; in most cases, the robustness gaps for the normal distribution are less than 1%. We thus conclude that, if a normal distribution is assumed for the demand risks, then the potential violation of this assumption has less effect on optimal revenues than does the choice of pricing policy.

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9 In particular, starting with parameter values inspired by the Avis data, we investigated ranges of (10, 50) for $a$, (0.02, 0.80) for $b$, (40, 120) for $\bar{a}$, (0.40, 3.00) for $\bar{b}$, (0.4, 5.0) for $\Lambda$, (0.5, 11.0) for $\sigma$, and (2.00, 20.00) for $\bar{\sigma}$.

10 For consistency with the rest of this section, the model parameters are set at the same values ($a = 30$, $\bar{a} = 80$, $b = 0.215$, $\bar{b} = 2$, $\Lambda = 2$) and the normal distributions have mean 0 and standard deviations $\sigma = 2$ and $\bar{\sigma} = 12$. The gamma distributions have scale parameters $c = 1.30$ and $\bar{c} = 2.30$ and shape parameters $d = 1.30$ and $\bar{d} = 2.30$; and the uniform distributions are defined on the intervals $(-h, h)$ and $(-\bar{h}, \bar{h})$, where $h = 2.50$ and $\bar{h} = 6.00$. 
Table 8   Robustness tests: Percentage gaps in revenues under different true and assumed demand models

<table>
<thead>
<tr>
<th>True model</th>
<th>Pitted model</th>
<th>(F)</th>
<th>(HS)</th>
<th>(CD)</th>
<th>(HD)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>Gamma</td>
<td>1.09</td>
<td>1.13</td>
<td>1.94</td>
<td>2.04</td>
</tr>
<tr>
<td>Uniform</td>
<td>Gamma</td>
<td>3.63</td>
<td>5.63</td>
<td>21.74</td>
<td>33.11</td>
</tr>
<tr>
<td>Gamma</td>
<td>Normal</td>
<td>0.30</td>
<td>0.26</td>
<td>0.22</td>
<td>0.14</td>
</tr>
<tr>
<td>Uniform</td>
<td>Normal</td>
<td>7.50</td>
<td>7.47</td>
<td>10.10</td>
<td>10.81</td>
</tr>
<tr>
<td>Uniform</td>
<td>Gamma</td>
<td>0.58</td>
<td>0.55</td>
<td>1.08</td>
<td>1.44</td>
</tr>
<tr>
<td>Gamma</td>
<td>Gamma</td>
<td>1.05</td>
<td>1.10</td>
<td>1.32</td>
<td>1.45</td>
</tr>
</tbody>
</table>

7. Summary and Conclusions

We have investigated the value of coordinating price–allocation decisions in a framework of static, two-fare-class revenue management. In this context we considered two pairs of pricing and revenue management models that differ in their approach to pricing and degree of coordination. The first part of the paper characterized the associated demand conditions under which these models could admit unique solutions with natural sensitivity properties. The second part of the paper employed these demand models within several hierarchical and simultaneous revenue management processes in order to assess numerically the benefits of integrating the decisions on price and revenue management. Here we summarize our main findings and insights along these two dimensions.

First, from a methodological perspective, we have identified a broad class of demand models for which the hierarchical and coordinated pricing and revenue management models (HD), (HS), (CD), and (CS) can be solved efficiently as concave univariate problems. Our approach is valid if LSR elasticity is increasing in both $x$ and $p$, a condition satisfied by most demand models of practical interest. This condition also led to sensitivity results—for instance, that the optimal protection level is decreasing in price for sequential decision processes and that the joint optimal price–allocation solution is monotonic with respect to capacity.

Second, in terms of assessing benefits, our numerical experiments on data from the car rental industry demonstrated that a fully integrated pricing and revenue management system (F) can yield significant value relative to a hierarchical approach that employs deterministic pricing (HD). The value of coordination is higher when capacity is tighter or demand is more variable. Yet this value can be nearly achieved, surprisingly, by adjusting prices to capture demand uncertainty in a hierarchical process (HS). This policy can be operationalized via Proposition 3(a) as long as demand is stochastically decreasing in price in the hazard rate order (i.e., if LSR elasticity is increasing in $x$) and the policy’s performance closely replicates a fully integrated system (F), which is difficult to solve. Figure 5 illustrates the relative performance ranking of the four policies summarized in Table 1 and suggests that, for firms operating in an uncoordinated environment, stochastic pricing is more important than improving the integration of price and allocation decisions at the segment.
level. These insights underscore the importance of stochastic pricing and also (implicitly) that of modeling and estimating price-sensitive demand uncertainty.

Our model and results have several limitations. First, the analytical results presented here do not optimize the low-end price, a modeling choice motivated by the practical considerations discussed in the Introduction and in Section 4. From a technical standpoint, preliminary analysis suggests that the general demand conditions used in this paper would need to be strengthened in order for the fully coordinated model (F) to be tractable. However, our numerical experiments indicate that the practical benefits of such full coordination are negligible relative to a hierarchical process with stochastic pricing: the difference in performance between (F) and (HS) is less than 0.06%.

These results pertain to a static, two-fare-class monopolistic model. This stylized model is limited because it ignores dynamic, multidimensional, and competitive aspects of the revenue management context; whether our results extend in such settings remains to be investigated. Nevertheless, static pricing is common in practice, and supported by theoretical considerations, as discussed in the Introduction. Preliminary analysis suggests that our two-class insights extend to three or more fare classes. A fully coordinated system, which is clearly intractable in this case, is closely approximated by a (provably tractable) hierarchical policy that makes static pricing decisions based on a multiproduct extension of the stochastic model (S) and then uses those prices to make nested allocation decisions (cf. Brumelle and McGill 1993). This hierarchical process with stochastic pricing is a natural extension of our two-fare-class model (HS) and outperforms semicoordinated processes based on deterministic pricing, confirming our main insights.

In sum, our insights from this research emphasize the importance of stochastic pricing—in particular, of enhancing pricing analytics and forecasting (including uncensoring) techniques to capture price-sensitive demand uncertainty. In a static, uncoordinated environment, we propose that stochastic pricing can provide capacitated firms with a technically tractable and practically compelling alternative to coordination without affecting process flow or organizational structures.
References


To prove that the right-hand side is a linear program, so \( \hat{R}(p, \tilde{p}, x, q, \tilde{q}) = \tilde{p}[\min\{q, K-x]\} + p[\min\{q, \max\{K-\tilde{q}, x\}\}] \) (13).

Using \( Q \) and \( \tilde{Q} \) to denote (respectively) the number of units actually sold to high- and low-fare customers, so that \( \tilde{Q} = \min\{\tilde{q}, K-x\} \) and \( Q = \min\{q, \max\{K-\tilde{q}, x\}\} = \min\{k, K-\tilde{Q}\} \), we obtain

\[
\hat{R}(p, \tilde{p}, x, q, \tilde{q}) = \max_{Q, \tilde{Q}} pQ + \tilde{p}\tilde{Q}
\]

s.t. \( Q \leq q \), \( \tilde{Q} \leq \tilde{q} \),
\[
Q + \tilde{Q} \leq K,
\]
\[
\tilde{Q} \leq K-x.
\]

The right-hand side is a linear program, so \( \hat{R} \) is concave in \( q \) and \( \tilde{q} \). This allows us to apply Jensen’s inequality to show that \( \mathcal{R}[F] = \max_{p, \tilde{p}, x} \mathbb{E}[\hat{R}(p, \tilde{p}, x, D(p), \tilde{D}(\tilde{p}))] \leq \max_{p, \tilde{p}, x} \hat{R}(p, \tilde{p}, x, E[D(p)], E[\tilde{D}(\tilde{p})]) = \mathbb{V}[D] \).
The objective of the stochastic pricing model (S) can be written as
\[ V(p, \bar{p}, x) = r(p, x) + \bar{r}(\bar{p}, x), \]  
(15)
where \( r(p, x) = p\mathbb{E}[\min\{D(p), x\}] \) and \( \bar{r}(\bar{p}, x) = \bar{p}\mathbb{E}[\min\{\bar{D}(\bar{p}), K - x\}] \) are the respective expected revenues from the high and low segments. Because \( r(p, x) \) is increasing in \( x \), we obtain \( \mathbb{V}[S] = r(p^h, x^h) + \bar{r}(\bar{p}^h, x^h) \leq \mathbb{E}[r(p^h, \max\{K - \bar{D}(\bar{p}^h), x^h\}) + \bar{r}(\bar{p}^h, x^h)] = \mathbb{E}[r(p^h, \bar{p}^h, x^h)] \leq \max_x R(p^h, \bar{p}^h, x) = \mathbb{V}[HS]. \)

Finally, for the lower bound, observe that \( \mathbb{V}[HS] \geq \mathbb{V}[S] \geq \max_x \mathbb{V}(p^D, \bar{p}^D, x) \), which also bounds \( \mathbb{V}[HD] = \max_x R(p^D, \bar{p}^D, x) \geq \max_x \mathbb{V}(p^D, \bar{p}^D, x) \). Moreover, \( \max_x \mathbb{V}(p^D, \bar{p}^D, x) \geq r(p^D, \mu(p^D)) + \bar{r}(\bar{p}^D, \mu(p^D)) \geq r(p^D, \mu(p^D)) + r(p^D, K - \bar{p}(\bar{p}^D)) \geq p^D \mu(p^D)(1 - \frac{c_D}{2}) + \bar{p}^D \mu(\bar{p}^D)(1 - \frac{c_D}{2}) \geq (1 - \frac{1}{2} \max\{c_D, c_D\}) \mathbb{V}[D] \), where the third inequality follows e.g. from Bitran and Caldentey (2003, Prop. 6) for each segment. □

Proofs of the Results in Section 4

The proofs in this section rely on results for the price-setting newsvendor problem with objective \( \Pi(p, x) = \Pi(p, x; c, D) = p\mathbb{E}[\min\{D(p), x\}] - cx = r(p, x) - cx. \) The newsvendor-optimal price path and quantity path are defined as \( p^N(x) = \arg \max_{p \in \mathbb{N}} \Pi(p, x) \) and \( x^N(p) = \arg \max_{x \in \mathbb{X}} \Pi(p, x) \), respectively. The next lemma combines results from Theorems 1 and 2 of Kocabıyıkolu and Popescu (2011) under our demand model and assumptions. These results are used to obtain structural results for the more complex models (S) and (H).

Lemma 1. (Kocabıyıkolu and Popescu 2011) (a) If \( \mathcal{E}(p, x) \) is increasing in \( x \), then \( \Pi(p^N(x), x) \) is concave in \( x \) and \( p^N(x) \) is decreasing in \( x \). The latter holds if and only if \( \mathbb{E}(p^N(x), x) \geq 1 \). (b) If \( \mathcal{E}(p, x) \) is increasing in \( p \), then \( \Pi(p, x^N(p)) \) is concave in \( p \) and \( x^N(p) \) is decreasing in \( x \). The latter holds if and only if \( \mathcal{E}(p, x^N(p)) \geq 1 \). □

Proof of Proposition 2. Use (15) to define the optimal price paths for each product as \( p^F(x) = \arg \max_x \mathbb{V}(p, \bar{p}, x) = \arg \max_x r(p, x) \) and \( \bar{p}^F(x) = \arg \max_x \mathbb{V}(\bar{p}, p, x) = \arg \max_x \bar{r}(\bar{p}, x). \) In particular, these are independent of the price of the other product, proving the last part of (b). The first-order conditions can be written as \( r_p(p, x) = \bar{r}_{\bar{p}}(p, x) = 0 \) and \( r_x(p, x) = \bar{r}_x(p, x) \). Specifically, (9) and (10) follow by writing \( r(p, x) = p\mathbb{E}[\min\{D(p, x)\}] = p \int_0^x L(p, v) \, dv \), so \( r_x(p, x) = pL(p, x) \), and \( r_{\bar{p}}(p, x) = \int_0^x L(p, v) + pL_{\bar{p}}(p, v) \, dv = \int_0^x L(p, v)(1 - \mathcal{E}(p, v)) \, dv \). Analogous reasoning applies to \( \bar{p} \).

We next argue that \( r(p^F(x), x) \) and \( \bar{r}(\bar{p}^F(x), x) \) are both concave in \( x \) under the assumptions of the proposition, implying that \( V(p^F(x), \bar{p}^F(x), x) = r(p^F(x), x) + \bar{r}(\bar{p}^F(x), x) \) is concave in \( x \). Indeed, both \( r(p, x) \) and \( \bar{r}(\bar{p}, x) \) can be viewed as newsvendor pricing problems with zero cost. Formally, \( r(p, x) = \Pi(p, x; c = 0) \) and \( \bar{p}^F(x) = p^N(x) \). Hence, by Lemma 1(a), \( r(p^F(x), x) = \Pi(p^F(x), x; c = 0) \) is concave in \( x \) if \( \mathcal{E}(p, x) \) is increasing in \( x \), and similarly for \( \bar{r}(\bar{p}, x) = \Pi(\bar{p}, K - x; c = 0) \); this proves part (a). Finally, Lemma 1(a) also implies that \( p^N(x) = p^N(x) \) is decreasing in \( x \) and that \( \bar{p}^N(x) = \bar{p}^N(K - x) \) is increasing in \( x \), which proves part (b). □

Proof of Proposition 3. Part (a) follows by combining the results in Section 4.1, in particular Proposition 2, and Littlewood’s rule (11). The latter implies that the optimal protection level of problem (H) coincides with the optimal quantity decision of a (price-sensitive) newsvendor facing unit cost \( c = \bar{p} \) because both satisfy the same critical fractile condition (even though their objectives are quite different). Formally, \( x^*(p) = x^N(p, c = \bar{p}) \). By Lemma 1(b), this decreases in \( p \) whenever \( 1 \leq \mathcal{E}(p, x^N(p)) = \mathcal{E}(p, x^*(p)) = \mathcal{E}^*(p) \) and, in particular, if \( \mathcal{E}(p, x) \) is increasing in \( p \). Finally, condition (b)(ii) is sufficient because \( \mathcal{E}^*(p) \) increases by the definition of \( p_{\text{min}} \), so \( \mathcal{E}^*(p) \) increasing implies that \( \mathcal{E}^*(p) \geq 1 \) on \( P \). For an alternative, direct proof that computes the marginal effect of price on allocation, see equation (20) to follow. □
Proofs of the Results in Section 5

To simplify notation, in all remaining proofs we set \( \bar{p} = 1 \), so \( p \) can be interpreted as the percentage markup over \( \bar{p} \) and (11) becomes \( L^*(p) = L(p, x^*(p)) = 1/p; \) this is without loss of generality. Define \( z(p,x) \) so that \( d(p, z(p,x)) = x; \) the inverse is unique because \( d(p, z) \) is monotone in \( z \). Denote \( \Omega(x) = \bar{\Omega}(p, x) = (D(p) \leq x) = (Z \leq z(p,x)) \) and \( \bar{\Omega} = \bar{\Omega}(x) = (\bar{D} \leq \bar{K} - x) = (K \geq x) \) the events that high- and low-fare demand fall short of their allocations; here \( \bar{K} = K - \bar{D} \) is the uncertain excess capacity after all low-fare demand has been served.

**Proof of Proposition 4.** We write \( R(p) = \mathbb{E}[\min\{\bar{D}, K - x\}] + \mathbb{E}[r(p, \text{max}\{x, \bar{K}\})] \), which is concave in \( p \) because so is \( r(p,x) = p\mathbb{E}[\min\{\bar{D}(p), x\}] = \mathbb{E}[\min\{\pi(p, z), px\}] \), by concavity of \( \pi \) (the revenue from the low-fare class is independent of \( p \)). By the envelope theorem we have

\[
\frac{\partial R^*_p}{\partial p} = R^*_p(p) = R_p(p,x)_{x=x^*_p(p)} = \mathbb{E}[r_p(p, \text{max}\{x^*(p), \bar{K}\})] = L^*(p)r^*_p(p) + \mathbb{E}[r_p(p, \bar{K}); \bar{\Omega}^*(p)];
\]

(16)

here the second term is the marginal revenue in the event \( \bar{\Omega}^* = \bar{\Omega}^*(p) = (\bar{D} \leq K - x^*(p)) = (\bar{K} \geq x^*(p)) \) that low-fare demand is not constrained by the optimal booking limit, and \( L^*(p) = L(K - x^*(p)) = 1 - \mathbb{P}[\bar{\Omega}^*] \) is the lost sales rate for the low-fare class. If we put \( \bar{f}^*(p) = \bar{f}(K - x^*(p)) \), then the derivative of (16) is

\[
\frac{\partial R^*_p}{\partial p} = \frac{\partial L^*(p)}{\partial p} r^*_p(p) + \frac{\partial \Omega^*_p}{\partial p} + \mathbb{E}[r_p(p, \bar{K}); \bar{\Omega}^*] - r^*_p(p)\bar{f}^*(p) \frac{\partial x^*(p)}{\partial p}.
\]

(17)

We want to show that equation (17) is negative. Because \( \frac{\partial x^*_p}{\partial p} = \bar{f}^*(p)\frac{\partial x^*_p}{\partial \bar{p}} \), the first and last terms cancel. The third term (and hence \( r \)) is negative by concavity of \( \pi \), as argued previously. We next show that the second term is also negative. Evaluated at \( x^*(p) \), the derivative of \( r(p, x) = pxL(p, x) + \mathbb{E}[\pi(p, z); \Omega] \) is

\[
r^*_p(p) = r_p(p, x^*(p)) = \mathbb{E}[\pi_p(p, z); \Omega^*] + x^*(p)L^*(p) = \mathbb{E}[\pi_p(p, z); \Omega^*] + \frac{x^*(p)}{p},
\]

(18)

where \( \Omega^* = \Omega^*(p) = (D(p) \leq x^*(p)) \) and the second equality holds because \( L^*(p) = 1/p \). We obtain:

\[
\frac{\partial}{\partial p} r^*_p(p) = \mathbb{E}[\pi_p(p, z); \Omega^*] - \pi^*_p(p) \frac{\partial}{\partial p} L^*(p) + \frac{\partial}{\partial p} \left( \frac{x^*(p)}{p} \right) = \mathbb{E}[\pi_p(p, z); \Omega^*] + \left[ \frac{\pi^*_p(p)}{p^2} - \frac{x^*(p)}{p^2} + \frac{1}{p} \frac{\partial x^*(p)}{\partial p} \right].
\]

(19)

The first term is negative because \( \pi \) is concave. Since \( \pi^*_p(p) = pd^*_p(p) + d^*(p) \) and \( d^*(p) = d(p, z, p, x^*(p)) = x^*(p) \), the term in brackets equals \( d^*_p(p) + \frac{\partial x^*_p}{\partial p} \). This value is negative if \( x^*(p) \) is decreasing—in particular, under the conditions of Proposition 3(b)—which proves parts (b) and (c). To prove part (a), we argue that

\[
\frac{\partial x^*_p}{\partial p} = -\frac{L^*_p(p)}{L^*_p(p)} \left( 1 + \frac{1}{p^2 L^*_p(p)} \right) = -\frac{L^*_p(p)}{L^*_p(p)} \left( 1 - \frac{1}{x^*(p)} \right) = d^*_p(p) \left( 1 - \frac{1}{x^*(p)} \right).
\]

(20)

Indeed, the first equality obtains from differentiating both sides of \( L(p, x^*(p)) = 1/p \) with respect to \( p \). The second equality is derived from \( x^*(p) = -pL^*_p(p)/L^*_p(p) = -p^2 L^*_p(p) \) by (11). Finally, the last equality follows by differentiating \( L(p, d(p, z)) = 1 - \Phi(z) \) with respect to \( p \) to obtain \( L_p/L_z = -d_p \). This proves (20) and so provides an alternative, direct proof of Proposition (3b). It also implies that \( d^*_p(p) + \frac{\partial x^*_p}{\partial p} = d^*_p(p) \left( 2 - 1/x^*(p) \right) \leq 0 \) whenever \( x^*(p) \geq 1/2 \), which completes the proof.

**Proof of Proposition 5.** Because here \( D(p) \) is a function of \( p \), the expressions involving low-fare demand are different from those in the previous proofs. By abuse of notation, we keep the same letters for the same concepts even though their mathematical expressions are changed. We use \( R \) to denote the more complex objective in (12); we use \( \bar{r}(p, x) = \bar{r}(p, x; K) = \mathbb{E}[\min\{K - x, \bar{D}(p)\}] \) for the low-class expected revenue,
\( \pi(p, z) = \tilde{p}(p, z) \) for the pathwise revenue, \( \hat{L}(p, x) = \mathbb{E}[\mathcal{D}(p) \geq K - x] \) for the lost sales rate, \( \mathcal{K}(p) = K - \mathcal{D}(p) \), and so on. We next show that \( R^*(p) = R(p, x^*(p)) \) is concave in \( p \).

By the envelope theorem, we have
\[
\frac{\partial R^*(p)}{\partial p} = R_p^*(p) = r_p^*(p) + L^*(p)r_p^*(p) + \mathbb{E}[r_p(p, \mathcal{K}(p))]; \mathfrak{F}^*].
\]
The first term is the marginal revenue from the low-fare class, and the other two terms give the marginal revenues from the high-fare class depending on whether low-fare demand does or does not exceed the optimal booking limit; the latter event is \( \mathfrak{F}^* = \mathfrak{F}^*(p) = (\mathcal{D}(p) \leq K - x^*(p)) \). The derivative of each term is:
\[
\frac{\partial r_p^*(p)}{\partial p} = \frac{\partial}{\partial p} \mathbb{E}[\tilde{d}_p(p, z); \mathfrak{F}^*] = \mathbb{E}[\tilde{d}_{pp}(p, z); \mathfrak{F}^*] - \tilde{d}_p^*(p)f^*(p)\frac{\partial x^*(p)}{\partial p};
\]
\[
\frac{\partial}{\partial p} L^*(p)r_p^*(p) = L^*(p)\frac{\partial r_p^*(p)}{\partial p} + \frac{\partial L^*(p)}{\partial p}r_p^*(p) = L^*(p)\frac{\partial r_p^*(p)}{\partial p} + \hat{f}^*(p)\frac{\partial x^*(p)}{\partial p} r_p^*(p). \tag{22}
\]
We write the third term of \( \frac{\partial R^*(p)}{\partial p} \) as \( \mathbb{E}[r_p(p, \mathcal{K}(p)); \mathfrak{F}^*] = A + B \), where \( A = \mathbb{E}[\pi_p(p, z); \mathcal{D}(p) \leq \mathcal{K}(p); \mathfrak{F}^*] \) and \( B = \mathbb{E}[(K - \pi_p(p, z))L(p, \mathcal{K}(p)); \mathfrak{F}^*] \). Here \( A \) corresponds to the unconstrained case in which there is excess capacity and the booking limit is nonbinding; \( B \) is the same except parameter \( K \) is binding. The respective derivatives of these terms are (we omit some functional arguments for readability):
\[
A_p = \mathbb{E}[\pi_{pp}; \mathcal{D}(p) \leq \mathcal{K}(p); \mathfrak{F}^*] = \mathbb{E}[(K - d + pd_p)(d_p + \hat{d}_p)f; \mathfrak{F}^*] - \mathbb{E}[\pi_p; \mathfrak{F}^*] f^*(p)\frac{\partial x^*(p)}{\partial p};
\]
\[
B_p = -\mathbb{E}[\pi_{pp}L(p, \mathcal{K}(p)); \mathfrak{F}^*] = \mathbb{E}[(K - d + pd_p)(d_p + \hat{d}_p)f; \mathfrak{F}^*] - (x^*(p) - pd_p^*)L^*(p)f^*(p)\frac{\partial x^*(p)}{\partial p}.
\]
Combining the second terms of each expression, regrouping the last terms, and using (18) we obtain:
\[
\frac{\partial}{\partial p} \mathbb{E}[r_p(p, \mathcal{K}(p)); \mathfrak{F}^*] = \mathbb{E}[\pi_{pp}; \mathcal{D}(p) \leq \mathcal{K}(p); \mathfrak{F}^*] - \mathbb{E}[\pi_{pp}L(p, \mathcal{K}(p)); \mathfrak{F}^*] - p\mathbb{E}[(d_p + \hat{d}_p)^2f; \mathfrak{F}^*] - \hat{f}^*(p)r_p^*(p)\frac{\partial x^*(p)}{\partial p} + f^*(p)\hat{d}_p(p)L^*(p)\frac{\partial x^*(p)}{\partial p}. \tag{23}
\]
If we combine (21), (22), and (23), then the last term of (22) and the fourth term of (23) cancel; the last terms of (21) and (23) also cancel because \( pL^*(p) = 1 \). Furthermore, we can use (19) to obtain
\[
\frac{\partial R^*_p(p)}{\partial p} = \mathbb{E}[\pi_{pp}; \mathcal{D}(p) \leq \mathcal{K}(p); \mathfrak{F}^*] - 2\mathbb{E}[\tilde{d}_pL(p, \mathcal{K}(p)); \mathfrak{F}^*] - \mathbb{E}[\tilde{d}_{pp}(pL(p, \mathcal{K}(p) - 1)); \mathfrak{F}^*] - \mathbb{E}[p(d_p + \hat{d}_p)^2f; \mathfrak{F}^*] - L^*(p)\frac{\partial r_p^*(p)}{\partial p}. \tag{24}
\]
The second and third terms are obtained by writing \( \tilde{p}_{pp} = 2\tilde{d}_p + pd_{pp} \). The first term is negative by concavity of \( \pi(p, z) \) and the second because \( d_p(p, z) \geq 0 \). Negativity of the third term follows because \( d_{pp}(p, z) \leq 0 \) and \( L(p, \mathcal{K}(p)) \leq L^*(p) = 1/p \) on \( \mathfrak{F}^* = (\mathcal{K}(p) \geq x^*(p)) \). The fourth term is obviously negative. From the proof of Proposition 4, the last term (which is unchanged because it does not involve \( \mathcal{D} \)) is also negative under the assumptions of this proposition. It follows that \( R^*_p(p) \) is decreasing, so \( R^*(p) \) is concave in \( p \). \hfill \Box

**Proof of Proposition 6.** (a) For the first part, by definition we have \( p^*(K) = \text{arg max}_p R(p, x^*(p; K); K) = \text{arg max}_p R^*(p; K) \). By the envelope theorem, \( \frac{\partial R^*(p; K)}{\partial p} = R_p^*(p, x^*(p; K); K) \). From (11) it follows that the optimal protection level for a given price, \( x^*(p; K) \equiv x^*(p) \), is independent of capacity. This allows us to write
\[
\frac{\partial}{\partial K} \frac{\partial R^*(p; K)}{\partial p} = \frac{\partial}{\partial K} R_p^*(p, x^*(p; K); K) = R_{pK}(p, x^*(p); K).
\]
Furthermore,
\[
R_{pK}(p, x; K) = \mathbb{E}[r_{pK}(p, \mathcal{K}); \mathfrak{F}] = \mathbb{E}[(1 - \mathcal{E}(p, \mathcal{K})L(p, \mathcal{K}); \mathfrak{F}] \leq (1 - \mathcal{E}(p, x))\mathbb{E}[L(p, \mathcal{K}); \mathfrak{F}]
\]
and its derivative (after canceling boundary terms) is
The first term is equal to 0 because
The last term of (25) can be written as
The derivative of the high-fare revenue with respect to
Section 6.1) and models. Recall that the linear model implies
Numerical assessment of the lower bound
amounts to a markup of approximately 1.4 on the low-fare price; for \( Z \) with a gamma distribution we have

because \( \mathcal{E}(p, x) \) is increasing in \( x \) and \( \bar{\Omega} = (\bar{K} \geq x) \). We thus obtain
For the second part, \( x^{**}(K) = x^{*}(p^{**}(K); K) = x^{*}(p^{**}(K)) \) by (11); that is, the optimal protection level for a given price is affected by changes in \( K \) only through \( p^{**}(K) \). The result follows since \( p^{**}(K) \) is decreasing in \( K \) (from part (a)) and since \( x^{*}(p) \) is decreasing in \( p \) if \( \mathcal{E}(p, x) \) is increasing in \( p \) (by Proposition 3).

(b) We show by a sample path argument that \( R(p, x; K) \), and hence \( R^{**}(K) \), is increasing and concave in \( K \). It is sufficient to show these properties for each sample path revenue for a given policy and demand realization—in other words, for \( \hat{R}(p, \hat{p}, x, q, \hat{q}) \) as defined in (13). Based on (14), this can be formulated as a linear program parametrized by \( K \), so it is increasing and concave in \( K \).

For the second part, we show that \( \frac{\partial}{\partial K} \left( \frac{R^{**}(K)}{K} \right) \leq 0 \). We write:

\[
R(K) = \hat{r}(x; K) + \mathbb{E}[r(p, \max(x, K - \bar{D})]] = \hat{r}(x; K) + \hat{L}(K-x)r(p, x) + \mathbb{E} \left[ r(p, K - \bar{D}); \bar{\Omega} \right].
\]

(25)
The derivative of the low-fare revenue \( \hat{r}(x; K) = \mathbb{E} \left[ \min(\bar{D}, K-x) \right] = \int_0^{K-x} L(v) \text{dv} \) is \( \frac{\partial \hat{r}(x; K)}{\partial K} = \hat{L}(K-x) \).
The derivative of the high-fare revenue with respect to \( K \) is \( p\mathbb{E} \left[ L(p, K - \bar{D}); \bar{\Omega} \right], \) as we show next.

Indeed, the derivative of the second term of equation (25) is \( \frac{\partial}{\partial K} (\hat{L}(K-x)r(p, x)) = -\hat{f}(K-x)r(p, x). \)
The last term of (25) can be written as \( \mathbb{E} \left[ r(p, \bar{K}) \right] = \mathbb{E} \left[ p(K - \bar{D})L(p, \bar{K}); \bar{\Omega} \right] + \mathbb{E} \left[ r(p, \bar{Z}); D(p) \leq \bar{K}; \bar{\Omega} \right], \) and its derivative (after canceling boundary terms) is \( \frac{\partial}{\partial K} \mathbb{E}[r(p, \bar{K})\bar{\Omega}] = \mathbb{E} \left[ pL(p, \bar{K}); \bar{\Omega} \right] + \hat{f}(K-x)r(p, x). \)
We obtain \( R_K(K) = \hat{L}(K-x) + \mathbb{E} \left[ pL(p, K); \bar{\Omega} \right]. \) From the envelope theorem, \( K R_K^*(K) - R^{**}(K) = -\hat{L}(K-x)pL(p, x-1) - \hat{L}(K-x)\mathbb{E} \left[ r(p, Z); \bar{\Omega} \right] + \mathbb{E} \left[ (pL(p, \bar{K}) - 1) \bar{D}; \bar{\Omega} \right] - \mathbb{E} \left[ r(p, Z); D(p) \leq \bar{K}; \bar{\Omega} \right] \mid_{x = x^{**}, p = p^{**}}. \)
The first term is equal to 0 because \( p^{**}L(p^{**}, x^{**}) = 1 \), and negativity of the third term follows because \( L(p^{**}, \bar{K}) \leq 1/p^{**} \) on \( \bar{\Omega}^{**} = (\bar{K} \geq x^{**}). \) Hence \( K R_K^*(K) - R^{**}(K) \leq 0, \) concluding the proof. \( \square \)

### Numerical assessment of the lower bound \( p_{\text{min}} \)

The regularity conditions derived for partially coordinated models (CS) and (CD) rely on a lower bound \( p_{\text{min}} \) on high-end prices, defined in Section 4. We conclude by evaluating \( p_{\text{min}} \) for the three demand models presented in Section 6.3 and for a wider range of low-end prices \( \bar{p} \) than would be suggested by the data (cf. Section 6.1) and models. Recall that the linear model implies \( \bar{p} \leq \bar{a}/\bar{b} = 40. \) The values for \( p_{\text{min}} \) reported in Table 10 are consistently close to the low-end prices \( \bar{p} \), which suggests that the technical assumption \( p \geq p_{\text{min}} \) for optimizing the (C) model is practically unrestrictive. The largest difference for normally distributed \( Z \) amounts to a markup of approximately 1.4 on the low-fare price; for \( Z \) with a gamma distribution we have \( p_{\text{min}} = \bar{p} \), which (in effect) imposes no additional constraints on the high-fare price.

<table>
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<th>( Z ) distribution</th>
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