An Intertemporal CAPM with Stochastic Volatility

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Abstract

This paper extends the approximate closed-form intertemporal capital asset pricing model of Campbell (1993) to allow for stochastic volatility. The return on the aggregate stock market is modelled as one element of a vector autoregressive (VAR) system, and the volatility of all shocks to the VAR is another element of the system. The paper presents evidence that growth stocks underperform value stocks because they hedge two types of deterioration in investment opportunities: declining expected stock returns, and increasing volatility.

JEL classification: G12, N22

1 Introduction

The fundamental insight of intertemporal asset pricing theory is that long-term investors should care just as much about the returns they earn on their invested wealth as about the level of that wealth. In a simple model with a constant rate of return, for example, the sustainable level of consumption is the return on wealth multiplied by the level of wealth, and both terms in this product are equally important. In a more realistic model with time-varying investment opportunities, conservative long-term investors will seek to hold "intertemporal hedges", assets that perform well when investment opportunities deteriorate. Such assets should deliver lower average returns in equilibrium if they are priced from conservative long-term investors' first-order conditions.

Since the seminal work of Merton (1973) on the intertemporal capital asset pricing model (ICAPM), a large empirical literature has explored the relevance of intertemporal considerations for the pricing of financial assets in general, and the cross-sectional pricing of stocks in particular. One strand of this literature uses the approximate accounting identity of Campbell and Shiller (1988a) and the assumption that a representative investor has Epstein-Zin utility (Epstein and Zin 1989) to obtain approximate closed-form solutions for the ICAPM's risk prices (Campbell 1993). These solutions can be implemented empirically if they are combined with vector autoregressive (VAR) estimates of asset return dynamics (Campbell 1996). Campbell and Vuolteenaho (2004) and Campbell, Polk, and Vuolteenaho (2010) use this approach to argue that value stocks outperform growth stocks on average because growth stocks do well when the expected return on the aggregate stock market declines; in other words, growth stocks have low risk premia because they are intertemporal hedges for long-term investors.

A weakness of the papers cited above is that they ignore time-variation in the volatility of stock returns. In general, investment opportunities may deteriorate either because expected stock returns decline or because the volatility of stock returns increases, and it is an empirical question which of these two types of intertemporal risk have a greater effect on asset returns. We address this weakness in this paper by extending the approximate closed-form ICAPM to allow for stochastic volatility. The resulting model explains risk premia in the stock market using three priced risk factors corresponding to three important attributes of aggregate market returns: revisions in expected future cash flows, discount rates, and volatility. An attractive characteristic of the model is that the prices of these three risk factors depend on only one free parameter, the long-horizon investor's coefficient of risk aversion.

Our work can be regarded as complementary to recent research on the "long-run risk model" of asset prices (Bansal and Yaron 2004). Both the approximate closed-form ICAPM and the long-run risk model start with the first-order conditions of an infinitely lived Epstein-Zin representative investor. As originally stated by Epstein and Zin (1989), these first-order conditions involve both aggregate consumption growth and the return on the market portfolio of aggregate wealth. Campbell (1993) pointed out that the intertemporal budget constraint could be used to substitute out consumption growth, turning the model into a Merton-style ICAPM. Restoy and Weil (1998, 2011) used the same logic to substitute out the market portfolio return, turning the model into a generalized consumption CAPM in the style of Breeden (1979). Bansal and Yaron (2004) added stochastic volatility to the Restoy-Weil model, and subsequent research on the long-run risk model has increasingly emphasized the importance of stochastic volatility for generating empirically plausible implications from this model (Bansal, Kiku, and Yaron 2011, Beeler and Campbell 2011). In this paper we give the approximate closed-form ICAPM the same capability to handle stochastic volatility that its cousin, the long-run risk model, already possesses.

One might ask whether there is any reason to work with an ICAPM rather than a consumption-based model given that these models are derived from the same set of assumptions. The ICAPM developed in this paper has several advantages. First, it describes risks as they appear to an investor who takes asset prices as given and chooses consumption to satisfy his budget constraint. This is the way risks appear to individual agents in the economy, and it seems important for economists to understand risks in the same way that market participants do rather than having an exclusively macroeconomic perspective. Second, the ICAPM allows an empirical analysis based on financial proxies for the aggregate market portfolio rather than on accurate measurement of aggregate consumption. While there are certainly challenges to the accurate measurement of financial wealth, financial time series are generally available on a more timely basis and over longer sample periods than consumption series. Third, the ICAPM in this paper is flexible enough to allow multiple state variables that can be estimated in a VAR system; it does not require low-dimensional calibration of the sort used in the long-run risk literature. Finally, the stochastic volatility process used here governs the volatility of all state variables, including itself. We show that this assumption fits financial data reasonably well, and it guarantees that stochastic volatility would always remain positive in a continuous-time version of the model, a property that does not hold in current implementations of the long-run risk model.

The closest precursors to our work are unpublished papers by Chen (2003) and Sohn (2010). Both papers explore the effects of stochastic volatility on asset prices in an ICAPM setting but make strong assumptions about the covariance structure of various news terms when deriving their pricing equations. Chen (2003) assumes constant covariances between shocks to the market return (and powers of those shocks) and news about future expected market return variance. Sohn (2010) makes two strong assumptions about asset returns and consumption growth, specifically that all assets have zero covariance with news about future consumption growth volatility and that the conditional contemporaneous correlation between the market return and consumption growth is constant through time. Duffee (2005) presents evidence against the latter assumption. It is in any case unattractive to make assumptions about consumption growth in an ICAPM that does not require accurate measurement of consumption.

Chen estimates a VAR with a GARCH model to allow for time variation in the volatility of return shocks, restricting market volatility to depend only on its past realizations and not those of the other state variables. His empirical analysis has little success in explaining the cross-section of stock returns. Sohn uses a similar but more sophisticated GARCH model for market volatility and tests how well short-run and long-run risk components from the GARCH estimation can explain the returns of various stock portfolios, comparing the results to factors previously shown to be empirically successful. In contrast, our paper incorporates the volatility process directly in the ICAPM, allowing heteroskedasticity to affect and to be predicted by all state variables, and showing how the price of volatility risk is pinned down by the time-series structure of the model along with the investor's coefficient of risk aversion.

Stochastic volatility has, of course, been explored in other branches of the finance literature. For example, Chacko and Viceira (2005) and Liu (2007) show how stochastic volatility affects the optimal portfolio choice of long-term investors. Chacko and Viceira argue that movements in volatility are not persistent enough to generate large intertemporal hedging demands. Coval and Shumway (2001), Ang, Hodrick, Xing, and Zhang (2006), and Adrian and Rosenberg (2008) present evidence that shocks to market volatility are priced risk factors in the cross-section of stock returns, but they do not develop any theory to explain the risk prices for these factors. There is also a large literature in financial econometrics describing how to use realized volatility from high-frequency data to estimate stochastic volatility processes (Barndorff-Nielsen and Shephard 2002, Andersen, Bollerslev, Diebold, and Labys 2003). We follow this literature by including a measure of realized volatility in our VAR system.

The empirical implementation of our model is a success. We find that growth stocks have low average returns because they outperform not only when the expected stock return declines, but also when stock market volatility increases. Thus growth stocks hedge two types of deterioration in investment opportunities, not just one. In the period since 1963 that creates the greatest empirical difficulties for the standard CAPM, we find that the three-beta model explains over 63% percent of the cross-sectional variation in average returns of 25 portfolios sorted by size and book-to-market ratios. The model is not rejected at the 5% level while the CAPM is strongly rejected. The implied coefficient of relative risk aversion is an economically reasonable 7.21, in contrast to the much larger estimate of risk aversion (28.75) in Campbell and Vuolteenaho's (2004) two-beta ICAPM. This success is due in large part to the inclusion of volatility betas in the specification. In particular, the spread in volatility betas in the cross section generates an annualized spread in average returns of 5.17% compared to a comparable spread of 2.75% and 2.20% for cash-flow and discount-rate betas.

The organization of our paper is as follows. Section 2 lays out the approximate closed-form ICAPM and shows how to extend it to incorporate stochastic volatility. Section 3 presents data, econometrics, and VAR estimates of the dynamic process for stock returns and realized volatility. Section 4 turns to cross-sectional asset pricing and estimates a representative investor's preference parameters to fit the cross-section, taking the dynamics of stock returns as given. Section 5 concludes.

2 An Intertemporal Model with Stochastic Volatility

2.1 Asset Pricing with Time Varying Risk

Preferences

We begin by assuming a representative agent with Epstein-Zin preferences. We write the value function as

$$V_t = \left[(1-\delta) C_t^{\frac{1-\gamma}{\theta}} + \delta \left(\mathrm{E}_t \left[V_{t+1}^{1-\gamma} \right] \right)^{1/\theta} \right]^{\frac{\theta}{1-\gamma}}, \tag{1}$$

where C_t is consumption and the preference parameters are the discount factor δ, risk aversion γ, and the elasticity of intertemporal substitution ψ. For convenience, we define $\theta = (1-\gamma)/(1-1/\psi)$.

The corresponding stochastic discount factor (SDF) can be written as

$$M_{t+1} = \left(\delta \left(\frac{C_t}{C_{t+1}} \right)^{1/\psi} \right)^{\theta} \left(\frac{W_t - C_t}{W_{t+1}} \right)^{1-\theta}, \tag{2}$$

where W_t is the market value of the consumption stream owned by the agent, including current consumption C_t.[2] The log return on wealth is $r_{t+1} = \ln \left(W_{t+1} / \left(W_t - C_t \right) \right)$, the log value of wealth tomorrow divided by reinvested wealth today. The log SDF is therefore

$$m_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1) \, r_{t+1}. \tag{3}$$

A convenient identity

The gross return to wealth can be written

$$1 + R_{t+1} = \frac{W_{t+1}}{W_t - C_t} = \left(\frac{C_t}{W_t - C_t} \right) \left(\frac{C_{t+1}}{C_t} \right) \left(\frac{W_{t+1}}{C_{t+1}} \right), \tag{4}$$

expressing it as the product of the current consumption payout, the growth in consumption, and the future price of a unit of consumption.

We find it convenient to work in logs. We define the log value of reinvested wealth per unit of consumption as $z_t = \ln \left(\left(W_t - C_t \right) / C_t \right)$, and the future value of a consumption claim as $h_{t+1} = \ln \left(W_{t+1} / C_{t+1} \right)$, so that the log return is:

$$r_{t+1} = -z_t + \Delta c_{t+1} + h_{t+1}. \tag{5}$$

[2] This notational convention is not consistent in the literature. Some authors exclude current consumption from the definition of current wealth.

Heuristically, the return on wealth is negatively related to the current value of reinvested wealth and positively related to consumption growth and the future value of wealth. The last term in equation (5) will capture the effects of intertemporal hedging on asset prices, hence the choice of the notation h_{t+1} for this term.

The ICAPM

We assume that asset returns are jointly conditionally lognormal, but we allow changing conditional volatility so we are careful to write second moments with time subscripts to indicate that they can vary over time. Under this standard assumption, the expected return on any asset must satisfy

$$0 = \ln \mathrm{E}_t \exp\{m_{t+1} + r_{i,t+1}\} = \mathrm{E}_t \left[m_{t+1} + r_{i,t+1}\right] + \frac{1}{2}\mathrm{Var}_t \left[m_{t+1} + r_{i,t+1}\right], \tag{6}$$

and the risk premium on any asset is given by

$$\mathrm{E}_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\mathrm{Var}_t r_{t+1} = -\mathrm{Cov}_t \left[m_{t+1}, r_{i,t+1}\right]. \tag{7}$$

The convenient identity (5) can be used to write the log SDF (3) without reference to consumption:

$$m_{t+1} = \theta \ln \delta - \frac{\theta}{\psi} z_t + \frac{\theta}{\psi} h_{t+1} - \gamma r_{t+1}. \tag{8}$$

Since the first two terms in (5) are known at time t, only the latter two terms appear in the conditional covariance in (7). We obtain an ICAPM pricing equation that relates the risk premium on any asset to the asset's covariance with the wealth return and with shocks to future consumption claim values:

$$\mathrm{E}_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\mathrm{Var}_t r_{t+1} = \gamma \mathrm{Cov}_t \left[r_{i,t+1}, r_{t+1}\right] - \frac{\theta}{\psi}\mathrm{Cov}_t \left[r_{i,t+1}, h_{t+1}\right] \tag{9}$$

Return and Risk Shocks in the ICAPM

To better understand the intertemporal hedging component h_{t+1}, we proceed in two steps. First, we approximate the relationship of h_{t+1} and z_{t+1} by taking a loglinear approximation about $\bar{z}$:

$$h_{t+1} \approx \kappa + \rho z_{t+1} \tag{10}$$

where the loglinearization parameter $\rho = \exp(\bar{z})/(1 + \exp(\bar{z})) \approx 1 - C/W$.

Second, we apply the general pricing equation (6) to the wealth portfolio itself (setting $r_{i,t+1} = r_{t+1}$), and use the convenient identity (5) to substitute out consumption from this expression. Rearranging, we can write the variable z_t as

$$z_t = \psi \ln \delta + (\psi - 1)\mathrm{E}_t r_{t+1} + \mathrm{E}_t h_{t+1} + \frac{\psi}{\theta}\frac{1}{2}\mathrm{Var}_t \left[m_{t+1} + r_{t+1}\right]. \tag{11}$$

Third, we combine these expressions to obtain the innovation in h_{t+1}:

$$\begin{aligned} h_{t+1} - \mathrm{E}_t h_{t+1} &= \rho(z_{t+1} - \mathrm{E}_t z_{t+1}) \\ &= (\mathrm{E}_{t+1} - \mathrm{E}_t)\rho\left((\psi - 1)r_{t+2} + h_{t+2} + \frac{\psi}{\theta}\frac{1}{2}\mathrm{Var}_{t+1}\left[m_{t+2} + r_{t+2}\right]\right). \end{aligned} \quad (12)$$

Solving forward to an infinite horizon,

$$\begin{aligned} h_{t+1} - \mathrm{E}_t h_{t+1} &= (\psi - 1)(\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j r_{t+1+j} \\ &\quad + \frac{1}{2}\frac{\psi}{\theta}(\mathrm{E}_{t+1} - \mathrm{E}_t)\sum_{j=1}^{\infty}\rho^j \mathrm{Var}_{t+j}\left[m_{t+1+j} + r_{t+1+j}\right] \\ &= (\psi - 1)N_{DR,t+1} + \frac{1}{2}\frac{\psi}{\theta}N_{RISK,t+1}. \end{aligned} \quad (13)$$

The second equality follows Campbell and Vuolteenaho (2004) and uses the notation N_{DR} ("news about discount rates") for revisions in expected future returns. In a similar spirit we write revisions in expectations of future risk (the variance of the future log return plus the log stochastic discount factor) as N_{RISK}.

Finally, we substitute back into the intertemporal model (9):

$$\begin{aligned} &\mathrm{E}_t r_{i,t+1} - r_{f,t} + \frac{1}{2}\mathrm{Var}_t r_{i,t+1} \\ = \;& \gamma\mathrm{Cov}_t\left[r_{i,t+1}, r_{t+1}\right] + (\gamma - 1)\,\mathrm{Cov}_t\left[r_{i,t+1}, N_{DR,t+1}\right] - \frac{1}{2}\mathrm{Cov}_t\left[r_{i,t+1}, N_{RISK,t+1}\right] \\ = \;& \gamma\mathrm{Cov}_t\left[r_{i,t+1}, N_{CF,t+1}\right] + \mathrm{Cov}_t\left[r_{i,t+1}, -N_{DR,t+1}\right] - \frac{1}{2}\mathrm{Cov}_t\left[r_{i,t+1}, N_{RISK,t+1}\right]. \end{aligned} \quad (14)$$

The first equality expresses the risk premium as risk aversion γ times covariance with the current market return, plus $(\gamma - 1)$ times covariance with news about future market returns, minus one half covariance with risk. This is an extension of the ICAPM as written by Campbell (1993), with no reference to consumption or the elasticity of intertemporal substitution ψ.[3]

The second equality rewrites the model, following Campbell and Vuolteenaho (2004), by breaking the market return into cash-flow news and discount-rate news. Cash-flow news N_{CF} is defined by $N_{CF} = r_{t+1} - \mathrm{E}_t r_{t+1} + N_{DR}$. The price of risk for cash-flow news is γ times greater than the price of risk for discount-rate news, hence Campbell and Vuolteenaho call betas with cash-flow news "bad betas" and those with discount-rate news "good betas" since

[3]Campbell (1993) briefly considers the heteroskedastic case, noting that when $\gamma = 1$, $\mathrm{Var}_t\left[m_{t+1} + r_{t+1}\right]$ is a constant. This implies that N_{RISK} does not vary over time so the stochastic volatility term disappears. Campbell claims that the stochastic volatility term also disappears when $\psi = 1$, but this is incorrect. When limits are taken correctly, N_{RISK} does not depend on ψ (except indirectly through the loglinearization parameter, ρ).

they have lower risk prices in equilibrium. The third term in (14) shows the risk premium associated with exposure to news about future risks and did not appear in Campbell and Vuolteenaho's model, which assumed homoskedasticity. Not surprisingly, the coefficient is negative, indicating that an asset providing positive returns when risk expectations increase will offer a lower return on average.

2.2 From Risk to Volatility

The risk shocks defined in the previous subsection are shocks to the conditional volatility of returns plus the stochastic discount factor, that is, the conditional volatility of risk-neutralized returns. We now make additional assumptions to derive a model in which this conditional volatility is proportional to the conditional volatility of returns themselves.

Suppose the economy is described by a first-order VAR

$$\mathbf{x}_{t+1} = \bar{\mathbf{x}} + \mathbf{\Gamma}\left(\mathbf{x}_t - \bar{\mathbf{x}}\right) + \sigma_t \mathbf{u}_{t+1}, \tag{15}$$

where $\mathbf{x}_{t+1}$ is an $n \times 1$ vector of state variables that has r_{t+1} as its first element, σ^2_{t+1} as its second element, and $n-2$ other variables that help to predict the first and second moments of returns. $\boldsymbol{\phi}$ and $\mathbf{\Gamma}$ are an $n \times 1$ vector and an $n \times n$ matrix of constant parameters, and $\mathbf{u}_{t+1}$ is a vector of shocks to the state variables. The key assumption here is that a scalar random variable, σ^2_t, governs time-variation in the variance of all shocks to this system. Both market returns and state variables, including volatility itself, have innovations whose variances move in proportion to one another.

Given this structure, news about discount rates can be written as

$$\begin{aligned} N_{DR,t+1} &= \left(E_{t+1} - E_t\right) \sum_{j=1}^{\infty} \rho^j r_{t+1+j} \\ &= \mathbf{e}_1' \sum_{j=1}^{\infty} \rho^j \mathbf{\Gamma}^j \sigma_t \mathbf{u}_{t+1} \\ &= \mathbf{e}_1' \rho \mathbf{\Gamma}\left(\mathbf{I} - \rho\mathbf{\Gamma}\right)^{-1} \sigma_t \mathbf{u}_{t+1} \end{aligned} \tag{16}$$

Furthermore, our log-linear model will make the log SDF m_{t+1} a linear function of the state variables. Since all shocks to the SDF are then proportional to σ_t, $\mathrm{Var}_t\left[m_{t+1} + r_{t+1}\right] \propto \sigma^2_t$. As a result, without knowing the parameters of the utility function, we can write $\mathrm{Var}_t\left[m_{t+1} + r_{t+1}\right] = \omega\sigma^2_t$ for some constant parameter $\omega > 0$ so that the news about risk,

N_{RISK}, is proportional to news about market return variance, N_V.

$$\begin{aligned} N_{RISK,t+1} &= (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \text{Var}_{t+j} \left[r_{t+1+j} + m_{t+1+j} \right] \\ &= (E_{t+1} - E_t) \sum_{j=1}^{\infty} \rho^j \left(\omega \sigma_{t+j}^2 \right) \\ &= \omega \rho \mathbf{e}_2' \sum_{j=0}^{\infty} \rho^j \mathbf{\Gamma}^j \sigma_t \mathbf{u}_{t+1} \\ &= \omega \rho \mathbf{e}_2' \left(\mathbf{I} - \rho \mathbf{\Gamma} \right)^{-1} \sigma_t \mathbf{u}_{t+1} = \omega N_{V,t+1}. \end{aligned} \tag{17}$$

Substituting (17) into (14), we obtain an empirically testable intertemporal CAPM with stochastic volatility:

$$\begin{aligned} & \text{E}_t r_{i,t+1} - r_{f,t} + \frac{1}{2} \text{Var}_t r_{i,t+1} \\ = \; & \gamma \text{Cov}_t \left[r_{i,t+1}, N_{CF,t+1} \right] + \text{Cov}_t \left[r_{i,t+1}, -N_{DR,t+1} \right] - \frac{1}{2} \omega \text{Cov}_t \left[r_{i,t+1}, N_{V,t+1} \right], \end{aligned} \tag{18}$$

where covariances with news about three key attributes of the market portfolio (cash flows, discount rates, and volatility) describe the cross section of average returns.

The parameter ω is a nonlinear function of the coefficient of relative risk aversion γ, as well as the VAR parameters and the loglinearization coefficient ρ, but it does not depend on the elasticity of intertemporal substitution ψ except indirectly through the influence of ψ on ρ. In the appendix, we show that ω solves:

$$\omega \sigma_t^2 = (1-\gamma)^2 \text{Var}_t \left[N_{CF_{t+1}} \right] + \omega (1-\gamma) \text{Cov}_t \left[N_{CF_{t+1}}, N_{V_{t+1},} \right] + \omega^2 \frac{1}{4} \text{Var}_t \left[N_{V_{t+1}} \right]. \tag{19}$$

We can see two main channels through which γ affects ω. First, a higher risk aversion—given the underlying volatilities of all shocks—implies a more volatile stochastic discount factor m, and therefore a higher RISK. This effect is proportional to $(1-\gamma)^2$, so it increases rapidly with γ. Second, there is a feedback effect on RISK through future risk: ω appears on the right-hand side of the eqution as well. Given that in our estimation we find $\text{Cov}_t \left[N_{CF_{t+1}}, N_{V_{t+1},} \right] < 0$, this second effect makes ω increase even faster with γ.

This equation can also be written directly in terms of the VAR parameters. If we define x_{CF} and x_V as the error-to-news vectors such that

$$\frac{1}{\sigma_t} N_{CF,t+1} = x_{CF} u_{t+1} = \left(e_1' + e_1' \rho \Gamma (I - \rho \Gamma)^{-1} \right) u_{t+1} \tag{20}$$

$$\frac{1}{\sigma_t} N_{V,t+1} = x_V u_{t+1} = \left(e_2' \rho (I - \rho \Gamma)^{-1} \right) u_{t+1} \tag{21}$$

and define the covariance matrix of the residuals (scaled to eliminate stochastic volatility) as $\mathbf{\Sigma} = \text{Var}[u_{t+1}]$, then ω solves

$$0 = \omega^2 \frac{1}{4} x_V \mathbf{\Sigma} x_V' - \omega \left(1 - (1-\gamma)\, x_{CF} \mathbf{\Sigma} x_V'\right) + (1-\gamma)^2\, x_{CF} \mathbf{\Sigma} x_{CF}' \tag{22}$$

This quadratic equation for ω has two solutions. The appendix shows that one of them can be discarded because it wrongly implies that ω becomes infinite as volatility shocks become small. The correct solution is

$$\omega = \frac{1 - (1-\gamma)\, x_{CF} \mathbf{\Sigma} x_V' - \sqrt{\left(1 - (1-\gamma)\, x_{CF} \mathbf{\Sigma} x_V'\right)^2 - (1-\gamma)^2 \left(x_V \mathbf{\Sigma} x_V'\right)\left(x_{CF} \mathbf{\Sigma} x_{CF}'\right)}}{\frac{1}{2} x_V \mathbf{\Sigma} x_V'}. \tag{23}$$

If risk aversion, volatility shocks, and cash flow shocks are all very large, as measured by the product $(1-\gamma)^2 \left(x_V \mathbf{\Sigma} x_V'\right)\left(x_{CF} \mathbf{\Sigma} x_{CF}'\right)$, equation (23) may deliver a complex rather than a real value for ω. In this case there is no linear relationship between σ_t^2 and the price of a consumption claim that satisfies the loglinearized Euler equation exactly. This problem highlights the limitations of loglinear approximations to asset pricing models.

Given our VAR estimates of the variance and covariance terms, we find that there is an exact positive solution to (23) as γ ranges from zero to 6.39.[4] Since this range for γ is quite restrictive, we look for an approximate solution that relaxes the problem described above. In particular, for all γ high enough that no real solution exists for ω, we choose ω to be the real part of the complex solutions. In other words, we set:

$$\omega = \frac{1 - (1-\gamma)\, x_{CF} \mathbf{\Sigma} x_V'}{\frac{1}{2} x_V \mathbf{\Sigma} x_V'} \tag{24}$$

This choice is intuitively appealing because it corresponds to choosing the value of ω that, while not achieving a zero value, minimizes the value of the quadratic equation (22). Therefore, it is the real value of ω that gives the best approximation of the loglinearized Euler equation (11), where $\omega\sigma_t^2$ appears in the last term. A downside of this approach is that in this region ω is a decreasing function of γ, since $x_{CF}\Sigma x_V' < 0$. Figure 1 plots ω against γ conditional on our estimated VAR parameters and $\rho = 0.95$ per year. Up to $\gamma = 6.39$, an exact solution exists. For higher values of γ up to 19.95, there is an approximate positive solution using the real part of (23). In our empirical work we allow γ to range from zero to 19.95 and use the corresponding ω values shown in Figure 1.

[4] In this range, the solution is roughly proportional to $(\gamma - 1)^2$. To understand this, recall that ω corresponds to the amount by which the variance of the log SDF plus the equity return scales with the variance of the equity return. In a homoskedastic model with constant discount rates, the log SDF can be written $(m_{t+1} - E_t m_{t+1}) = -\gamma\,(r_{t+1} - \text{E}_t r_{t+1})$, which would imply $\omega = (\gamma-1)^2$. This equality does not hold under heteroskedasticity, but still provides intuition about the form of the relationship between ω and γ.

3 Data and Econometrics

3.1 Data and volatility estimation

Our full VAR specification of the vector $\mathbf{x}_{t+1}$ includes six variables, five of which are the same as in Campbell, Giglio and Polk (2011). To those five variables, we add an estimate of conditional volatility. The data are all quarterly, from 1926:2 to 2010:4. In addition to these six state variables, our analysis also requires returns on a cross section of test assets. Our primary cross section consists of the excess returns on the 25 ME- and BE/ME-sorted portfolios, studied in Fama and French (1993), extended in Davis, Fama, and French (2000), and made available by Professor Kenneth French on his web site.[5] In the empirical analysis, we consider two main subsamples: early (1936:3-1963:3) and modern (1963:4-2010:4) due to the findings in Campbell and Vuolteenaho (2004) of dramatic differences in the risks of these portfolios between the early and modern period. The first subsample is shorter than that in Campbell and Vuolteenaho (2004) as we require each of the 25 portfolios to have at least two stocks as of the time of formation in June.

The first variable in the VAR is the log real return on the market, r_M, the difference between the log return on the Center for Research in Securities Prices (CRSP) value-weighted stock index and the log return on the Consumer Price Index.

The second variable is expected market variance ($EVAR$). This variable is meant to capture the volatility of market returns, σ_t, conditional on information available at time t, so that innovations to this variable can be mapped to the N_V term described above. To construct $EVAR_t$, we proceed as follows. We first construct a series of within-quarter *realized* variance of daily returns for each time t, $RVAR_t$. We then run a regression of $RVAR_{t+1}$ on lagged realized variance ($RVAR_t$) as well as the other five state variables at time t. This regression then generates a series of predicted values for $RVAR$ at each time $t+1$, that depend on information available at time t: $\widehat{RVAR}_{t+1}$. Finally, we define our expected variance at time t to be exactly this predicted value at $t+1$:

$$EVAR_t \equiv \widehat{RVAR}_{t+1}.$$

The third variable is the price-earnings ratio (PE) from Shiller (2000), constructed as the price of the S&P 500 index divided by a ten-year trailing moving average of aggregate earnings of companies in the S&P 500 index. Following Graham and Dodd (1934), Campbell and Shiller (1988b, 1998) advocate averaging earnings over several years to avoid temporary spikes in the price-earnings ratio caused by cyclical declines in earnings. We avoid any interpolation of earnings as well as lag the moving average by one quarter in order to ensure that all components of the time-t price-earnings ratio are contemporaneously observable by time t. The ratio is log transformed.

[5] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/

Fourth, the term yield spread (TY) is obtained from Global Financial Data. We compute the TY series as the difference between the log yield on the 10-Year US Constant Maturity Bond (IGUSA10D) and the log yield on the 3-Month US Treasury Bill (ITUSA3D).

Fifth, the small-stock value spread (VS) is constructed from data on the six "elementary" equity portfolios also obtained from Professor French's website. These elementary portfolios, which are constructed at the end of each June, are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on the ratio of book equity to market equity (BE/ME). The size breakpoint for year t is the median NYSE market equity at the end of June of year t. BE/ME for June of year t is the book equity for the last fiscal year end in $t-1$ divided by ME for December of $t-1$. The BE/ME breakpoints are the 30th and 70th NYSE percentiles.

At the end of June of year t, we construct the small-stock value spread as the difference between the $\ln(BE/ME)$ of the small high-book-to-market portfolio and the $\ln(BE/ME)$ of the small low-book-to-market portfolio, where BE and ME are measured at the end of December of year $t-1$. For months from July to May, the small-stock value spread is constructed by adding the cumulative log return (from the previous June) on the small low-book-to-market portfolio to, and subtracting the cumulative log return on the small high-book-to-market portfolio from, the end-of-June small-stock value spread. The construction of this series follows Campbell and Vuolteenaho (2004) closely.

The sixth variable in our VAR is the default spread (DEF), defined as the difference between the log yield on Moody's BAA and AAA bonds. The series is obtained from the Federal Reserve Bank of St. Louis. Campbell, Giglio and Polk (2011) add the default spread to the Campbell and Vuolteenaho (2004) VAR specification in part because that variable is known to track time-series variation in expected real returns on the market portfolio (Fama and French, 1989), but mostly because shocks to the default spread should to some degree reflect news about aggregate default probabilities. Of course, news about aggregate default probabilities should in turn reflect news about the market's future cash flows.

In order for the regression model that generates $EVAR_t$ to be consistent with a reasonable data-generating process for market variance, we deviate from standard OLS in two ways. First, given that we explicitly consider heteroskedasticity of the innovations to our variables, we estimate this regression using Weighted Least Squares (WLS), where the weight of each observation pair ($RVAR_{t+1}, \mathbf{x}_t$) is initially based on the time-t value of $(RVAR)^{-1}$. However, to ensure that the ratio of weights across observations is not extreme, we shrink these initial weights towards equal weights. In particular, we set our shrinkage factor large enough so that the ratio of the largest observation weight to the smallest observation weight is always bounded by the corresponding ratio observed for the VIX index.[6]

Second, we deviate from OLS by constraining the regression coefficients to produce fitted

[6] According to the CBOE website, the VIX reached a minimum of 9.31% annualized volatility on 12/22/1993 and a maximum of 80.86 annualized volatility on 11/20/2008.

values (i.e. expected market return variance) that lie between reasonable ex-ante bounds. To be consistent with our constraints on the weights in WLS, we again use the observed range on the VIX index to inform our priors. Both the constraint on observation weights and the constraint on the regression's fitted values bind in the sample we study.

The results of the first stage regression generating the state variable $EVAR_t$ are reported in Panel A of Table 1. Perhaps not surprisingly, past realized variance strongly predicts future realized variance. In addition, Panel A of Table 1 documents that an increase in either PE or DEF predicts higher future realized volatility. Both of these results are statistically significant. The result that higher PE predicts *higher* $RVAR$ might seem surprising at first, but one has to remember that the coefficient indicates the effect of a change in PE holding constant the other variables, including the return on the market; therefore, it captures the effect of a decrease in earnings on future volatility. The R^2 of this regression is just over 10%. The low R^2 masks the fact that the fit is indeed quite good, as we can see from Figure 2, in which $RVAR$ and $EVAR$ are plotted together. The R^2 is heavily influenced by the occasional spikes in realized variance, which the simple linear model we use is not able to capture.

Table 1 also reports descriptive statistics for these variables for the full sample (Panel B), the early sample (Panel C), and the modern sample (Panel D). Consistent with Campbell, Giglio and Polk (2011), we document high correlation between DEF and both PE and VS. The table also documents the high persistence of both $RVAR$ and $EVAR$ (autocorrelations of 0.525 and 0.757 respectively) and the high correlation between these variance measures and the default spread. Perhaps the most notable difference between the two subsamples is the correlation between PE and $EVAR$. In the early sample, this correlation is strongly negative, with a value of -0.573. This strong negative correlation reflects the high volatility that occurred during the Great Depression when prices were relatively low. In the modern sample, the correlation is much closer to zero, -0.047. This estimate reflects a mix of episodes with high volatility and high stock prices, such as the technology boom of the late 1990s, and episodes with high volatility and low stock prices, such as the recession of the early 1980s.

3.2 Estimation of the VAR and the news terms

Following Campbell (1993), we estimate a first-order VAR as in equation (15), where $\mathbf{x}_{t+1}$ is a 6×1 vector of state variables ordered as follows:

$$\mathbf{x}_{t+1} = [r_{M,t+1}\ EVAR_{t+1}\ PE_{t+1}\ TY_{t+1}\ DEF_{t+1}\ VS_{t+1}]$$

so that the real market return $r_{M,t+1}$ is the first element and $EVAR$ is the second element. $\bar{\mathbf{x}}$ is a 6×1 vector of the means of the variables, and $\mathbf{\Gamma}$ is a 6×6 matrix of constant parameters. Finally, $\sigma_t \mathbf{u}_{t+1}$ is a 6×1 vector of innovations, with the conditional variance-covariance matrix of $\mathbf{u}_{t+1}$ a constant:

$$\mathbf{\Sigma} = \text{Var}(\mathbf{u}_{t+1})$$

so that the parameter σ_t^2 scales the entire variance-covariance matrix of the vector of innovations.

The first-stage regression forecasting realized market return variance described in the previous section generates the variable $EVAR$. The theory in Section 2 assumes that σ_t^2, proxied for by $EVAR$, scales the variance-covariance matrix of state variable shocks. Thus, as in the first stage, we estimate the second-stage VAR using WLS, where the weight of each observation pair $(\mathbf{x}_{t+1}, \mathbf{x}_t)$ is initially based on $(EVAR_t)^{-1}$. We continue to constrain both the weights across observations and the fitted values of the regression forecasting $EVAR$ to be consistent with the historical properties of the VIX index.

Table 2 Panel A presents the results of the VAR estimation for the full sample (1926:2 to 2010:4). We report both Newey-West standard errors, estimated with a lag length of four quarters, and bootstrap standard errors for the parameter estimates of the VAR. The bootstrap standard errors for our second-stage regression allow us to take into account the uncertainty generated by forecasting variance in the first stage. Consistent with previous research, we find that PE and DEF negatively predict future returns, though DEF is only marginally significant. The value spread, which is highly correlated with both EVAR and the default spread, has a positive but not statistically significant effect on future returns. In our specification, a higher conditional variance, $EVAR$, is associated with higher future returns, though the effect is not statistically significant. Indeed, once the uncertainty generated by the first stage is taken into account, no variable is statistically significant. As for the other novel aspects of the transition matrix, both high PE and high DEF predict higher future conditional variance of returns.

Panel B of Table 2 reports the sample correlation and autocorrelation matrices of both unscaled and scaled residuals. The correlation matrices report standard deviations on the diagonals. There are a couple of aspects of these results to note. For one thing, a comparison of the standard deviations of the unscaled and scaled residuals provides a rough indication of the effectiveness of our empirical solution to the heteroskedasticity of the VAR. In general, the standard deviations of the scaled residuals are several times larger than their unscaled counterparts. Our approach implies that the scaled return residuals should have unit standard deviation. Our implementation results in a sample standard deviation smaller than this at 0.552.

Additionally, a comparison of the unscaled and scaled autocorrelation matrices reveals that much of the sample autocorrelation in the unscaled residuals is eliminated by our WLS approach. For example, the unscaled residuals in the regression forecasting the log real return have an autocorrelation of -0.135. The corresponding autocorrelation of the scaled return residuals is essentially zero, -0.003. Similarly, the autocorrelation in $EVAR$ is reduced from -0.088 to -0.002. Though the scaled residuals in the PE and DEF regression still display significant negative autocorrelation, the unscaled residuals are much more negatively autocorrelated.

Panel C of Table 2 reports the coefficients of a regression of the squared unscaled residuals of each VAR equation on a constant and EVAR. These results are consistent with our assumption that EVAR captures the conditional volatility of market returns (the coefficient of EVAR for the squared residuals of r_M is 0.649 and not significantly different from one, and the intercept is not significantly different from zero). The fact that EVAR significantly predicts with a positive sign all the squared errors of the VAR supports our underlying assumption that one parameter (σ_t^2) drives the volatility of all innovations.

The top panel of Table 3 presents the variance-covariance matrix and the standard deviation/correlation matrix of the news terms, estimated as described above. Consistent with previous research, we find that discount-rate news is twice as volatile as cash-flow news.

The interesting new results in this table concern the variance news term N_V. First, it is about as volatile as the discount-rate news. Second, it is highly negatively correlated with cash-flow news: as one might expect from the literature on the "leverage effect" (Black 1976, Christie 1982), news about low cash flows is associated with news about higher future volatility. Third, N_V correlates positively with discount-rate news, indicating that news of high volatility tends to coincide with news of high future real returns. This correlation has been called the "volatility feedback effect" (Campbell and Hentschel 1992, Calvet and Fisher 2007). Both these correlations contribute to a strong negative correlation between volatility shocks and contemporaneous market returns (French, Schwert, and Stambaugh 1987).

The lower right panel of Table 3 reports the decomposition of the vector of innovations $\sigma_t^2 u_{t+1}$ into the three terms $N_{CF,t+1}, N_{DR,t+1}$ and $N_{V,t+1}$. As shocks to $EVAR$ are just a linear combination of shocks to the underlying state variables, which includes $RVAR$, we "unpack" $EVAR$ to express the news terms as a function of r_M, PE, TY, VS, DEF, and $RVAR$. The panel shows that innovations to $RVAR$ are mapped almost one-to-one to news about future volatility. However, several of the other state variables also drive news about volatility. We find that innovations in DEF and VS are associated with news of higher future volatility. Finally, a positive shock to PE (with no change in returns) corresponds to a negative shock to earnings and predicts higher future volatility.

Figures 3, 4, and 5 plot the smoothed series for N_{CF}, $-N_{DR}$ and N_V using an exponentially-weighted moving average with a quarterly decay parameter of 0.08. This decay parameter implies a half-life of six years. The pattern of N_{CF} and $-N_{DR}$ we find is consistent with previous research. As a consequence, we focus on the smoothed series for market variance news. There is considerable time variation in N_V, and in particular we find episodes of news of high future volatility during the Great Depression and just before the beginning of World War 2, followed by a period of little news until the late 1960s. From then on, periods of positive volatility news alternate with periods of negative volatility news in cycles of 3 to 5 years. Spikes in news about future volatility are found in the early 1970s (following the oil shocks), in the late 1970s and again following the 1987 crash of the stock market. The late 1990s are characterized by strongly negative news about future returns, and at the same time higher expected future volatility. The recession of the late 2000s is instead characterized by a strong negative cash-flow news, together with a spike in volatility of the highest magnitude in our sample. The recovery from the financial crisis has brought positive cash-flow news together with news about lower future volatility.

4 Measuring and Pricing Cash-flow, Discount-Rate, and Risk Betas

4.1 Beta Measurement

We next examine the validity of an unconditional version of the first-order condition in equation (14). We modify equation (14) in three ways. First, we use simple expected returns on the left-hand side to make our results easier to compare with previous empirical studies. Second, we condition down equation (14) to avoid having to estimate all required conditional moments. Finally, we cosmetically multiply and divide all three covariances by the sample variance of the unexpected log real return on the market portfolio. By doing so, we can express our pricing equation in terms of betas, facilitating comparison to previous research. These modifications result in the following asset-pricing equation

$$E[R_i - R_f] = \gamma \sigma_M^2 \beta_{i,CF_M} + \sigma_M^2 \beta_{i,DR_M} - \frac{1}{2}\omega \sigma_M^2 \beta_{i,V_M}, \tag{25}$$

where

$$\begin{aligned} \beta_{i,CF_M} &\equiv \frac{Cov(r_{i,t}, N_{CF,t})}{Var(r_{M,t} - E_{t-1} r_{M,t})}, \\ \beta_{i,DR_M} &\equiv \frac{Cov(r_{i,t}, -N_{DR,t})}{Var(r_{M,t} - E_{t-1} r_{M,t})}, \\ \text{and } \beta_{i,V_M} &\equiv \frac{Cov(r_{i,t}, N_{V,t})}{Var(r_{M,t} - E_{t-1} r_{M,t})}. \end{aligned}$$

We price the average excess returns on the 25 size- and book-to-market-sorted portfolios using the unconditional first-order condition in equation (25) and the quadratic relationship between the parameters ω and γ given by (19) or equivalently (23). As a first step, we estimate cash-flow and discount-rate betas using the fitted values of the market's cash flow, discount-rate, and variance news estimated in the previous section. Specifically, we estimate simple WLS regressions of each portfolio's log returns on each news term, weighting each time-$t+1$ observation pair by $(EVAR_t)^{-1}$. We then scale the regression loadings by the ratio of the sample variance of the news term in question to the sample variance of the unexpected log real return on the market portfolio to generate estimates for our three-beta model.

Table 4 shows the estimated betas for the 25 size- and book-to-market portfolios over the 1936-1963 period. The portfolios are organized in a square matrix with growth stocks at the left, value stocks at the right, small stocks at the top, and large stocks at the bottom. At the right edge of the matrix we report the differences between the extreme growth and extreme value portfolios in each size group; along the bottom of the matrix we report the differences between the extreme small and extreme large portfolios in each BE/ME category. The top matrix displays cash flow betas, the middle matrix displays discount-rate betas, while the bottom matrix displays variance betas. In square brackets after each beta estimate we report a standard error, calculated conditional on the realizations of the news series from the aggregate VAR model.

In the pre-1963 sample period, value stocks have both higher cash-flow and higher discount-rate betas than growth stocks. An equal-weighted average of the extreme value stocks across size quintiles has a cash-flow beta 0.13 higher than an equal-weighted average of the extreme growth stocks. The difference in estimated discount-rate betas is also 0.13 and in the same direction. Similar to value stocks, small stocks have higher cash-flow betas and discount-rate betas than large stocks in this sample (by 0.13 and 0.25, respectively, for an equal-weighted average of the smallest stocks across value quintiles relative to an equal-weighted average of the largest stocks). These differences are extremely similar to those in Campbell and Vuolteenaho (2004), despite the exclusion of the 1929-1936 subperiod, the replacement of the excess log market return with the log real return, and the use of a richer, heteroskedastic VAR.

The new finding in Table 4 is that value stocks and small stocks are also riskier in terms of variance-news betas. An equal-weighted average of the extreme value stocks across size quintiles has a variance beta 0.39 lower than an equal-weighted average of the extreme growth stocks. Similarly, an equal-weighted average of the smallest stocks across value quintiles has a variance beta that is 0.36 lower than an equal-weighted average of the largest stocks. In summary, value and small stocks were unambiguously riskier than growth and large stocks over the 1936-1963 period.

Table 5 reports the corresponding estimates for the post-1963 period. As documented in this subsample by Campbell and Vuolteenaho (2004), value stocks still have slightly higher cash-flow betas than growth stocks, but much lower discount-rate betas. Our new finding here is that value stocks continue to have much lower variance betas, and the spread in variance betas is even greater than in the early period. The variance beta for the equal-weighted average of the extreme value stocks across size quintiles is 0.52 lower than the variance beta of an equal-weighted average of the extreme growth stocks, a difference that is more than 30% higher than the corresponding difference in the early period.

These results imply that in the post-1963 period where the CAPM has difficulty explaining the low returns on growth stocks relative to value stocks, growth stocks hedge two key aspects of the investment opportunity set. Consistent with Campbell and Vuolteenaho (2004), growth stocks hedge news about future real stock returns. The novel finding of this paper is that growth stocks also hedge news about the variance of the market return.

4.2 Beta Pricing

We next turn to pricing the cross section with these three betas. We evaluate the performance of three asset-pricing models: 1) the traditional CAPM that restricts cash-flow and discount-rate betas to have the same price of risk and sets the price of variance risk equal to zero; 2) our three-beta intertemporal asset pricing model that restricts the price of discount-rate risk to equal the variance of the market return and constrains the price of cash-flow and variance risk to be related by equation (23), with $\rho = 0.95$ per year; and 3) an unrestricted three-beta model that allows free risk prices for cash-flow, discount-rate, and variance betas. Each model is estimated in two different forms: one with a restricted zero-beta rate equal to the Treasury-bill rate, and one with an unrestricted zero-beta rate following Black (1972).

Table 6 reports results for the early sample period 1936-1963. The table has six columns, two specifications for each of our three asset pricing models. The first 16 rows of Table 6 are divided into four sets of four rows. The first set of four rows corresponds to the zero-beta rate (in excess of the Treasury-bill rate), the second set to the premium on cash-flow beta, the third set to the premium on discount-rate beta, and the fourth set to the premium on variance beta. Within each set, the first row reports the point estimate in fractions per quarter, and the second row annualizes this estimate, multiplying by 400 to aid in interpretation. These parameters are estimated from a cross-sectional regression

$$\overline{R}_i^e = g_0 + g_1\widehat{\beta}_{i,CF_M} + g_2\widehat{\beta}_{i,DR_M} + g_3\widehat{\beta}_{i,V_M} + e_i, \tag{26}$$

where a bar denotes time-series mean and $\overline{R}_i^e \equiv \overline{R}_i - \overline{R}_{rf}$ denotes the sample average simple excess return on asset i. The third and fourth rows present two alternative standard errors of the monthly estimate, described below.

Below the premia estimates, we report the R^2 statistic for a cross-sectional regression of average returns on our test assets onto the fitted values from the model. We also report a composite pricing error, computed as a quadratic form of the pricing errors. The weighting matrix in the quadratic form is a diagonal matrix with the inverse of the sample test asset return volatilities on the main diagonal.

Standard errors are produced with a bootstrap from 2,500 simulated realizations. Our bootstrap experiment samples test-asset returns and first-stage VAR errors, and uses the first-stage and second-stage WLS VAR estimates in Table 2 to generate the state-variable data. We partition the VAR errors and test-asset returns into two groups, one for 1936 to 1963 and another for 1963 to 2010, which enables us to use the same simulated realizations in subperiod analyses. The first set of standard errors (labelled A) conditions on estimated news terms and generates betas and return premia separately for each simulated realization, while the second set (labelled B) also estimates the first-stage and second-stage VAR and the news terms separately for each simulated realization. Standard errors B thus incorporate the considerable additional sampling uncertainty due to the fact that the news terms as well as betas are generated regressors.

Two alternative 5-percent critical values for the composite pricing error are produced with a bootstrap method similar to the one we have described above, except that the test-asset returns are adjusted to be consistent with the pricing model before the random samples are generated. Critical values A condition on estimated news terms, while critical values B take account of the fact that news terms must be estimated.

Finally, Table 6 reports the implied risk-aversion coefficient, γ, which can be recovered as g_1/g_2, as well as the sensitivity of news about risk to news about market variance, ω, which can be recovered as $-2 * g_3/g_2$. The three-beta ICAPM estimates are constrained so that both γ and the implied ω are strictly positive.

Table 6 shows that in the 1936-1963 period, the restricted three-beta model explains the cross-section of stock returns reasonably well. The cross-sectional R^2 statistics are about 60% for both forms of this model. Both the Sharpe-Lintner and Black versions of the CAPM do a slightly poorer job describing the cross section (R^2 statistics are 52% and 53% respectively). None of the models considered are rejected by the data based on the composite pricing test.

Figure 6 provides a visual summary of these results. The figure plots the predicted average excess return on the horizontal axis and the actual sample average excess return on the vertical axis. In summary, we find that the three-beta ICAPM improves pricing relative to both the Sharpe-Lintner and Black versions of the CAPM.

This success is due in part to the inclusion of variance betas in the specification. For the Black version of the three-beta ICAPM, the spread in variance betas across the 25 size- and book-to-market-sorted portfolios generates an annualized spread in average returns of 2.95% compared to a comparable spread of 7.46% and 2.45% for cash-flow and discount-rate betas. Variation in volatility betas accounts for 6% of the variation in explained returns compared to 34% and 4% for cash-flow and discount-rate betas respectively. The remaining 66% of the explained variation in average returns is due of course to the covariation among the three types of betas.

Results are very different in the 1963-2010 period. Table 7 shows that in this period, both versions of the CAPM do a very poor job of explaining cross-sectional variation in average returns on the test assets. When the zero-beta rate is left as a free parameter, the cross-sectional regression picks a negative premium for the CAPM beta and implies an R^2 of slightly under 6%. When the zero-beta rate is constrained to the risk-free rate, the CAPM R^2 falls to roughly -39%. Both versions of the static CAPM are easily rejected at the five-percent level by both sets of critical values.

The three-beta model with the restricted zero-beta rate explains over 56% of the cross-sectional variation in average returns across our test assets. If we continue to restrict the risk price for discount-rate and variance news but allow an unrestricted zero-beta rate, the explained variation increases to roughly 63% percent. The estimated risk price for cash-flow beta is an economically reasonable 22 percent per year with an implied coefficient of relative risk aversion of 7.21. Both versions of our intertemporal CAPM with stochastic volatility are not rejected at the 5-percent level by either set of critical values.

Figure 7 provides a visual summary of these results. For the Black version of the three-beta ICAPM, spread in volatility betas across the 25 size- and book-to-market-sorted portfolios generates an annualized spread in average returns of 5.17% compared to a comparable spread of 2.75% and 2.20% for cash-flow and discount-rate betas. Variation in volatility betas accounts for 97% of the variation in explained returns compared to 20% for cash-flow betas as well as 14% for discount-rate betas. Covariation among the three types of betas is responsible for the remaining -31% of explained variation in average returns.

Finally, we note that our key findings are robust to a variety of methodological changes. These changes include forecasting excess rather than real returns in the VAR, using OLS rather than WLS, using a unconstrained WLS approach rather than a constrained WLS approach, and setting the parameter ρ to any value between 0.88 and 0.97.

5 Conclusion

We extend the approximate closed-form intertemporal capital asset pricing model of Campbell (1993) to allow for stochastic volatility. Our model recognizes that an investor's investment opportunities may deteriorate either because expected stock returns decline or because the volatility of stock returns increases. A conservative long-term investor will wish to hedge against both types of changes in investment opportunities; thus, a stock's risk is determined not only by its beta with unexpected market returns and news about future returns (or equivalently, news about market cash flows and discount rates), but also by its beta with news about future market volatility. Although our model has three dimensions of risk, the prices of all these risks are determined by a single free parameter, the coefficient of relative risk aversion.

Our implementation models the return on the aggregate stock market as one element of a vector autoregressive (VAR) system; the volatility of all shocks to the VAR is another element of the system. We show that the negative post-1963 CAPM alphas of growth stocks are justified because these stocks hedge long-term investors against both declining expected stock returns, and increasing volatility. The addition of volatility risk to the model helps it to deliver a moderate, economically reasonable value of risk aversion.

Our empirical work is limited in two important respects. First, we test our model using only the returns on the market portfolio and portfolios of stocks sorted by size and market-book ratios. In the next version of the paper we plan to add risk-sorted portfolios to the set of test assets.

Second, we test only the unconditional implications of the model and do not evaluate its conditional implications. A full conditional test is likely to be a challenging hurdle for the model. To see why, recall that we assume a rational long-term investor always holds 100% of his or her assets in equities. However, time-variation in real stock returns generally gives the long-term investor an incentive to shift the relative weights on cash and equity, unless real interest rates and market volatility move in exactly the right way to make the equity premium proportional to market volatility. Although we do not explicitly test whether this is the case, previous work by Campbell (1987) and Harvey (1989, 1991) rejects this proportionality restriction.

One way to support the assumption of constant 100% equity investment is to invoke binding leverage constraints. Indeed, in the modern sample, the Black (1972) version of our three-beta model is consistent with this interpretation as the estimated difference between the zero-beta and risk-free rates is positive, statistically significant, and economically large. However, the risk aversion coefficient we estimate may be too large to explain why leverage constraints should bind.

Nevertheless, our model does directly answer the interesting microeconomic question: Are there reasonable preference parameters that would make a long-term investor, constrained to invest 100% in equity, content to hold the market rather than tilting towards value stocks or other high-return stock portfolios? Our answer is clearly yes.

Appendix

Deriving the equation for ω

Here we show how to solve for the unknown parameter ω as discussed in section 2. We start from the definition of ω

$$\begin{aligned}
\omega\sigma_t^2 &= \text{Var}_t\left[m_{t+1}+r_{t+1}\right] \\
&= \text{Var}_t\left[\frac{\theta}{\psi}h_{t+1}+(1-\gamma)r_{t+1}\right] \\
&= \text{Var}_t\left[\frac{\theta}{\psi}\left((\psi-1)N_{DR,t+1}+\frac{1}{2}\frac{\psi}{\theta}\omega N_{V,t+1}\right)+(1-\gamma)r_{t+1}\right] \\
&= \text{Var}_t\left[(1-\gamma)N_{DR,t+1}+\frac{1}{2}\omega N_{V,t+1}+(1-\gamma)r_{t+1}\right] \\
&= \text{Var}_t\left[(1-\gamma)N_{CF,t+1}+\frac{1}{2}\omega N_{V,t+1}\right] \\
&= (1-\gamma)^2\text{Var}_t\left[N_{CF_{t+1}}\right]+\frac{\omega}{2}(1-\gamma)\text{Cov}_t\left[N_{CF_{t+1},}N_{V_{t+1},}\right]+\frac{\omega^2}{4}\text{Var}_t\left[N_{V_{t+1}}\right],
\end{aligned}$$

deriving equation (19). Since cash flow and volatility news can be expressed in terms of the VAR parameters as

$$\begin{aligned}
N_{V,t+1} &= e_2'\rho(I-\rho\Gamma)^{-1}\sigma_t u_{t+1} \\
N_{CF,t+1} &= (e_1'+e_1'\rho\Gamma(I-\rho\Gamma)^{-1})\sigma_t u_{t+1}
\end{aligned}$$

we can define the covariance matrix of VAR shocks as $\mathbf{\Sigma}$ =$\text{Var}_t\left[u_{t+1}\right]$ =$\text{Var}[u_{t+1}]$ and the error-to-news vectors x_{CF} and x_V, defined in equations (20) and (21), to write ω as the solution to

$$0=\omega^2\frac{1}{4}x_V\mathbf{\Sigma}x_V'-\omega\left(1-(1-\gamma)\,x_{CF}\mathbf{\Sigma}x_V'\right)+(1-\gamma)^2\,x_{CF}\mathbf{\Sigma}x_{CF}'$$

as was presented in equation (22).

Selecting the correct root of the quadratic equation

The equation defining ω will generally have two solutions

$$\omega=\frac{1-(1-\gamma)\,x_{CF}\mathbf{\Sigma}x_V'\pm\sqrt{\left(1-(1-\gamma)\,x_{CF}\mathbf{\Sigma}x_V'\right)^2-(1-\gamma)^2\left(x_V\mathbf{\Sigma}x_V'\right)\left(x_{CF}\mathbf{\Sigma}x_{CF}'\right)}}{\frac{1}{2}x_V\mathbf{\Sigma}x_V'}.$$

As was discussed in the paper, this is an artifact of the loglinear approximation. While the (approximate) Euler equation holds for both roots, the correct solution is the one with the negative sign on the radical shown in equation (23).

This can be confirmed from numerical computation, and it can also be easily seen by observing the behavior of the solutions in the limit as volatility news goes to zero and the model become homoskedastic. With the false solution, ω becomes infinitely large as $x_V \to 0$. This corresponds to the log value of invested wealth going to negative infinity. On the other hand, we can use the correct solution for ω converges to $(1-\gamma)^2 x_{CF} \mathbf{\Sigma} x_{CF}'$. This is what we would expect, since in that case $\omega = \frac{1}{\sigma_t} \mathrm{Var}_t \left[(1-\gamma) N_{CF,t+1} \right]$.

Connection to the price of the consumption claim

Consider z_t, the value of reinvested wealth, which we priced in equation (11). With our loglinear assumption that $h_{t+1} = \kappa + \rho z_{t+1}$

$$
\begin{aligned}
z_t &= \psi \ln \delta + (\psi - 1) \mathrm{E}_t \left[r_{t+1} \right] + \mathrm{E}_t \left[\kappa + \rho z_{t+1} \right] + \frac{\psi}{\theta} \frac{1}{2} \mathrm{Var}_t \left[m_{t+1} + r_{t+1} \right] \\
&= \psi \ln \delta + (\psi - 1) \mathrm{E}_t \left[r_{t+1} \right] + \mathrm{E}_t \left[\kappa + \rho z_{t+1} \right] + \frac{\psi}{\theta} \frac{1}{2} \mathrm{Var}_t \left[\frac{\theta}{\psi} \left(\kappa + \rho z_{t+1} \right) + (1-\gamma) r_{t+1} \right]
\end{aligned}
$$

Using the implication that z_{t+1} will be linear in the state variables $\mathbf{x}_{t+1}$, we can write this linear relationship as $z_{t+1} = \bar{z} + \mathbf{a}' \left(\mathbf{x}_{t+1} - \bar{\mathbf{x}} \right)$, where $\bar{z}$ and $\bar{\mathbf{x}}$ represent unconditional means, and solve for $\mathbf{a}$ using the method of undetermined coefficients. The variance term would generate a quadratic equation for the price of the state variables and yield two solutions. However, we already solved the corresponding quadratic equation for $\omega = \frac{1}{\sigma_t^2} \mathrm{Var}_t \left[m_{t+1} + r_{t+1} \right]$. We can substitute in the value for ω we previously found and see that

$$
z_t = \psi \ln \delta + (\psi - 1) \mathrm{E}_t \left[r_{t+1} \right] + \mathrm{E}_t \left[\kappa + \rho \left(\bar{z} + \mathbf{a}' \left(\mathbf{x}_{t+1} - \bar{\mathbf{x}} \right) \right) \right] + \frac{\psi}{2\theta} \omega \sigma_t^2
$$

Returns and variance are the first and second elements of the state variable vector, allowing us to take the expectations and then write z_t as a function of $\left(\mathbf{x}_t - \bar{\mathbf{x}} \right)$

$$
\begin{aligned}
z_t &= \psi \ln \delta + (\psi - 1) e_1' \left(\bar{\mathbf{x}} + \mathbf{\Gamma} \left(\mathbf{x}_t - \bar{\mathbf{x}} \right) \right) + \kappa + \rho \bar{z} + \rho \mathbf{a}' \mathbf{\Gamma} \left(\mathbf{x}_t - \bar{\mathbf{x}} \right) + \frac{\psi}{2\theta} \omega e_2' \mathbf{x}_t \\
z_t &= \bar{z} + \left((\psi - 1) e_1' \mathbf{\Gamma} + \rho \mathbf{a}' \mathbf{\Gamma} + \frac{\psi}{2\theta} \omega e_2' \right) \left(\mathbf{x}_t - \bar{\mathbf{x}} \right)
\end{aligned}
$$

The price $z_t = \bar{z} + \mathbf{a}' \left(\mathbf{x}_t - \bar{\mathbf{x}} \right)$ is now defined by

$$
\begin{aligned}
\bar{z} &= \frac{1}{1-\rho} \left(\psi \ln \delta + (\psi - 1) e_1' \bar{\mathbf{x}} + \kappa \frac{\psi}{2\theta} \omega e_2' \bar{\mathbf{x}} \right) \\
\mathbf{a}' &= \frac{\psi}{\theta} \left((1-\gamma) e_1' \mathbf{\Gamma} + \frac{1}{2} \omega e_2' \right) (1 - \rho \mathbf{\Gamma})^{-1}
\end{aligned}
$$

Note that if we had not previously solved for ω, we could write

$$
\omega = \left(\frac{\theta}{\psi} \rho \mathbf{a}' + (1-\gamma) e_1' \right) \mathbf{\Sigma} \left(\frac{\theta}{\psi} \rho \mathbf{a} + (1-\gamma) e_1 \right)
$$

and have a quadratic equation defining **a**. This would yield two solutions, each corresponding to a solution for the quadratic equation defining ω that we previously solved.

Building intuition through a simple example

To develop our intuition regarding how stochastic volatility affects asset prices, we can show how this formula specializes in particular cases of interest. First, consider the textbook model without time variation in volatility ($\sigma_t^2 = \bar{\sigma}^2$) and with constant expected returns, so that $r_{t+1} = \bar{r} + \bar{\sigma} u_{t+1}$. The price of the consumption claim is constant over time, which we will label

$$z = z_0 + \left(\frac{(\psi - 1)(1 - \gamma)}{2(1 - \rho)} \right) \bar{\sigma}^2,$$

where $z_0 = \frac{\psi}{\theta} \frac{\theta \ln \delta + \frac{\theta}{\psi} \kappa + (1-\gamma)\bar{r}}{1-\rho}$, and the associated stochastic discount factor is

$$m_{t+1} = -\bar{r} - \frac{(1-\gamma)^2}{2} \bar{\sigma}^2 - \gamma \bar{\sigma} u_{t+1}.$$

Note that the level of variance may increase or decrease the value of z, depending on the preference parameters, but higher volatility always increases discount rates given $\bar{r}$.

Now consider how things change when we add time variation in volatility so that

$$\begin{aligned} r_{t+1} &= \bar{r} + \Gamma_r \left(\sigma_t^2 - \bar{\sigma}^2 \right) + \sigma_t u_{t+1} \\ \sigma_{t+1}^2 &= \bar{\sigma}^2 + \Gamma_\sigma \left(\sigma_t^2 - \bar{\sigma}^2 \right) + \sigma_t w_{t+1} \end{aligned}$$

where $\mathrm{Var}[u_{t+1}] = 1$, volatility shocks are $\mathrm{Var}_t \left[\sigma_t w_{t+1} \right] = \sigma_t^2 \sigma_v^2$, and for simplicity assume $\mathrm{Cov}[u_{t+1}, w_{t+1}] = 0$.

The price of a consumption claim, z_t, will no longer be constant but will be of the form $z_t = \bar{z} + a_\sigma \left(\sigma_t^2 - \bar{\sigma} \right)$. Applying equation (11) and solving for coefficients $\bar{z}$ and a_σ gives a solution for the conditional price z_t that we can compare to the price in the homoskedastic case shown above:

$$z_t = z_0 + \frac{\omega}{(1-\gamma)^2} \left(\frac{(\psi - 1)(1 - \gamma)}{2(1 - \rho)} \right) \bar{\sigma}^2 + a_\sigma \left(\sigma_t^2 - \bar{\sigma}^2 \right).$$

Relative to the homoskedastic case, the effect of average volatility $\bar{\sigma}^2$ is multiplied by a factor of $\frac{\omega}{(1-\gamma)^2}$. In the text of the paper, we suggested $\omega \approx (1-\gamma)^2$. Now we can be more explicit.

$$\begin{aligned} \omega &= \frac{1}{\sigma_t^2} \mathrm{Var}_t \left[(1-\gamma) r_{t+1} + \frac{\theta}{\psi} h_{t+1} \right] \\ &= \mathrm{Var}_t \left[(1-\gamma) u_{t+1} + \frac{\theta}{\psi} \rho a_\sigma w_{t+1} \right] \\ &= (1-\gamma)^2 + \frac{(1-\gamma)^2}{(\psi - 1)^2} \rho^2 a_\sigma^2 \sigma_V^2 \end{aligned}$$

Of course, this always greater than $(1-\gamma)^2$, so ω amplifies the effect of $\bar{\sigma}^2$ on z_t. Looking at the two components in ω, the first term is the direct effect of risk aversion, which will tend to dominate when volatility does not vary much over time. The second term reflects the fact that an investor must also account for the volatility of volatility. Its importance grows with the magnitude of risk aversion (γ), the importance of future shocks (ρ), and the impact of conditional volatility on the value of wealth (a_σ). The effect attenuates as an investor is more willing to substitute consumption across time (when ψ is large).

We can calculate the innovations to discount rates, cash flows and volatility

$$\begin{aligned}
N_{DR,t+1} &= (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j r_{t+1+j} = \sigma_t \frac{\rho}{1-\rho\Gamma_\sigma} \Gamma_r w_{t+1} \\
N_{CF,t+1} &= (\mathrm{E}_{t+1} - \mathrm{E}_t)\, r_{t+1} + N_{DR,t+1} = \sigma_t u_{t+1} + \sigma_t \frac{\rho}{1-\rho\Gamma_\sigma} \Gamma_r w_{t+1} \\
N_V &= (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j \sigma_{t+j}^2 = (\mathrm{E}_{t+1} - \mathrm{E}_t) \sum_{j=1}^{\infty} \rho^j \Gamma_\sigma^{j-1} \sigma_t w_{t+1} = \sigma_t \frac{\rho}{1-\rho\Gamma_\sigma} w_{t+1}
\end{aligned}$$

which gives us the quadratic equation for ω referenced in (22)

$$\begin{aligned}
\omega &= \mathrm{Var}_t \left[(1-\gamma) \frac{N_{CF,t+1}}{\sigma_t} + \frac{\omega}{2} \frac{N_{V,t+1}}{\sigma_t} \right] \\
&= (1-\gamma)^2 \left(1 + \left(\frac{\rho\Gamma_r}{1-\rho\Gamma_\sigma} \right)^2 \sigma_V^2 \right) + \frac{\omega^2}{4} \left(\frac{\rho}{1-\rho\Gamma_\sigma} \right)^2 \sigma_V^2
\end{aligned}$$

with solution

$$\omega = \frac{1 - \sqrt{1 - (1-\gamma)^2 \left(1 + \left(\frac{\rho\Gamma_r}{1-\rho\Gamma_\sigma} \right)^2 \sigma_V^2 \right) \left(\frac{\rho}{1-\rho\Gamma_\sigma} \right)^2 \sigma_V^2}}{\frac{1}{2} \left(\frac{\rho}{1-\rho\Gamma_\sigma} \right)^2 \sigma_V^2}.$$

Knowing ω, we can express the effect of conditional volatility on the price of wealth as

$$a_\sigma = \frac{\psi}{\theta} \frac{(1-\gamma)\Gamma_r + \frac{1}{2}\omega}{1-\rho\Gamma_\sigma}$$

and write the SDF as

$$m_{t+1} = \bar{m} - \left(\gamma\Gamma_r + \frac{\theta}{\psi} a_\sigma (1-\rho\Gamma_\sigma) \right) \left(\sigma_t^2 - \bar{\sigma}^2 \right) + \sigma_t \left(\frac{\theta}{\psi} \rho a_\sigma w_{t+1} - \gamma u_{t+1} \right),$$

where $\bar{m} = \theta \ln\delta + \frac{\theta}{\psi}\kappa - \frac{\theta}{\psi}(1-\rho)\bar{z} - \gamma\bar{r}$.

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Table 1: Descriptive Statistics of the VAR State Variables

The table reports descriptive statistics for quarterly observations of the state variables included in the VAR. r_M is the log real return on the CRSP value-weight index. $RVAR$ is the realized variance of within-quarter daily returns on the CRSP value-weight index. PE is the log ratio of the S&P 500's price to the S&P 500's ten-year moving average of earnings. TY is the term yield spread in percentage points, measured as the difference between the log yield on the ten-year US constant-maturity bond and the log yield on the three-month US Treasury Bill. DEF is the default yield spread in percentage points, measured as the difference between the log yield on Moody's BAA bonds and the log yield on Moody's AAA bonds. VS is the small-stock value-spread, the difference in the log book-to-market ratios of small value and small growth stocks. The small-value and small-growth portfolios are two of the six elementary portfolios constructed by Davis et al. (2000). Panel A reports the WLS parameter estimates of a constrained regression forecasting RVAR with lagged values of these state variables; the forecasted values from that regression are the state variable EVAR used in the second stage of the estimation. Initial WLS weights of each observation are inversely proportional to $RVAR_t$. These weights are then shrunk towards equal weights so that the maximum ratio of actual weights used is bounded by the corresponding historical ratio for the VIX index. Similarly, the regression estimates are constrained to generate fitted values that fall within the historical range of the VIX. The first seven columns report coefficients on the seven explanatory variables, and the remaining column shows the R^2 and F statistics. Newey-West standard errors estimated with four lags are in square brackets. Panel B reports descriptive statistics of these state variables over the full sample period 1926.2-2010.4, 339 quarterly data points. Panel C reports descriptive statistics of these state variables over the early sample period 1926.2-1963.2, 149 quarterly data points. Panel D reports descriptive statistics of these state variables over the modern sample period 1963.3-2010.4, 190 quarterly data points. "Stdev." denotes standard deviation and "Autocorr." the first-order autocorrelation of the series.

Panel A: Variance Estimation

First stage	Constant	$r_{M,t}$	$RVAR_t$	PE_t	TY_t	DEF_t	VS_t	$R^2\%/F$
$RVAR_{t+1}$	-0.037	-0.017	0.298	0.013	-0.002	0.024	0.001	10.09%
	[0.025]	[0.021]	[0.061]	[0.007]	[0.002]	[0.006]	[0.008]	6.19

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Panel B: Full-Sample Summary Statistics

Variable	Mean	Median	Stdev.	Min	Max	Autocorr.	
r_M	0.016	0.027	0.107	-0.406	0.635	-0.036	
$RVAR$	0.028	0.013	0.047	0.001	0.452	0.525	
$EVAR$	0.031	0.024	0.025	0.009	0.209	0.757	
PE	2.921	2.914	0.380	1.508	3.910	0.965	
TY	1.411	1.365	1.071	-1.650	3.748	0.846	
DEF	1.073	0.851	0.674	0.324	5.167	0.902	
VS	1.639	1.511	0.358	1.180	2.685	0.969	
Correlations	$r_{M,t+1}$	$RVAR_{t+1}$	$EVAR_{t+1}$	PE_{t+1}	TY_{t+1}	DEF_{t+1}	VS_{t+1}
$r_{M,t+1}$	1	-0.303	-0.320	0.085	0.023	-0.140	-0.038
$RVAR_{t+1}$	-0.303	1	0.910	-0.220	0.193	0.602	0.339
$EVAR_{t+1}$	-0.320	0.910	1	-0.304	0.230	0.845	0.524
PE_{t+1}	0.085	-0.220	-0.304	1	-0.233	-0.597	-0.358
TY_{t+1}	0.023	0.193	0.230	-0.233	1	0.399	0.301
DEF_{t+1}	-0.140	0.602	0.845	-0.597	0.399	1	0.646
VS_{t+1}	-0.038	0.339	0.524	-0.358	0.301	0.646	1
$r_{M,t}$	-0.036	-0.151	-0.166	0.096	-0.002	-0.162	-0.023
$RVAR_t$	0.025	0.525	0.597	-0.219	0.236	0.569	0.358
$EVAR_t$	0.030	0.607	0.757	-0.302	0.291	0.778	0.532
PE_t	-0.155	-0.119	-0.201	0.965	-0.240	-0.542	-0.352
TY_t	0.040	0.169	0.212	-0.225	0.846	0.369	0.290
DEF_t	0.075	0.526	0.723	-0.582	0.433	0.902	0.641
VS_t	-0.032	0.340	0.505	-0.359	0.316	0.619	0.969
Covariances	r_M	$RVAR$	$EVAR$	PE	TY	DEF	VS
r_M	0.0115	-0.0015	-0.0009	0.0033	0.0024	-0.0101	-0.0015
$RVAR$	-0.0015	0.0022	0.0011	-0.0039	0.0097	0.0189	0.0057
$EVAR$	-0.0009	0.0011	0.0006	-0.0029	0.0062	0.0142	0.0047
PE	0.0033	-0.0039	-0.0029	0.1443	-0.0937	-0.1526	-0.0484
TY	0.0024	0.0097	0.0062	-0.0937	1.1470	0.2876	0.1157
DEF	-0.0101	0.0189	0.0142	-0.1526	0.2876	0.4543	0.1558
VS	-0.0015	0.0057	0.0047	-0.0484	0.1157	0.1558	0.1281

Panel C: 1926-1963 Summary Statistics

Variable	Mean	Median	Stdev.	Min	Max	Autocorr.	
r_M	0.020	0.029	0.128	-0.406	0.635	-0.105	
$RVAR$	0.032	0.013	0.052	0.003	0.363	0.568	
$EVAR$	0.034	0.020	0.031	0.009	0.163	0.820	
PE	2.715	2.723	0.300	1.508	3.502	0.914	
TY	1.379	1.318	0.897	-1.067	3.284	0.892	
DEF	1.214	0.820	0.879	0.435	5.167	0.910	
VS	1.838	1.730	0.441	1.236	2.685	0.981	
Correlations	$r_{M,t+1}$	$RVAR_{t+1}$	$EVAR_{t+1}$	PE_{t+1}	TY_{t+1}	DEF_{t+1}	VS_{t+1}
$r_{M,t+1}$	1	-0.233	-0.315	0.128	-0.065	-0.217	-0.100
$RVAR_{t+1}$	-0.233	1	0.905	-0.458	0.339	0.684	0.411
$EVAR_{t+1}$	-0.315	0.905	1	-0.573	0.456	0.914	0.645
PE_{t+1}	0.128	-0.458	-0.573	1	-0.643	-0.727	-0.501
TY_{t+1}	-0.065	0.339	0.456	-0.643	1	0.612	0.699
DEF_{t+1}	-0.217	0.684	0.914	-0.727	0.612	1	0.777
VS_{t+1}	-0.100	0.411	0.645	-0.501	0.699	0.777	1
$r_{M,t}$	-0.105	-0.107	-0.143	0.131	-0.035	-0.170	-0.061
$RVAR_t$	0.046	0.568	0.648	-0.447	0.365	0.652	0.428
$EVAR_t$	0.050	0.687	0.820	-0.561	0.493	0.848	0.649
PE_t	-0.239	-0.332	-0.420	0.914	-0.621	-0.615	-0.480
TY_t	-0.046	0.272	0.421	-0.647	0.892	0.602	0.694
DEF_t	0.068	0.667	0.826	-0.704	0.638	0.910	0.771
VS_t	-0.039	0.410	0.622	-0.494	0.695	0.749	0.981
Covariances	r_M	$RVAR$	$EVAR$	PE	TY	DEF	VS
r_M	0.0163	-0.0016	-0.0013	0.0048	-0.0078	-0.0243	-0.0058
$RVAR$	-0.0016	0.0027	0.0015	-0.0071	0.0159	0.0313	0.0095
$EVAR$	-0.0013	0.0015	0.0010	-0.0053	0.0127	0.0249	0.0088
PE	0.0048	-0.0071	-0.0053	0.0898	-0.1716	-0.1911	-0.0657
TY	-0.0078	0.0159	0.0127	-0.1716	0.8047	0.4825	0.2771
DEF	-0.0243	0.0313	0.0249	-0.1911	0.4825	0.7723	0.3009
VS	-0.0058	0.0095	0.0088	-0.0657	0.2771	0.3009	0.1945

Panel D: 1963-2010 Summary Statistics

Variable	Mean	Median	Stdev.	Min	Max	Autocorr.	
r_M	0.013	0.025	0.088	-0.314	0.204	0.074	
$RVAR$	0.024	0.014	0.042	0.001	0.452	0.465	
$EVAR$	0.030	0.025	0.019	0.012	0.209	0.617	
PE	3.083	3.107	0.358	2.331	3.910	0.976	
TY	1.436	1.445	1.192	-1.650	3.748	0.825	
DEF	0.962	0.855	0.424	0.324	3.167	0.856	
VS	1.484	1.480	0.146	1.180	2.045	0.804	
Correlations	$r_{M,t+1}$	$RVAR_{t+1}$	$EVAR_{t+1}$	PE_{t+1}	TY_{t+1}	DEF_{t+1}	VS_{t+1}
$r_{M,t+1}$	1	-0.411	-0.339	0.103	0.102	0.008	0.066
$RVAR_{t+1}$	-0.411	1	0.940	-0.007	0.099	0.480	0.248
$EVAR_{t+1}$	-0.339	0.940	1	-0.047	0.049	0.683	0.304
PE_{t+1}	0.103	-0.007	-0.047	1	-0.099	-0.544	0.429
TY_{t+1}	0.102	0.099	0.049	-0.099	1	0.280	-0.003
DEF_{t+1}	0.008	0.480	0.683	-0.544	0.280	1	0.063
VS_{t+1}	0.066	0.248	0.304	0.429	-0.003	0.063	1
$r_{M,t}$	0.074	-0.221	-0.219	0.132	0.029	-0.177	-0.002
$RVAR_t$	-0.010	0.465	0.521	-0.012	0.159	0.439	0.289
$EVAR_t$	-0.011	0.485	0.617	-0.052	0.144	0.607	0.332
PE_t	-0.101	0.104	0.046	0.976	-0.124	-0.534	0.420
TY_t	0.118	0.105	0.044	-0.079	0.825	0.213	-0.058
DEF_t	0.080	0.274	0.461	-0.525	0.338	0.856	0.049
VS_t	-0.111	0.260	0.286	0.407	0.060	0.015	0.804
Covariances	r_M	$RVAR$	$EVAR$	PE	TY	DEF	VS
r_M	0.0078	-0.0015	-0.0006	0.0033	0.0106	0.0002	0.0009
$RVAR$	-0.0015	0.0017	0.0007	-0.0001	0.0050	0.0085	0.0015
$EVAR$	-0.0006	0.0007	0.0004	-0.0003	0.0012	0.0054	0.0008
PE	0.0033	-0.0001	-0.0003	0.1278	-0.0423	-0.0823	0.0224
TY	0.0106	0.0050	0.0012	-0.0423	1.4197	0.1429	-0.0012
DEF	0.0002	0.0085	0.0054	-0.0823	0.1429	0.1796	0.0035
VS	0.0009	0.0015	0.0008	0.0224	-0.0012	0.0035	0.0213

Table 2: VAR Parameter Estimates

The table shows the WLS parameter estimates for a first-order VAR model including a constant, the log real market return (r_M), forecasted variance ($EVAR$), price-earnings ratio (PE), term yield spread (TY), default yield spread (DEF), and small-stock value spread (VS). Initial weights of each observation are inversely proportional to $EVAR_t$. These weights are then shrunk towards equal weights so that the maximum ratio of actual weights used is bounded by the corresponding historical ratio for the VIX index. Similarly, the regression estimates are constrained to generate fitted values that fall within the historical range of the VIX. In Panel A of the Table, each set of three rows corresponds to a different dependent variable. The first seven columns report coefficients on the seven explanatory variables, and the remaining column shows the R^2 and F statistics. Newey-West standard errors estimated with four lags are in square brackets and bootstrap standard errors in parentheses. Bootstrap standard errors are computed from 2,500 simulated sample realizations. Panel B of the table reports the correlation matrices of both the unscaled and scaled shocks with shock standard deviations on the diagonal, labeled "corr/std.", as well as the autocorrelation matrices of both the unscaled and scaled shocks, labeled "Autocorr." Panel C of the table reports the results of regressions forecasting the squared second-stage residuals with $EVAR_t$. Newey-West standard errors estimated with four lags are in square brackets. The sample period for the dependent variables is 1926.2-2010.4, 338 quarterly data points.

Panel A: VAR Estimates

Second stage	Constant	$r_{M,t}$	$EVAR_t$	PE_t	TY_t	DEF_t	VS_t	$R^2\%/F$
$r_{M,t+1}$	0.199	0.124	0.664	-0.054	0.007	-0.029	-0.017	4.59%
	[0.071]	[0.056]	[0.518]	[0.021]	[0.006]	[0.024]	[0.020]	2.65
	(0.129)	(0.082)	(0.927)	(0.039)	(0.009)	(0.028)	(0.047)	
$EVAR_{t+1}$	-0.040	-0.004	0.342	0.012	-0.001	0.018	0.005	25.03%
	[0.011]	[0.009]	[0.080]	[0.003]	[0.001]	[0.004]	[0.003]	18.42
	(0.023)	(0.005)	(0.085)	(0.007)	(0.001)	(0.004)	(0.008)	
PE_{t+1}	0.122	0.190	0.569	0.959	0.007	-0.024	-0.004	99.23%
	[0.067]	[0.053]	[0.494]	[0.020]	[0.005]	[0.023]	[0.019]	7142.61
	(0.122)	(0.079)	(0.878)	(0.037)	(0.008)	(0.027)	(0.044)	
TY_{t+1}	-0.046	-0.161	2.911	-0.002	0.851	0.099	0.044	75.94%
	[0.366]	[0.289]	[2.679]	[0.106]	[0.029]	[0.124]	[0.102]	174.08
	(0.526)	(0.374)	(4.011)	(0.160)	(0.039)	(0.125)	(0.198)	
DEF_{t+1}	0.125	-0.448	2.231	-0.033	-0.003	0.865	0.035	71.46%
	[0.163]	[0.129]	[1.193]	[0.047]	[0.013]	[0.055]	[0.045]	138.10
	(0.285)	(0.200)	(1.822)	(0.080)	(0.020)	(0.064)	(0.102)	
VS_{t+1}	0.122	0.066	0.970	-0.010	-0.005	-0.001	0.930	95.51%
	[0.057]	[0.045]	[0.417]	[0.017]	[0.005]	[0.019]	[0.016]	1174.01
	(0.115)	(0.073)	(0.743)	(0.033)	(0.008)	(0.025)	(0.041)	

Panel B: Correlations and Standard Deviations

corr/std	r_M	$EVAR$	PE	TY	DEF	VS
			unscaled			
r_M	0.107	-0.546	0.908	-0.031	-0.493	-0.046
$EVAR$	-0.546	0.015	-0.641	-0.086	0.698	0.126
PE	0.908	-0.641	0.100	-0.016	-0.599	-0.071
TY	-0.031	-0.086	-0.016	0.565	0.020	-0.021
DEF	-0.493	0.698	-0.599	0.020	0.295	0.328
VS	-0.046	0.126	-0.071	-0.021	0.328	0.087
scaled						
r_M	0.552	-0.523	0.901	-0.078	-0.289	0.045
$EVAR$	-0.523	0.067	-0.587	-0.077	0.646	0.063
PE	0.901	-0.587	0.499	-0.071	-0.382	0.030
TY	-0.078	-0.077	-0.071	3.156	0.027	-0.018
DEF	-0.289	0.646	-0.382	0.027	1.127	0.238
VS	0.045	0.063	0.030	-0.018	0.238	0.505
Autocorr.	$r_{M,t+1}$	$EVAR_{t+1}$	PE_{t+1}	TY_{t+1}	DEF_{t+1}	VS_{t+1}
unscaled						
$r_{M,t}$	-0.135	0.044	-0.123	0.079	0.149	0.066
$EVAR_t$	0.110	-0.088	0.117	-0.142	-0.264	-0.126
PE_t	-0.144	0.133	-0.200	0.099	0.271	0.115
TY_t	-0.051	0.078	-0.034	-0.117	0.097	0.059
DEF_t	0.191	-0.130	0.218	-0.170	-0.349	-0.169
VS_t	0.030	-0.059	0.026	-0.077	-0.085	-0.089
scaled						
$r_{M,t}$	-0.003	-0.043	-0.014	0.018	0.016	-0.014
$EVAR_t$	0.060	-0.002	0.070	-0.062	-0.127	-0.063
PE_t	-0.019	0.033	-0.075	0.022	0.093	0.007
TY_t	-0.042	0.054	-0.037	-0.033	0.073	0.031
DEF_t	0.084	-0.056	0.105	-0.084	-0.208	-0.105
VS_t	0.033	-0.064	0.016	-0.022	-0.073	-0.068

Panel C: Heteroskedastic Shocks

Squared, second-stage, unscaled residual	Constant	$EVAR_t$	R^2%
$r_{M_{t+1}}$	-0.0090	0.6486	23.78%
	[0.0066]	[0.2525]	
$EVAR_{t+1}$	-0.0002	0.0138	5.54%
	[0.0001]	[0.0032]	
PE_{t+1}	-0.0101	0.6405	23.68%
	[0.0069]	[0.2675]	
TY_{t+1}	0.1151	6.4643	3.73%
	[0.0706]	[2.4644]	
DEF_{t+1}	-0.2128	9.5320	32.09%
	[0.0934]	[3.5951]	
VS_{t+1}	0.0028	0.1522	6.67%
	[0.0013]	[0.0392]	

Table 3: Cash-flow, Discount-rate, and Variance News for the Market Portfolio

The table shows the properties of cash-flow news (N_{CF}), discount-rate news (N_{DR}), and volatility news (N_V) implied by the VAR model of Table 2. The upper-left section of the table shows the covariance matrix of the news terms. The upper-right section shows the correlation matrix of the news terms with standard deviations on the diagonal. The lower-left section shows the correlation of shocks to individual state variables with the news terms. The lower-right section shows the functions ($\mathbf{e1}'+\mathbf{e1}'\lambda_{DR}$, $\mathbf{e1}'\lambda_{DR}$, $\mathbf{e2}'\lambda_V$) that map the state-variable shocks to cash-flow, discount-rate, and variance news. We define $\lambda_{DR} \equiv \rho\mathbf{\Gamma}(\mathbf{I}-\rho\mathbf{\Gamma})^{-1}$ and $\lambda_V \equiv \rho(\mathbf{I}-\rho\mathbf{\Gamma})^{-1}$, where $\mathbf{\Gamma}$ is the estimated VAR transition matrix from Table 2 and ρ is set to 0.95 per annum. r_M is the log real return on the CRSP value-weight index. $RVAR$ is the realized variance of daily returns on the CRSP value-weight index. PE is the log ratio of the S&P 500's price to the S&P 500's ten-year moving average of earnings. TY is the term yield spread in percentage points, measured as the difference between the log yield on the ten-year US constant-maturity bond and the log yield on the three-month US Treasury Bill. DEF is the default yield spread in percentage points, measured as the difference between the log yield on Moody's BAA bonds and the log yield on Moody's AAA bonds. VS is the small-stock value-spread, the difference in the log book-to-market ratios of small value and small growth stocks. Bootstrap standard errors (in parentheses) are computed from 2,500 simulated sample realizations.

News cov.	N_{CF}	N_{DR}	N_V	News corr/std	N_{CF}	N_{DR}	N_V
N_{CF}	0.00244	-0.00048	-0.00259	N_{CF}	0.049	-0.108	-0.446
	(0.00094)	(0.00121)	(0.00127)		(0.008)	(0.229)	(0.269)
N_{DR}	-0.00048	0.00802	0.00045	N_{DR}	-0.108	0.090	0.043
	(0.00121)	(0.00285)	(0.00246)		(0.229)	(0.015)	(0.370)
N_V	-0.00259	0.00045	0.01379	N_V	-0.446	0.043	0.117
	(0.00127)	(0.00246)	(0.00482)		(0.269)	(0.370)	(0.028)

Shock correlations	N_{CF}	N_{DR}	N_V	Functions	N_{CF}	N_{DR}	N_V
r_M shock	0.188	-1.640	-0.077	r_M shock	0.958	-0.042	-0.090
	(0.195)	(0.437)	(0.342)		(0.040)	(0.040)	(0.069)
$RVAR$ shock	-0.032	0.773	0.308	$RVAR$ shock	-0.084	-0.084	0.990
	(0.113)	(0.284)	(0.136)		(0.349)	(0.349)	(0.616)
PE shock	0.086	-1.698	-0.146	PE shock	-0.952	-0.952	0.581
	(0.175)	(0.448)	(0.352)		(0.178)	(0.178)	(0.321)
TY shock	0.048	0.281	-0.120	TY shock	0.013	0.013	-0.023
	(0.114)	(0.253)	(0.264)		(0.017)	(0.017)	(0.030)
DEF shock	-0.166	0.530	0.699	DEF shock	-0.069	-0.069	0.392
	(0.143)	(0.376)	(0.248)		(0.044)	(0.044)	(0.080)
VS shock	-0.180	-0.474	0.440	VS shock	-0.187	-0.187	0.304
	(0.125)	(0.269)	(0.264)		(0.132)	(0.132)	(0.232)

Table 4: Cash-flow, Discount-rate, and Variance Betas in the Early Sample

The table shows the estimated cash-flow ($\widehat{\beta}_{CF}$), discount-rate ($\widehat{\beta}_{DR}$), and variance betas ($\widehat{\beta}_{V}$) for the 25 ME- and BE/ME-sorted portfolios. "Growth" denotes the lowest BE/ME, "Value" the highest BE/ME, "Small" the lowest ME, and "Large" the highest ME stocks. "Diff." is the difference between the extreme cells. Standard errors [in brackets] are computed from 2,500 simulated sample realizations and are conditional on the estimated news series. Estimates are based on quarterly data for the 1936:3-1963:2 period using weighted least squares. Initial weights of each time-$t+1$ observation are inversely proportional to $EVAR_t$. These weights are then shrunk towards equal weights so that the maximum ratio of actual weights used is bounded by the corresponding historical ratio for the VIX index.

$\widehat{\beta}_{CF}$	Growth		2		3		4		Value		Diff	
Small	0.44	[0.13]	0.43	[0.10]	0.41	[0.10]	0.43	[0.10]	0.46	[0.10]	0.02	[0.06]
2	0.32	[0.07]	0.35	[0.09]	0.36	[0.09]	0.40	[0.09]	0.43	[0.10]	0.11	[0.04]
3	0.31	[0.08]	0.30	[0.08]	0.34	[0.09]	0.34	[0.08]	0.47	[0.12]	0.16	[0.05]
4	0.28	[0.07]	0.29	[0.07]	0.33	[0.08]	0.36	[0.08]	0.47	[0.11]	0.19	[0.05]
Large	0.24	[0.07]	0.24	[0.06]	0.28	[0.08]	0.35	[0.09]	0.41	[0.29]	0.17	[0.05]
Diff	-0.20	[0.07]	-0.20	[0.06]	-0.13	[0.05]	-0.08	[0.04]	-0.05	[0.04]		

$\widehat{\beta}_{DR}$	Growth		2		3		4		Value		Diff	
Small	0.89	[0.14]	0.88	[0.15]	0.82	[0.16]	0.81	[0.16]	0.83	[0.15]	-0.06	[0.07]
2	0.69	[0.11]	0.74	[0.13]	0.70	[0.14]	0.73	[0.15]	0.84	[0.12]	0.15	[0.07]
3	0.67	[0.13]	0.63	[0.09]	0.68	[0.11]	0.69	[0.11]	0.80	[0.14]	0.13	[0.08]
4	0.57	[0.07]	0.60	[0.09]	0.63	[0.09]	0.67	[0.12]	0.85	[0.14]	0.28	[0.12]
Large	0.56	[0.08]	0.52	[0.08]	0.52	[0.10]	0.66	[0.13]	0.71	[0.12]	0.15	[0.12]
Diff	-0.33	[0.13]	-0.36	[0.10]	-0.31	[0.15]	-0.15	[0.12]	-0.12	[0.08]		

$\widehat{\beta}_{V}$	Growth		2		3		4		Value		Diff	
Small	-0.72	[0.29]	-0.79	[0.24]	-0.85	[0.26]	-0.82	[0.25]	-0.88	[0.25]	-0.16	[0.13]
2	-0.50	[0.17]	-0.52	[0.22]	-0.60	[0.20]	-0.62	[0.22]	-0.82	[0.25]	-0.32	[0.12]
3	-0.48	[0.20]	-0.38	[0.15]	-0.53	[0.19]	-0.55	[0.19]	-0.85	[0.27]	-0.37	[0.13]
4	-0.21	[0.13]	-0.35	[0.17]	-0.43	[0.18]	-0.58	[0.24]	-0.87	[0.28]	-0.65	[0.19]
Large	-0.22	[0.14]	-0.23	[0.14]	-0.44	[0.21]	-0.67	[0.27]	-0.68	[0.18]	-0.46	[0.16]
Diff	0.50	[0.21]	0.56	[0.14]	0.41	[0.17]	0.16	[0.14]	0.19	[0.12]		

Table 5: Cash-flow, Discount-rate, and Variance Betas in the Modern Sample

The table shows the estimated cash-flow ($\widehat{\beta}_{CF}$), discount-rate ($\widehat{\beta}_{DR}$), and variance betas ($\widehat{\beta}_{V}$) for the 25 ME- and BE/ME-sorted portfolios. "Growth" denotes the lowest BE/ME, "Value" the highest BE/ME, "Small" the lowest ME, and "Large" the highest ME stocks. "Diff." is the difference between the extreme cells. Standard errors [in brackets] are computed from 2,500 simulated sample realizations and are conditional on the estimated news series. Estimates are based on quarterly data for the 1963:3-2010:4 period. Initial weights of each time-$t+1$ observation are inversely proportional to $EVAR_t$. These weights are then shrunk towards equal weights so that the maximum ratio of actual weights used is bounded by the corresponding historical ratio for the VIX index.

$\widehat{\beta}_{CF}$	Growth		2		3		4		Value		Diff	
Small	0.22	[0.06]	0.21	[0.05]	0.21	[0.04]	0.20	[0.04]	0.24	[0.05]	0.02	[0.04]
2	0.20	[0.05]	0.19	[0.04]	0.21	[0.04]	0.20	[0.04]	0.23	[0.05]	0.03	[0.04]
3	0.18	[0.05]	0.19	[0.04]	0.18	[0.04]	0.19	[0.04]	0.21	[0.04]	0.03	[0.04]
4	0.17	[0.04]	0.18	[0.04]	0.19	[0.04]	0.18	[0.04]	0.21	[0.05]	0.04	[0.03]
Large	0.11	[0.03]	0.14	[0.03]	0.13	[0.03]	0.14	[0.04]	0.16	[0.04]	0.05	[0.03]
Diff	-0.10	[0.04]	-0.08	[0.03]	-0.08	[0.03]	-0.06	[0.03]	-0.08	[0.03]		

$\widehat{\beta}_{DR}$	Growth		2		3		4		Value		Diff	
Small	1.33	[0.11]	1.09	[0.09]	0.92	[0.08]	0.86	[0.08]	0.89	[0.09]	-0.45	[0.09]
2	1.26	[0.09]	1.00	[0.08]	0.88	[0.07]	0.80	[0.07]	0.82	[0.09]	-0.44	[0.09]
3	1.18	[0.08]	0.92	[0.06]	0.80	[0.07]	0.74	[0.07]	0.76	[0.08]	-0.42	[0.09]
4	1.06	[0.07]	0.89	[0.06]	0.77	[0.06]	0.74	[0.06]	0.79	[0.08]	-0.28	[0.09]
Large	0.89	[0.05]	0.75	[0.05]	0.62	[0.05]	0.61	[0.06]	0.68	[0.06]	-0.21	[0.07]
Diff	-0.45	[0.11]	-0.34	[0.09]	-0.30	[0.07]	-0.24	[0.07]	-0.21	[0.09]		

$\widehat{\beta}_{V}$	Growth		2		3		4		Value		Diff	
Small	0.61	[0.32]	0.37	[0.26]	0.24	[0.24]	0.19	[0.22]	0.02	[0.32]	-0.59	[0.12]
2	0.69	[0.29]	0.42	[0.26]	0.24	[0.23]	0.19	[0.25]	0.08	[0.27]	-0.60	[0.12]
3	0.67	[0.28]	0.36	[0.24]	0.27	[0.22]	0.13	[0.25]	0.15	[0.19]	-0.52	[0.14]
4	0.63	[0.25]	0.36	[0.23]	0.19	[0.26]	0.17	[0.27]	0.11	[0.27]	-0.53	[0.12]
Large	0.47	[0.22]	0.36	[0.17]	0.18	[0.19]	0.12	[0.25]	0.14	[0.21]	-0.33	[0.09]
Diff	-0.14	[0.14]	-0.01	[0.13]	-0.06	[0.09]	-0.07	[0.09]	0.12	[0.14]		

Table 6: Asset Pricing Tests for the Early Sample

The table shows the premia estimated from the 1936:3-1963:2 sample for the CAPM, the three-beta ICAPM, and an unrestricted factor model. The test assets are the 25 ME- and BE/ME-sorted portfolios. The first column per model constrains the zero-beta rate (R_{zb}) to equal the risk-free rate (R_{rf}) while the second column allows R_{zb} to be a free parameter. Estimates are from a cross-sectional regression of average simple excess test-asset returns (quarterly in fractions) on an intercept and estimated cash-flow ($\widehat{\beta}_{CF}$), discount-rate ($\widehat{\beta}_{DR}$), and variance betas ($\widehat{\beta}_{V}$). Standard errors and critical values [A] are conditional on the estimated news series and (B) incorporate full estimation uncertainty of the news terms. The test rejects if the pricing error is higher than the listed 5 percent critical value.

Parameter	CAPM		Three-beta ICAPM		Factor Model	
R_{zb} less R_f (g_0)	0	-0.0060	0	-0.0004	0	0.0268
% per annum	0%	-2.41%	0%	-0.16%	0%	10.72%
Std. err. A	N/A	[0.0166]	N/A	[0.0128]	N/A	[0.0210]
Std. err. B	N/A	(0.0164)	N/A	(0.0142)	N/A	(0.0189)
$\widehat{\beta}_{CF}$ premium (g_1)	0.0437	0.0492	0.0789	0.0795	0.1164	0.0525
% per annum	17.50%	19.69%	31.56%	31.80%	46.57%	21.00%
Std. err. A	[0.0162]	[0.0258]	[0.0315]	[0.0422]	[0.1316]	[0.1374]
Std. err. B	(0.0161)	(0.0256)	(0.0740)	(0.0818)	(0.1396)	(0.1443)
$\widehat{\beta}_{DR}$ premium (g_2)	0.0437	0.0492	0.0166	0.0166	-0.0053	-0.0368
% per annum	17.50%	19.69%	6.64%	6.64%	-2.11%	-14.72%
Std. err. A	[0.0162]	[0.0258]	[0.0055]	[0.0055]	[0.0663]	[0.0811]
Std. err. B	(0.0161)	(0.0256)	(0.0055)	(0.0055)	(0.0965)	(0.1043)
$\widehat{\beta}_{V}$ premium (g_3)			-0.0109	-0.0111	-0.0137	-0.046
% per annum			-4.35%	-4.46%	-5.49%	-18.42%
Std. err. A			[0.0047]	[0.0072]	[0.0435]	[0.0594]
Std. err. B			(0.0348)	(0.0363)	(0.2122)	(0.2247)
$\widehat{R^2}$	51.99%	52.61%	59.15%	59.16%	60.34%	64.48%
Pricing error	0.0271	0.0241	0.0193	0.0192	0.0184	0.0200
5% critic. val. A	[0.073]	[0.034]	[0.070]	[0.045]	[0.041]	[0.038]
5% critic. val. B	(0.077)	(0.035)	(0.105)	(0.050)	(0.043)	(0.042)
Implied γ	N/A	N/A	4.75	4.79	N/A	N/A
Implied ω	N/A	N/A	1.31	1.34	N/A	N/A

Table 7: Asset Pricing Tests for the Modern Sample

The table shows the premia estimated from the 1963:3-2010:4 sample for the CAPM, the three-beta ICAPM, and an unrestricted factor model. The test assets are the 25 ME- and BE/ME-sorted portfolios. The first column per model constrains the zero-beta rate (R_{zb}) to equal the risk-free rate (R_{rf}) while the second column allows R_{zb} to be a free parameter. Estimates are from a cross-sectional regression of average simple excess test-asset returns (quarterly in fractions) on an intercept and estimated cash-flow ($\widehat{\beta}_{CF}$), discount-rate ($\widehat{\beta}_{DR}$), and variance betas ($\widehat{\beta}_{V}$). Standard errors and critical values [A] are conditional on the estimated news series and (B) incorporate full estimation uncertainty of the news terms. The test rejects if the pricing error is higher than the listed 5 percent critical value.

Parameter	CAPM		Three-beta ICAPM		Factor Model	
R_{zb} less R_f (g_0)	0	0.0277	0	0.0112	0	0.0026
% per annum	0%	11.06%	0%	4.47%	0%	1.02%
Std. err. A	N/A	[0.0151]	N/A	[0.0110]	N/A	[0.0132]
Std. err. B	N/A	(0.0149)	N/A	(0.0139)	N/A	(0.0157)
$\widehat{\beta}_{CF}$ premium (g_1)	0.0202	-0.0048	0.0991	0.0550	0.2161	0.2080
% per annum	8.08%	-1.93%	39.66%	22.00%	86.42%	83.22%
Std. err. A	[0.0089]	[0.0187]	[0.0322]	[0.0295]	[0.0963]	[0.1038]
Std. err. B	(0.0092)	(0.0189)	(0.0442)	(0.0553)	(0.1343)	(0.1358)
$\widehat{\beta}_{DR}$ premium (g_2)	0.0202	-0.0048	0.0076	0.0076	-0.0197	-0.021
% per annum	8.08%	-1.93%	3.05%	3.05%	-7.89%	-8.39%
Std. err. A	[0.0089]	[0.0187]	[0.0018]	[0.0018]	[0.0254]	[0.0262]
Std. err. B	(0.0092)	(0.0189)	(0.0018)	(0.0018)	(0.0533)	(0.0568)
$\widehat{\beta}_{V}$ premium (g_3)			-0.0106	-0.0194	-0.0018	-0.0015
% per annum			-4.23%	-7.74%	-0.71%	-0.60%
Std. err. A			[0.0082]	[0.0084]	[0.0283]	[0.0277]
Std. err. B			(0.0234)	(0.0381)	(0.1198)	(0.1203)
$\widehat{R^2}$	-38.86%	5.60%	56.36%	63.35%	78.47%	78.72%
Pricing error	0.1220	0.1158	0.0365	0.0412	0.0234	0.0241
5% critic. val. A	[0.059]	[0.039]	[0.200]	[0.092]	[0.052]	[0.036]
5% critic. val. B	(0.057)	(0.040)	(0.173)	(0.101)	(0.069)	(0.052)
Implied γ	N/A	N/A	12.99	7.21	N/A	N/A
Implied ω	N/A	N/A	2.77	5.07	N/A	N/A

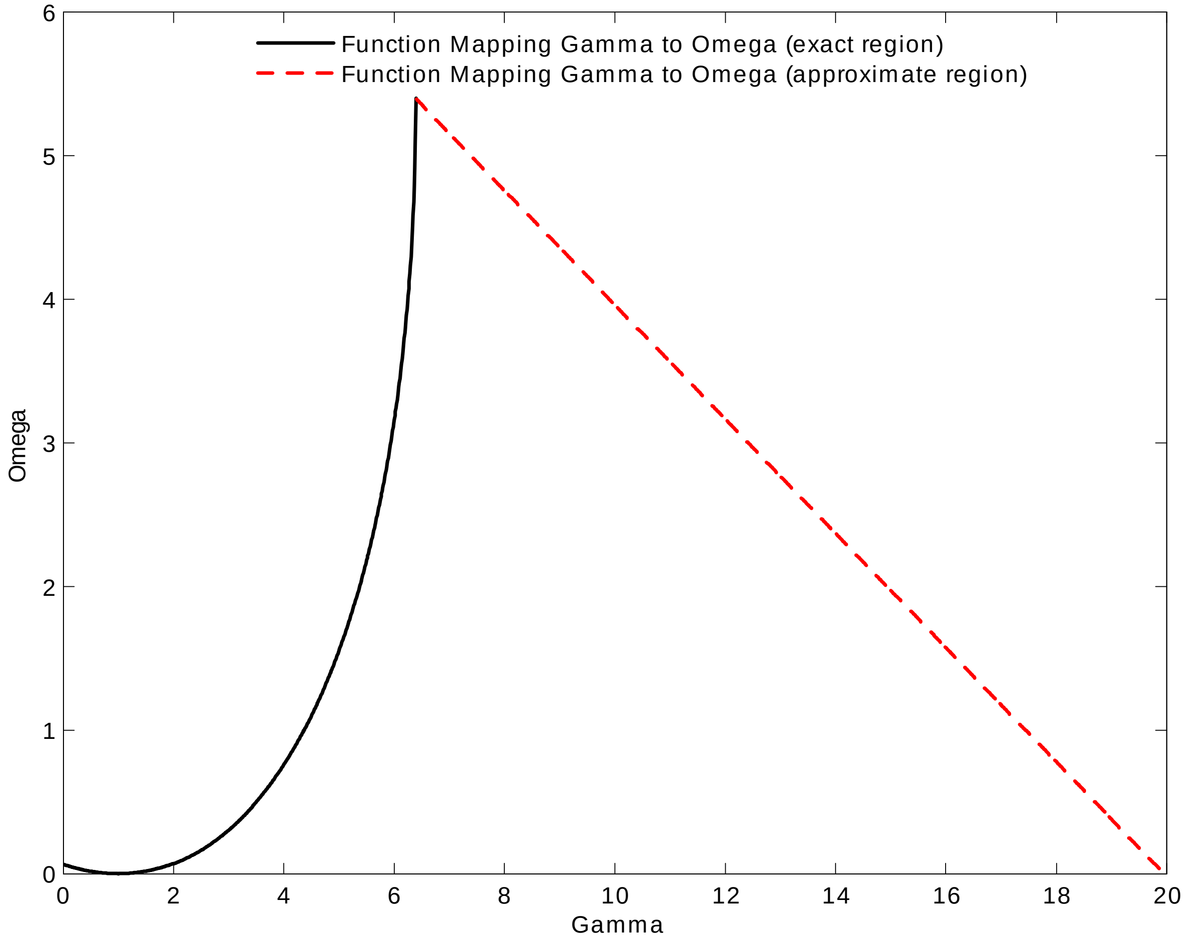

Figure 1: This figure graphs the function mapping the parameter γ to the parameter ω. This function depends on the loglinearization parameter ρ, set to 0.95 per year, and the empirically estimated VAR parameters of Table 2. γ is the investor's risk aversion while ω is the sensitivity of news about risk, N_{RISK}, to news about market variance, N_V. The solid curve represents the region where the log-linearized Euler equation holds exactly. The dotted line represents the region where the log-linearized Euler equation does not hold exactly. In this region the mapping provides the value of omega that minimizes the resulting error.

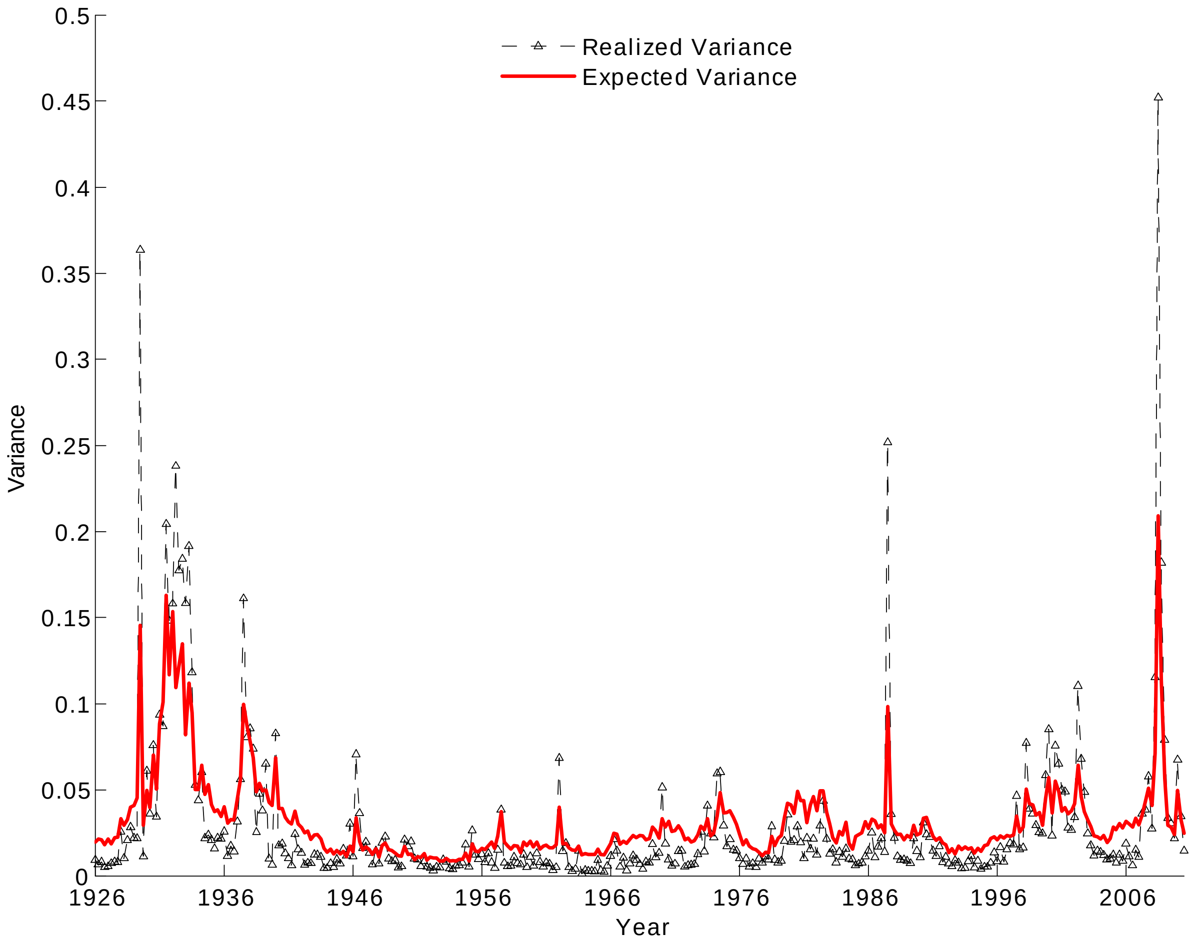

Figure 2: This figure plots quarterly observations of realized within-quarter daily return variance over the sample period 1926:2-2010:4 and the expected variance implied by the model estimated in Table 1 Panel A.

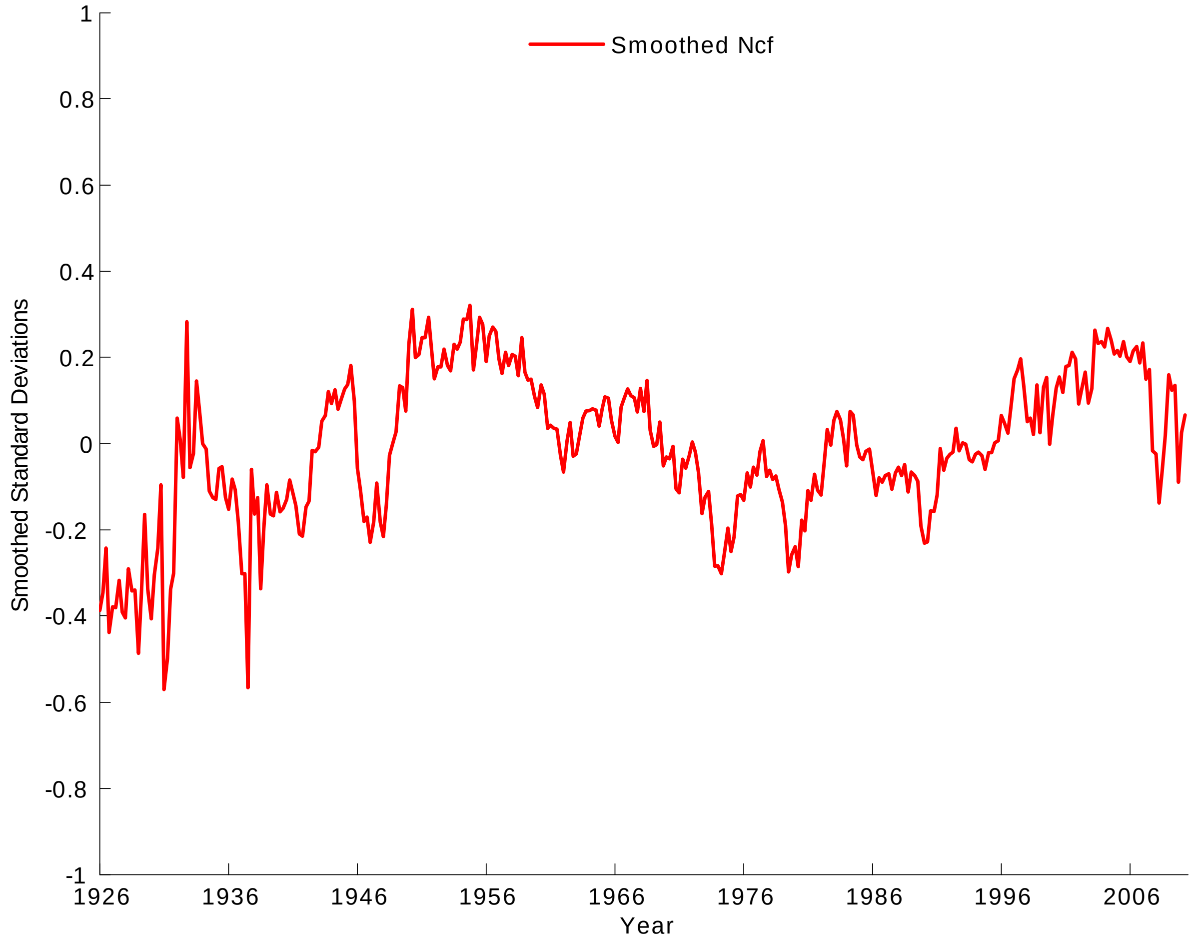

Figure 3: This figure plots normalized cash-flow news, smoothed with a trailing exponentially-weighted moving average. The decay parameter is set ot 0.08 per quarter, and the smoothed news series is generated as $MA_t(N) = 0.08N_t + (1 - 0.08)MA_{t-1}(N)$. This decay parameter implies a half-life of six years. The sample period is 1926:2-2010:4.

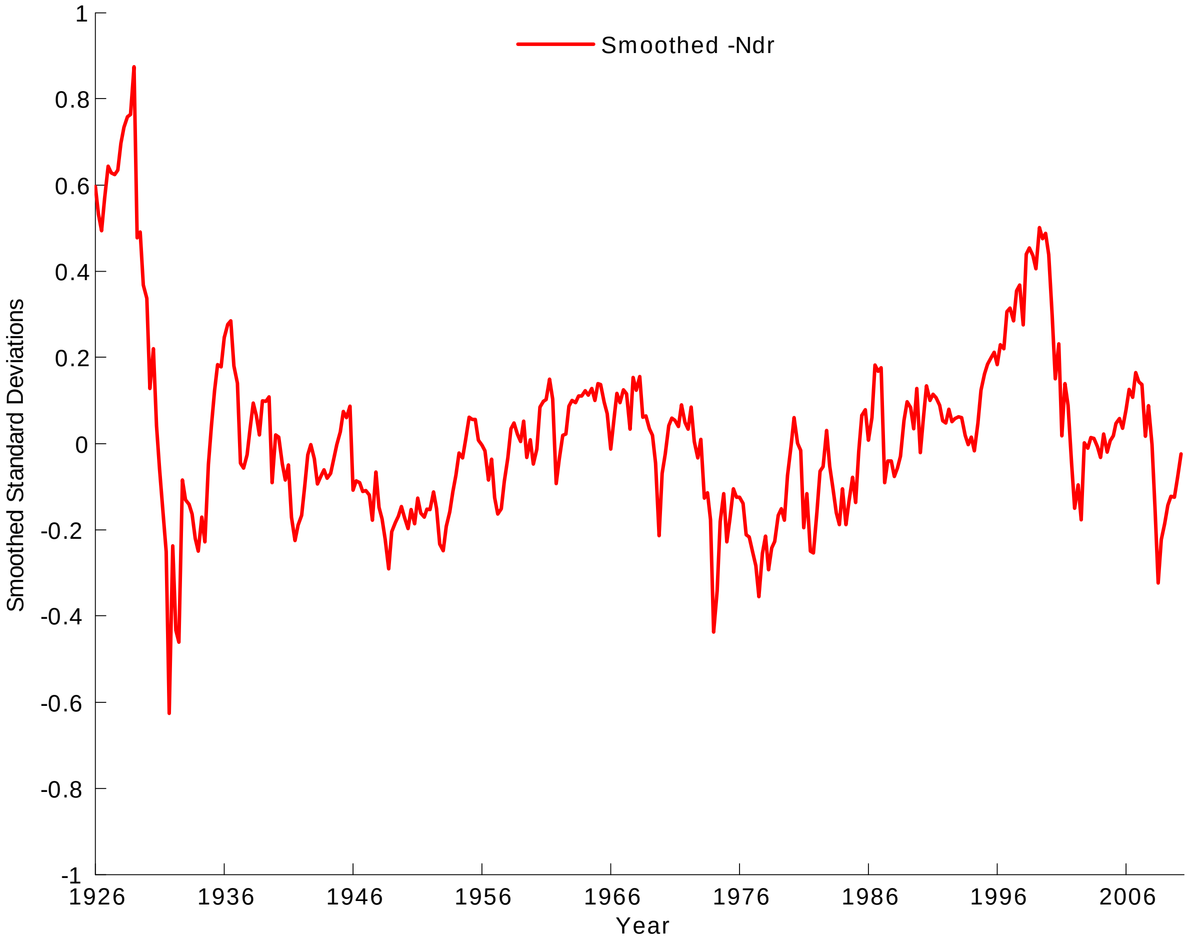

Figure 4: This figure plots the negative of normalized discount-rate news, smoothed with a trailing exponentially-weighted moving average. The decay parameter is set ot 0.08 per quarter, and the smoothed news series is generated as $MA_t(N) = 0.08N_t + (1 - 0.08)MA_{t-1}(N)$. This decay parameter implies a half-life of six years. The sample period is 1926:2-2010:4.

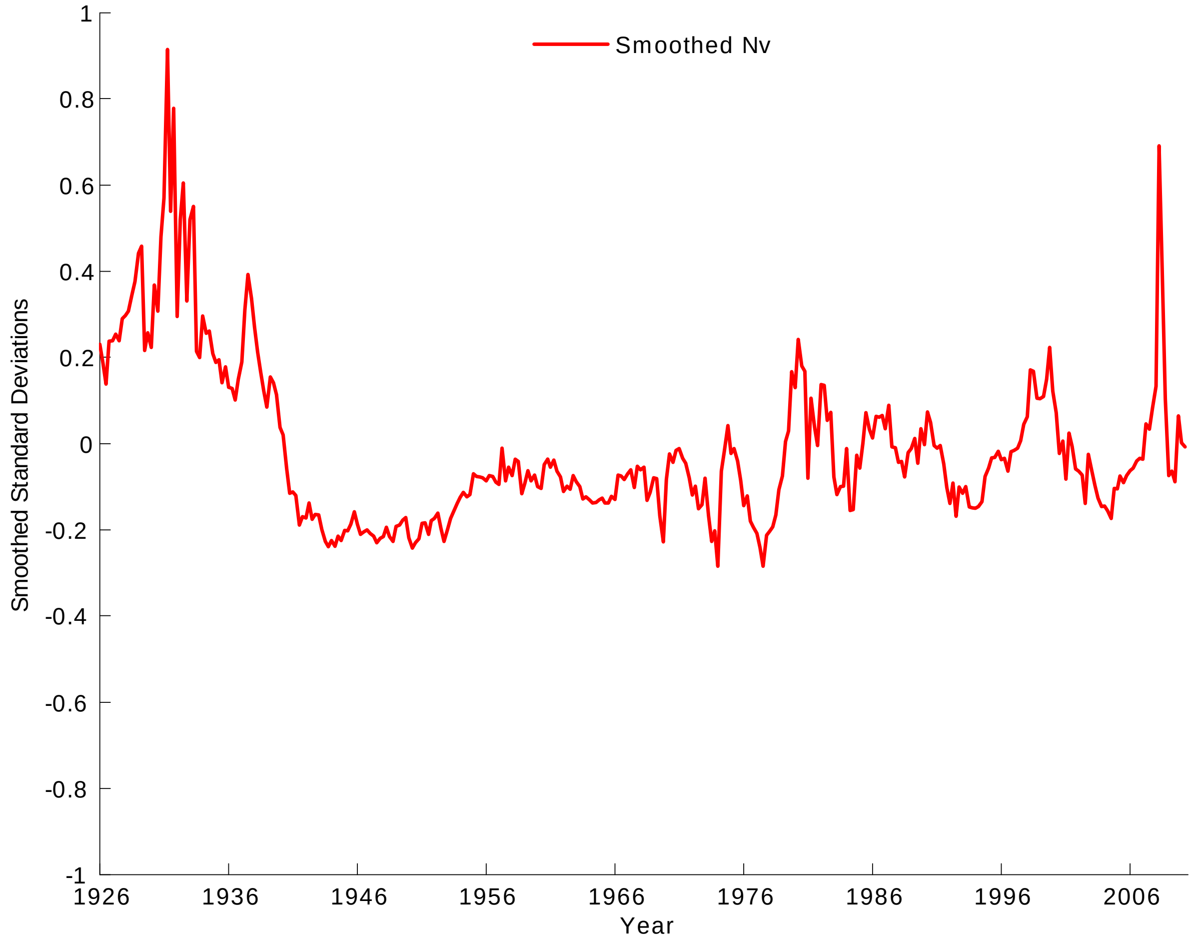

Figure 5: This figure plots normalized return variance news, smoothed with a trailing exponentially-weighted moving average. The decay parameter is set ot 0.08 per quarter, and the smoothed news series is generated as $MA_t(N) = 0.08N_t + (1 - 0.08)MA_{t-1}(N)$. This decay parameter implies a half-life of six years. The sample period is 1926:2-2010:4.

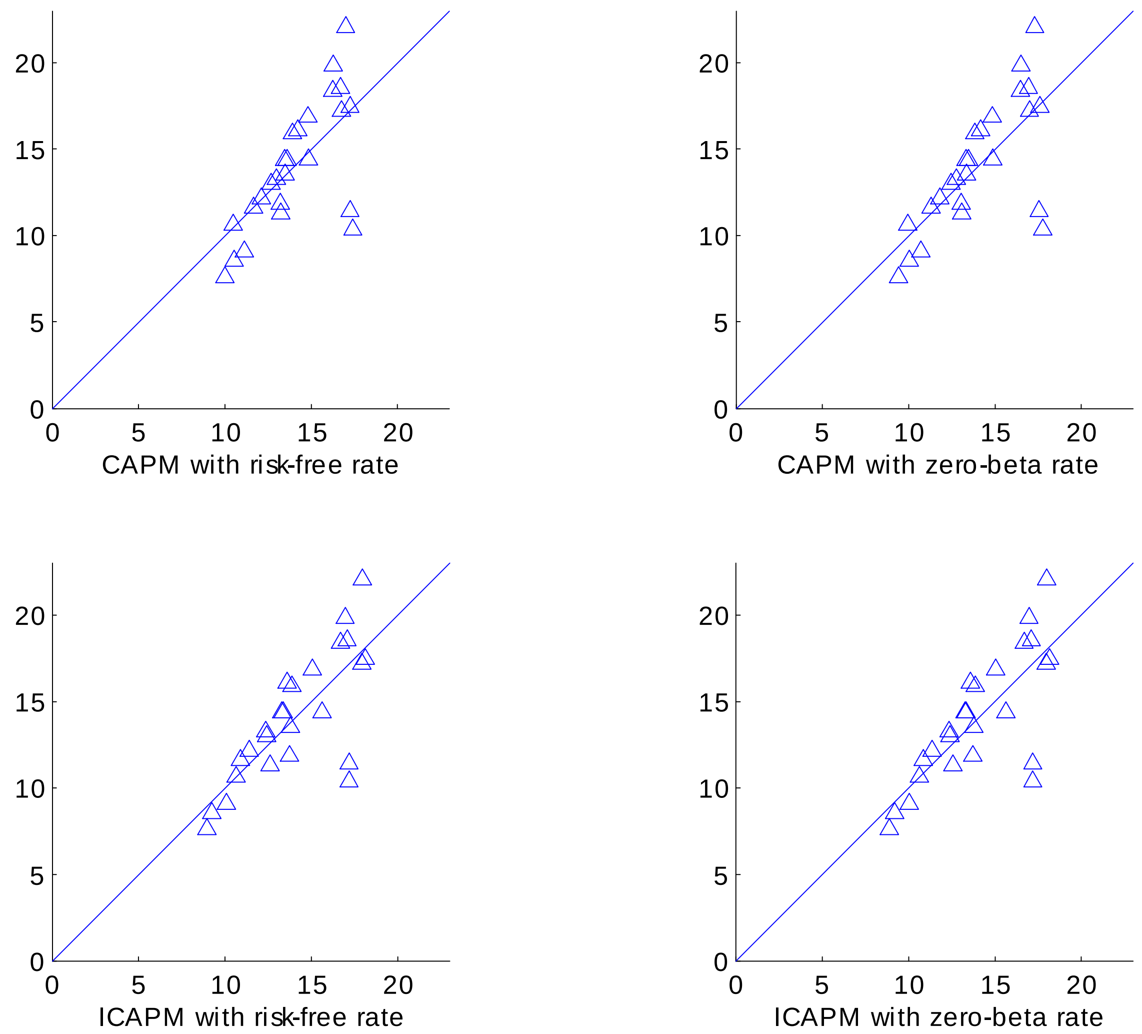

Figure 6: The four diagrams correspond to (clockwise from the top left) the CAPM with a constrained zero-beta rate, the CAPM with an unconstrained zero-beta rate, the three-factor ICAPM with a free zero-beta rate, and the three-factor ICAPM with the zero-beta rate constrained to the risk-freee rate. The horizontal axes correspond to the predicted average excess returns and the vertical axes to the sample average realized excess returns for the 25 ME- and BE/ME-sorted portfolios. The predicted values are from regressions presented in Table 6 for the sample period 1936:3-1963:2.

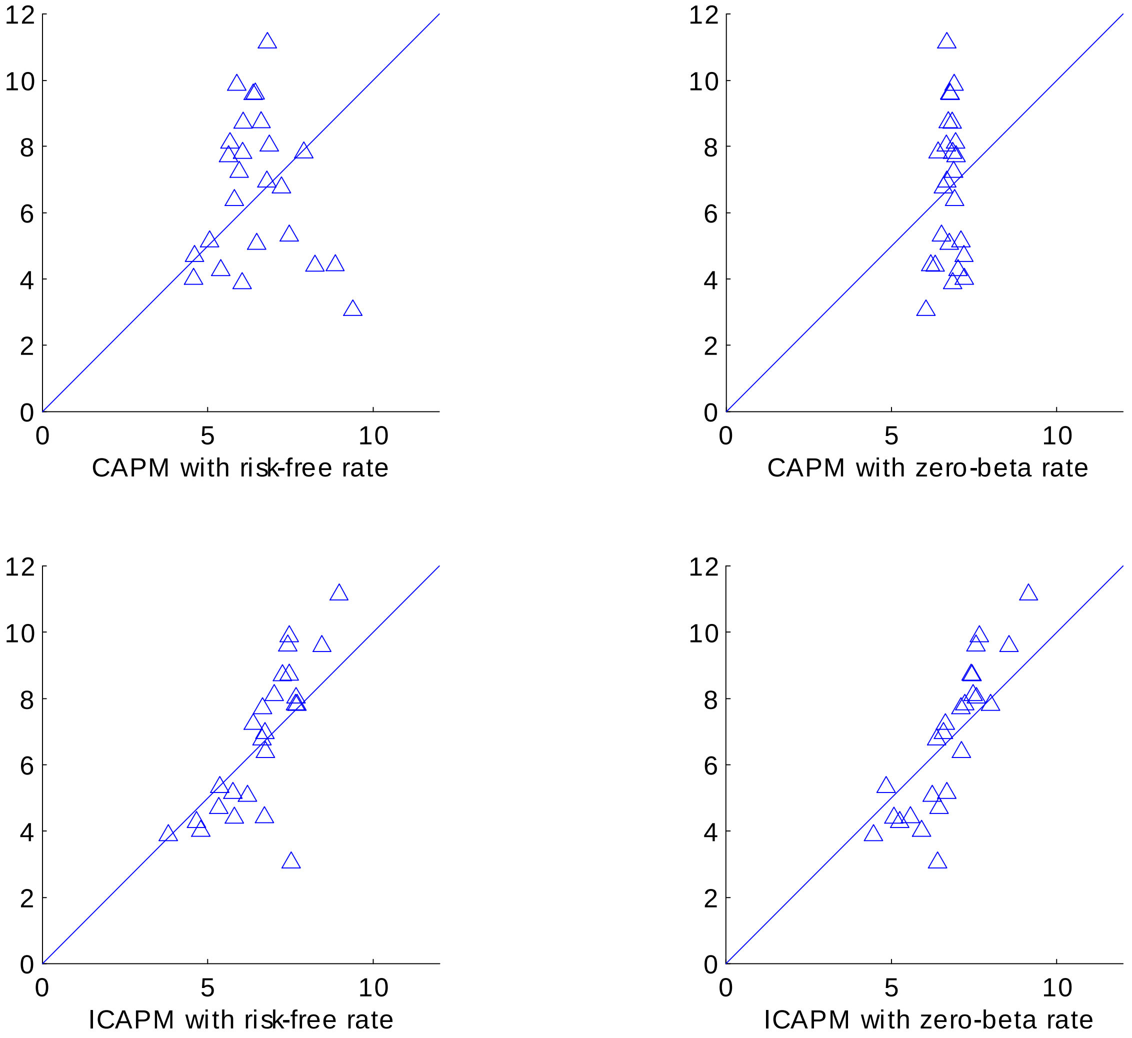

Figure 7: The four diagrams correspond to (clockwise from the top left) the CAPM with a constrained zero-beta rate, the CAPM with an unconstrained zero-beta rate, the three-factor ICAPM with a free zero-beta rate, and the three-factor ICAPM with the zero-beta rate constrained to the risk-freee rate. The horizontal axes correspond to the predicted average excess returns and the vertical axes to the sample average realized excess returns for the 25 ME- and BE/ME-sorted portfolios. The predicted values are from regressions presented in Table 7 for the sample period 1963:3-2010:4.